Generating functions

Ordinary generating functions

Def: Let $\{a_k\}_0^\infty$ be a series. The generating function corresponding to $\{a_k\}_0^\infty$ is defined by

$$g(x) = \sum_{k=0}^{\infty} a_k x^k$$

Ex: 1. Let $a_k = 1$, $k \geq 0$. Then the generating function of a_k is

$$g(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

provided that $|x| < 1$.

2. Let $a_k = 2^k$, $k \geq 0$. Then the generating function of a_n is

$$g(x) = \sum_{k=0}^{\infty} 2^k x^k = \sum_{k=0}^{\infty} (2x)^k = \frac{1}{1-2x}$$

with $|2x| < 1$

Common generating functions:

1. Let $a_k = c^k$, $k \geq 0$. Then $g(x) = \sum c^k x^k = \frac{1}{1-cx}$, $|cx| < 1$

2. Let $a_k = \binom{n}{k}$, $k \geq 0$ and n fixed. Then,

$$g(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$$

$$\boxed{\text{Recall that } (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}$$

3. Let $a_k = \binom{n+k}{k}$, k fixed and $n \geq 0$. Then,

$$g(x) = \sum_{n=0}^{\infty} \binom{n+k}{n} x^n = \frac{1}{(1-x)^{k+1}}.$$

Example of application of the generating function.

Suppose I have n apples that should be distributed to 3 people, Alice, John and Aya, such that Alice wants at most 2 apples, each of John and Aya wants at least one, John cannot eat more than 3 and Aya wants at most 2. In how many ways can we divide these apples.

The problem is equivalent to solve

$$Y_1 + Y_2 + Y_3 = n, \quad \begin{cases} 0 \leq Y_1 \leq 2 \\ 1 \leq Y_2 \leq 3 \\ 1 \leq Y_3 \leq 2 \end{cases}$$

where Y_1, Y_2, Y_3 is the number of apples that Alice, John, Aya can respectively have.

The answer is the coefficent of x^n in the following generating function:

$$f(x) = \underbrace{(x^0 + x^1 + x^2)}_{\text{the exponents are the possible values of } Y_1} \underbrace{(x^1 + x^2 + x^3)}_{\{1,2,3\} \text{ are the possible values of } Y_2} \underbrace{(x^1 + x^2)}_{\{1,2\} \text{ are the possible values of } Y_3}$$

$$f(x) = x^2 + 3x^3 + 5x^4 + 5x^5 + 3x^6 + x^7.$$

If $n=3$, there are 3 ways of dividing the apples, namely, $(0,1,2)$, $(0,2,1)$ and $(1,1,1)$.

Solve a recursive equation using generating functions.

Suppose we have the following recursive series

$$\begin{cases} a_1 = 0 \\ a_{n+1} = 8a_n + 9 \cdot 10^{n-1} \end{cases}$$ ← gives the total number of numbers of a certain length with odd number of zeros.

and we wish to find an explicit formula for a_n.

The generating function of a_n is:

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n+1} x^{n+1}$$

$$= \sum_{n=1}^{\infty} \left(8 a_n x^{n+1} + 9 \cdot 10^{n-1} x^{n+1}\right)$$

$$= \sum_{n=1}^{\infty} \left((8x) a_n x^n + (9x^2)\, 10^{n-1} x^{n-1}\right)$$

$$\Leftrightarrow \sum_{n=1}^{\infty} a_n x^n - \underbrace{a_1 x}_{=0} = 8x \sum_{n=1}^{\infty} a_n x^n + 9x^2 \sum_{n=1}^{\infty} (10x)^{n-1}$$

Let $S = \sum_{n=1}^{\infty} a_n x^n$, then

$$S = 8x S + 9x^2 \cdot \underbrace{\sum_{n=0}^{\infty} (10x)^n}_{\text{geometric series}}$$

$$\Rightarrow S - 8x S = 9x^2 \cdot \frac{1}{1-10x}$$

$$\Rightarrow (1-8x) S = \frac{9x^2}{1-10x} \Rightarrow S = \frac{9x^2}{(1-10x)(1-8x)}$$

$$S = 9x^2 \cdot \left(\frac{5}{1-10x} - \frac{4}{1-8x}\right) = 9x^2 \left(5 \sum_{n=0}^{\infty} (10x)^n - 4 \sum_{n=0}^{\infty} (8x)^n\right)$$

$$= 9 \sum_{n=2}^{\infty} \left(5 \cdot 10^{n-2} - 4 \cdot 8^{n-2}\right) x^n$$

Hence, $S = \sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} \underbrace{9\left(5 \cdot 10^{n-2} - 4 \cdot 8^{n-2}\right)}_{= a_n} x^n$