Exercises 13.2

Express the following as the sum of partial fractions:

1. $\frac{1}{(x+4)(x+3)}$
2. $\frac{2x-3}{(x-2)(x-3)}$
3. $\frac{x^2+2x+3}{(x-1)^2(x-2)}$
4. $\frac{x}{(x-1)(x-2)(x-3)}$
5. $\frac{3x^2+1}{(x-1)^2(x-2)^2}$
6. $\frac{2x+1}{(x-4)(x+2)}$
7. $\frac{2x-3}{(x-2)(x+4)}$
8. $\frac{x^2+2x+3}{(x+1)(x-2)^2}$
9. $\frac{x^2}{(x-1)(x+3)(x-5)}$
10. $\frac{3x^2+1}{(x-1)^3}$

Expand the following generating functions:

11. $\frac{1}{x+4}$
12. $\frac{1}{(x-2)^2}$
13. $\frac{1}{(x-1)^3}$
14. $\frac{x}{(x-1)(x-2)}$
15. $\frac{3x^2+1}{(x-1)^2(x-2)}$
16. $\frac{1}{x-3}$
17. $\frac{x}{(x-1)^2}$
18. $\frac{x}{(x-1)^3}$
19. $\frac{x}{(x+1)(x-2)(x+3)}$
20. $\frac{2x^2+6}{(x-1)^2(x+2)^2}$

Use generating functions to solve the following recursive functions:

21. $a_0 = 1$
    $a_n = 2a_{n-1} + 3^n$ for $n > 0$
22. $a_0 = 2$
    $a_n = 3a_{n-1} + n$ for $n > 0$
23. $a_0 = 8$
    $a_1 = 16$
    $a_n = 2a_{n-1} + 3a_{n-2}$ for $n > 1$
24. $a_0 = 1$
    $a_1 = 0$
    $a_n = 4a_{n-1} - 4a_{n-2}$ for $n > 1$
25. $a_0 = 4$
    $a_n = 2a_{n-1} - 3$ for $n > 0$
26. $a_0 = 1$
    $a_n = 2a_{n-1}$ for $n > 0$
27. $a_0 = 1$
    $a_n = 4a_{n-1} + 2n$ for $n > 0$
28. $a_0 = 1$
    $a_1 = 3$
    $a_n = 5a_{n-1} - 6a_{n-2}$ for $n > 1$
29. $a_0 = 1$
    $a_1 = 3$
    $a_n = 7a_{n-1} - 10a_{n-2}$ for $n > 1$
30. $a_0 = 1$
    $a_n = 2a_{n-1} + n - 1$ for $n > 0$
31. $a_0 = 1$
    $a_n = a_{n-1} + n$ for $n > 0$
32. $a_0 = 2$
    $a_n = 3a_{n-1} + 2^n$ for $n > 0$
33. $a_0 = 1$
    $a_n = 2a_{n-1} + 2^n + n$ for $n > 0$
34. $a_0 = 1$
    $a_1 = 8$
    $a_n = 6a_{n-1} - 9a_{n-2} + 2^n$ for $n > 1$

13.3 Generating Functions and Counting

As one might guess from the title of this section, we now explore the use of generating functions for counting. This is really a beautiful technique that one can really appreciate when and if one becomes accustomed to using it. As in the last section, we shall consider generating functions to be formal expressions and not be concerned with many of the concepts that one encounters in calculus, such as convergence, Taylor's series, integration, and derivatives.

We begin by looking at $(1+x)^n$. We know, by the binomial theorem, that

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

Chapter 13—Section 13.2

1. $\dfrac{1}{(x+3)(x+4)} = \dfrac{A}{x+4} + \dfrac{B}{x+3}$

Multiplying both sides by $(x+3)(x+4)$, $1 = A(x+3) + B(x+4)$

Letting $x = -3$, $B = 1$.

Letting $x = -4$, $A = -1$.

Therefore, $\dfrac{1}{(x+3)(x+4)} = \dfrac{-1}{x+4} + \dfrac{1}{x+3}$.

3. $\dfrac{x^2+2x+3}{(x-1)^2(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2}$

Multiplying both sides by $(x-1)^2(x-2)$,

$x^2+2x+3 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$

Letting $x = 1$, $B = -6$.

Letting $x = 2$, $C = 11$.

Letting $x = 0$, $3 = A(-1)(-2) - 6(-2) + 11$ and $A = -10$.

Therefore,

$$\frac{x^2+2x+3}{(x-1)^2(x-2)} = \frac{-10}{x-1} - \frac{6}{(x-1)^2} + \frac{11}{x-2}$$

5. $\dfrac{3x^2+1}{(x-1)^2(x-2)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2} + \dfrac{D}{(x-2)^2}$

Multiplying both sides by $(x-1)^2(x-2)^2$,

$$3x^2+1 = A(x-1)(x-2)^2 + B(x-2)^2 + C(x-1)^2(x-2) + D(x-1)^2$$

Letting $x = 1$, $B = 4$.

Letting $x = 2$, $D = 13$.

Letting $x = 0$, $2A + C = 14$.

Letting $x = 3$, $A + 2C = -14$.

Solving the last two equations simultaneously, $A = 14$ and $C = -14$.

Therefore,

$$\frac{3x^2+1}{(x-1)^2(x-2)^2} = \frac{14}{x-1} + \frac{4}{(x-1)^2} - \frac{14}{x-2} + \frac{13}{(x-2)^2}.$$

7. $\dfrac{2x-3}{(x+4)(x-2)} = \dfrac{A}{x-2} + \dfrac{B}{x+4}$

Multiplying both sides by $(x-2)(x+4)$,

$$2x-3 = A(x+4) + B(x-2)$$

Letting $x = -4$, $B = \dfrac{11}{6}$.

Letting $x = 2$, $A = \dfrac{1}{6}$.

Therefore,

$$\frac{2x-3}{(x+4)(x-2)} = \frac{\frac{1}{6}}{x-2} + \frac{\frac{11}{6}}{x+4}$$

9. $\dfrac{x^2}{(x-1)(x+3)(x-5)} = \dfrac{A}{x-1} + \dfrac{B}{x+3} + \dfrac{C}{x-5}$

Multiplying both sides by $(x-1)(x+3)(x-5)$,

$$x^2 = A(x+3)(x-5) + B(x-1)(x-5) + C(x-1)(x+3)$$

Letting $x = 1$, $A = -\dfrac{1}{16}$.

Letting $x = -3$, $B = \dfrac{9}{32}$.

Letting $x = 5$, $C = \dfrac{25}{32}$.

Therefore,

$$\frac{x^2}{(x-1)(x+3)(x-5)} = \frac{-\frac{1}{16}}{x-1} + \frac{\frac{9}{32}}{x+3} + \frac{\frac{25}{32}}{x-5}$$

11. $$\frac{1}{x+4} = \frac{\frac{1}{4}}{\frac{x}{4}+1} = \frac{1}{4}\frac{1}{1-(-\frac{x}{4})} = \frac{1}{4}\left(1 - \frac{1}{4}x + \frac{1}{16}x^2 - \frac{1}{64}x^3 + \cdots + (-\tfrac{1}{4})^n x^n + \cdots\right)$$

13. $$\frac{1}{(x-1)^3} = -\frac{1}{(1-x)^3} = -\left(1 + 3x + 6x^2 + 10x^3 + \cdots + \frac{n(n+1)}{2}x^n - \cdots\right) = -1 - 3x - 6x^2 - 10x^3 - \cdots - \frac{n(n+1)}{2}x^n - \cdots$$

15. $\dfrac{3x^2+1}{(x-1)^2(x-2)} = \dfrac{A}{(1-x)} + \dfrac{B}{(1-x)^2} + \dfrac{C}{(2-x)}$

Multiplying by $(x-1)^2(x-2)$,

$3x^2+1 = A(2-x)(1-x) + B(2-x) + C(1-x)^2$

Letting $x = 1$, $B = -4$.

Letting $x = 2$, $C = -13$.

Letting $x = 0$, $A = 10$.

$$\frac{3x^2+1}{(x-1)^2(x-2)}$$
$$= \frac{10}{(1-x)} - 4\cdot\frac{1}{(1-x)^2} - 13\frac{1}{(2-x)}$$
$$= \frac{10}{(1-x)} - 4\cdot\frac{1}{(1-x)^2} - \frac{13}{2}\frac{1}{\left(1-\frac{x}{2}\right)}$$
$$= 10\cdot(1+x+x^2+x^3+\cdots+x^n+\cdots)$$
$$-4\cdot(1+2x+3x^2+4x^3+\cdots+(n+1)x^n+\cdots)$$
$$-\frac{13}{2}\left(1+\frac{1}{2}x+\frac{1}{4}x^2+\cdots+\frac{1}{2^n}x^n+\cdots\right)$$
$$= -\tfrac{1}{2}-\tfrac{5}{4}x-\tfrac{29}{8}x^2-\cdots-\left(4n-6+\frac{13}{2^{n+1}}\right)x^n-\cdots$$

$$\frac{x}{(x-2)^2} = x\cdot\frac{1}{(2-x)^2}$$
$$= \frac{1}{4}\cdot x\cdot\frac{1}{\left(1-\frac{x}{2}\right)^2}$$
$$= \frac{1}{4}\cdot x\cdot\left(1+2\cdot\frac{1}{2}x+3\cdot\frac{1}{2^2}x^2+4\cdot\frac{1}{2^3}x^3 + \cdots+(n+1)\frac{1}{2^n}x^n+\cdots\right)$$
$$= \frac{1}{4}\left(x+x^2+\frac{3}{2^2}x^3+\frac{4}{2^3}x^4 + \cdots+\frac{n+1}{2^n}x^{n+1}+\cdots\right)$$

19. $$\frac{x}{(x+1)(x-2)(x+3)} = \frac{A}{1+x}+\frac{B}{2-x}+\frac{C}{3+x}$$

Multiplying by $(1+x)(2-x)(3+x)$,

$$-x = A(2-x)(3+x)+B(1+x)(3+x) + C(1+x)(2-x)$$

Letting $x=2$, $B = -\dfrac{2}{15}$.

Letting $x=-3$, $C = -\dfrac{3}{10}$.

Letting $x=-1$, $A = \dfrac{1}{6}$.

Therefore,

$$\frac{x}{(x-1)(x-2)(x+3)}$$
$$= \frac{1}{6}\cdot\frac{1}{1+x} - \frac{2}{15}\cdot\frac{1}{2-x} - \frac{3}{10}\cdot\frac{1}{3+x}$$
$$= \frac{1}{6}\cdot\frac{1}{1-(-x)} - \frac{1}{15}\cdot\frac{1}{\left(1-\frac{x}{2}\right)} - \frac{1}{10}\cdot\frac{1}{1-\left(-\frac{x}{3}\right)}$$
$$= \frac{1}{6}(1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots)$$
$$-\frac{1}{15}\cdot\left(1+\tfrac{1}{2}x+\tfrac{1}{4}x^2+\cdots+\tfrac{1}{2^n}x^n+\cdots\right)$$
$$-\frac{1}{10}\cdot\left(1-\frac{1}{3}x+\frac{1}{9}x^2+\cdots+\left(\frac{-1}{3}\right)^n x^n+\cdots\right)$$
$$= -\frac{1}{6}x+\frac{25}{90}x^2+\cdots + \left(\frac{(-1)^n}{6} - \frac{1}{15\cdot 2^n} - \frac{1}{10}\cdot\left(\frac{-1}{3}\right)^n\right)x^n+\cdots$$

21. $$f(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$$
$$2xf(x) = 2a_0x+2a_1x^2+2a_2x^3+\cdots+2a_nx^{n+1}+\cdots$$
$$\frac{1}{1-3x} = 1+3x+9x^2+\cdots+3^nx^n+\cdots$$
$$f(x)-2xf(x)-\frac{1}{1-3x} = a_0-1 = 0$$

Therefore,

$$(1-2x)f(x) = \frac{1}{1-3x}$$

and

$$f(x) = \frac{1}{(1-3x)(1-2x)} = \frac{A}{(1-3x)}+\frac{B}{(1-2x)}$$

Multiplying by $(1-3x)(1-2x)$, we have

$$1 = A(1-2x)+B(1-3x)$$

Letting $x=\frac{1}{2}$, we get $B=-2$.

Letting $x=\frac{1}{3}$, we get $A=3$.

Therefore,

$$f(x) = \frac{3}{(1-3x)}+\frac{-2}{(1-2x)}$$
$$= 3(1+3x+9x^2+\cdots+3^nx^n+\cdots)$$
$$-2(1+2x+4x^2+\cdots+2^nx^n+\cdots)$$

so

$$a_n = 3\cdot3^n-2\cdot2^n = 3^{n+1}-2^{n+1}$$

23. $$f(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$$
$$2xf(x) = 2a_0x+2a_1x^2+2a_2x^3+\cdots+2a_nx^{n+1}+\cdots$$
$$3x^2f(x) = 3a_0x^2+3a_1x^3+3a_2x^4+\cdots+3a_nx^{n+2}+\cdots$$
$$f(x)-2xf(x)-3x^2f(x) = a_0+a_1x-2a_0x = 8$$

so

$$(1-2x-3x^2)f(x) = 8$$

and

$$f(x) = \frac{8}{(1-3x)(1+x)} = \frac{A}{(1-3x)}+\frac{B}{1+x}$$

Multiplying by $(1-3x)(1+x)$, we have

$$8 = A(1+x)+B(1-3x)$$

Letting $x=-1$, we get $B=2$.

which is equal to

$$\frac{1}{1-x^5}\cdot\frac{1}{1-x^3}\cdot\frac{1-x^5}{1-x}\cdot\frac{x^3}{1-x}\cdot\frac{1-x^3}{1-x}=\frac{x^3}{(1-x)^3}$$

so that the generating function is

$$x^3\left(1+3x+\binom{4}{2}x^2+\cdots\binom{3+n-1}{n}x^n+\cdots\right)$$

The coefficient of x^r is $\binom{r-1}{r-3}$ and the coefficient of x^{20} is $\binom{19}{17}$.

EXAMPLE 13.14

Find a generating function whose nth coefficient gives the number of n
integral solutions of $e_1 + 4e_2 + 5e_3 + 3e_4 = n$. This is equivalent to t
number of ways of selecting n objects when objects of type two are sel
a time, objects of type three are selected 5 at a time, and objects of typ
selected 3 at a time. Therefore, the generating function is

$$(1+x+x^2+\cdots)\cdot(1+x^4+x^8+\cdots)$$
$$\cdot(1+x^5+x^{10}+\cdots)\cdot(1+x^3+x^{\ldots}$$

but this is equal to

$$\frac{1}{(1-x)(1-x^4)(1-x^5)(1-x^3)}$$

Exercises 13.3

Find generating functions whose rth coefficient gives the number of solutions of

1. $e_1 + e_2 + e_3 + e_4 = r$ where $0 \le e_2 \le 2$ and $0 \le e_4 \le 4$.
2. $e_1 + e_2 + e_3 = r$ where $0 \le e_1 \le 3$ and e_3 is even.
3. $e_1 + e_2 + e_3 + e_4 = r$ where e_1 is odd, e_2 is even, and $e_4 \ge 4$.
4. $e_1 + e_2 + e_3 + e_4 = r$ where $e_i \ge i$ for all i.
5. $e_1 + 2e_2 + 3e_3 + 4e_4 = r$.
6. $e_1 + e_2 + e_3 + e_4 = r$ where $e_1 \ge 2$ and $0 \le e_3 \le 3$.
7. $e_1 + e_2 + e_3 = r$ where $0 \le e_i \le 3$ for all i.
8. $e_1 + e_2 + e_3 + e_4 = r$ where e_1 is odd, e_2 and e_3 are even, and $e_4 \ge 2$.
9. $e_1 + e_2 + e_3 + e_4 = r$ where $e_i \ge 3$ for all i.
10. $e_1 + 2e_2 + e_3 + 2e_4 = r$.

Find a generating function used for finding the number of ways of selecting r items out of

11. 4 red, 3 blue, 6 orange, and 2 green balls.
12. 2 red, 5 green, 4 orange, and 3 green balls, i
ball of each kind must be selected.
13. 6 red, 12 black, 7 white, and 10 blue balls,
4 black balls must be selected, and an even
blue balls must be selected.
14. 6 red, 12 black, 7 white, and 10 blue balls, i
ball of each kind must be selected, an even
red balls must be selected, and an odd numb
balls must be selected.
15. 6 red, 5 blue, 4 orange, and 3 green balls.
16. 7 red, 5 green, 8 orange, and 4 white balls,
number of green and red balls must be selec
even number of orange and white balls must b
17. 5 red, 4 purple, 6 white, and 8 black balls
2 black balls, 1 purple ball, and 3 red ball
selected, and an even number of white ball
selected.
18. 9 red, 7 black, 6 white, and 11 blue balls if
balls of each kind must be selected, an odd n
red balls must be selected, and an odd numbe
balls must be selected.

Find the generating function used to determine the number of ways to get k cents, using pennies, nickels, dimes, and quarters.

In a small town of 50 registered voters, each person either votes twice or stays home, except for the mayor who votes either three or five times. Use a generating function to determine the number of ways n votes may be cast.

For $n \geq 8$, n cents postage can always be supplied using 5- and 3-cent stamps. Use a generating function to describe the number of ways n cents postage can be supplied using only 5- and 3-cent stamps.

Find a generating function to determine how many ways k pieces of chocolate can be distributed in 6 boxes if an even number of chocolates are placed in the first box, an odd number of chocolates in the second box, at least 3 in the third box, at most 3 in the fourth box, and any number in the fifth and sixth boxes.

Use a generating function to determine the number of ways k dice may thrown for a sum of n.

Find the coefficient of x^7 in the generating function formed by the expansion of $f(x) = (x + x^2 + x^3)^5$.

Find the coefficient of x^{10} in the generating function formed by the expansion of $f(x) = \dfrac{x^4}{(1-x)^5}$.

Find the coefficient of x^{17} in the generating function $f(x) = \dfrac{x^4}{1-x^5}$.

Find the coefficient of x^{19} in the generating function $f(x) = \dfrac{x^4}{1-x^5}$.

Find the coefficient of x^{20} in the generating function

$$f(x) = (x^4 + x^8 + x^{16} + \cdots)^3$$

Find the coefficient of x^{12} in the generating function

$$f(x) = (x^2 + x^3 + x^4 + \cdots)^4$$

Find the coefficient of x^{10} in the generating function

$$f(x) = (x^2 + x^3 + x^4 + x^5)(1 + x^5 + x^{10})$$

31. Find the coefficient of x^{12} in the generating function $f(x) = \dfrac{x+3}{1-2x+x^2}$.

32. Find the coefficient of x^{12} in the generating function $f(x) = \dfrac{x^3 - 3x^2}{(1-x)^4}$.

33. Find the coefficient of x^{12} in the generating function $f(x) = (1-3x)^{-4}$.

34. Find the coefficient of x^{12} in the generating function $f(x) = \dfrac{1}{(1-x^3)^4}$.

Use generating functions to find the number of ways to select 10 *balls from* 20 *red,* 20 *white, and* 20 *blue balls if*

35. At least one ball of each color is selected.

36. An even number of red balls and an even number of blue balls are chosen.

37. Use a generating function to determine how many ways there are to distribute 15 toys among 5 children, if no child can get more than 4 toys.

38. Use a generating function to determine how many ways there are to place 16 chocolates in a box if there are 4 types of chocolates and at most 4 of each of the first 3 types are selected.

39. Use a generating function to determine how many ways there to distribute 16 identical objects into 5 distinct boxes if each box contains not less than 2 or more that 6 objects.

40. Let $F(x)$ be the generating function whose coefficients give the Fibonacci numbers. Using the recurrence relation for $F(x)$, show that

$$F(x) = \frac{x}{(1-(x+x^2))}$$

Use this to expand the generating function $F(x)$. Show that the coefficient of x^{n+1} is

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

so that

$$\text{Fib}(n+1) = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

13.4 Partitions

In this section we use generating functions to describe the number of partitions of a set of n indistinguishable objects into a given number of indistinguishable boxes. This is equivalent to the number of ways of partitioning the integer n into collections of a given number of integers whose sum is n, where the collections are not distinguished by order.

Therefore,

$$\begin{aligned} a_n &= 2\cdot 2^n + (n+1)\cdot 2^n - 2 - (n+1)\\ &= (n+3)2^n - n - 3 \end{aligned}$$

Chapter 13—Section 13.3

1. $(1+x+x^2+\cdots)^2(1+x+x^2)(1+x+x^2+x^3+x^4)$

3. $(x+x^3+x^5+x^7+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x+x^2+\cdots)(x^4+x^5+x^6+x^7+\cdots)$

5. $(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)(1+x^4+x^8+\cdots)$

7. $(1+x+x^2+x^3)^3$

9. $(x^3+x^4+x^5+\cdots)^4$

11. $(1+x+x^2+x^3+x^4)(1+x+x^2+x^3)(1+x+x^2+\cdots+x^6)(1+x+x^2)$

13. $(1+x+x^2+\cdots+x^6)(x^4+x^5+\cdots+x^{11})(1+x+x^2+\cdots+x^7)(1+x^2+x^4+\cdots x^{10})$

15. $(1+x+x^2+\cdots+x^6)(1+x+x^2+\cdots+x^5)(1+x+x^2+x^3+x^4)(1+x+x^2+x^3)$

17. $(x^3+x^4+x^5)(x+x^2+x^3+x^4)(1+x^2+x^4+x^6)(x^2+x^3+x^4+x^5+x^6)$

19. $(1+x+x^2+\cdots)(1+x^5+x^{10}+\cdots)(1+x^{10}+x^{20}+\cdots)(1+x^{25}+x^{50}+\cdots)$

21. $(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)$ disregard first three terms.

23. $(x+x^2+x^3+x^4+x^5+x^6)^k$

25.
$$f(x) = \frac{x^4}{(1-x)^5} = x^4\left(1+\binom{5}{1}x + \binom{6}{2}x^2 + \cdots + \binom{5+n-1}{n}x^n + \cdots\right).$$

We want coefficient of x^6 in $(1+\binom{5}{1}x+\binom{6}{2}x^2+\cdots+\binom{5+n-1}{n}x^n+\cdots)$, which is $\binom{10}{6}$.

27. From the equation in the previous exercise, the coefficient of $x^{19} = 1$.

29.
$$\begin{aligned} &(x^2+x^3+x^4+\cdots)^4\\ &= x^8(1+x+x^2+x^3\cdots)^4\\ &= x^8\frac{1}{(1-x)^4}\\ &= x^8(1+4x+\binom{5}{2}x^2+\cdots+\binom{4+n-1}{2}x^n+\cdots) \end{aligned}$$

The coefficient of x^4 in $1+4x+\binom{5}{2}x^2+\cdots+\binom{4+n-1}{n}x^n+\cdots$ is $\binom{7}{4}$.

31.
$$\begin{aligned} \frac{x+3}{1-2x+x^2} &= \frac{x+3}{(1-x)^2}\\ &= (x+3)(1+2x+3x^2+\cdots+(n+1)x^n+\cdots). \end{aligned}$$

The coefficient of x^{12} is $12+3\cdot 13 = 51$.

33.
$$\frac{1}{(1-3x)^4} = \left(1+4\cdot 3x+\binom{5}{2}3^2x^2 + \cdots + \binom{4+n-1}{n}3^nx^n\right)+\cdots.$$

The coefficient of x^{12} is
$$\binom{4+12-1}{12}3^{12} = \binom{15}{12}\cdot 3^{12}$$

35. The generating function is
$$\begin{aligned} &f(x)\\ &= (x+x^2+x^3+\cdots)^3\\ &= x^3(1+x+x^2+x^3+\cdots)^3\\ &= x^3\frac{1}{(1-x)^3}\\ &= x^3\left(1+3x+6x^2+\cdots+\frac{(n+1)(n+2)}{2}x^n+\cdots\right) \end{aligned}$$

The coefficient of x^{10} is $\frac{(8)(9)}{2} = 36$.

37. The generating function is
$$\begin{aligned} f(x) &= (1+x+x^2+x^3+x^4)^5\\ &= \left(\frac{1-x^5}{1-x}\right)^5\\ &= (1-x^5)^5(1+5x+\cdots+\binom{n+4}{n}x^n+\cdots)\\ &= (1-5x^5+10x^{10}-10x^{15}\cdots)\\ &\quad\times(1+5x+\cdots+\binom{n+4}{n}x^n+\cdots) \end{aligned}$$

The coefficient of x^{15} is $\binom{15+4}{15} - 5\binom{10+4}{10} + 10\binom{5+4}{5} - 10 = 121$.

39. The generating function is
$$\begin{aligned} f(x) &= (x^2+x^3+x^4+x^5+x^6)^5\\ &= x^{10}(1+x+x^2+x^3+x^4)^5\\ &= x^{10}\left(\frac{1-x^5}{1-x}\right)^5\\ &x^{10}(1-x^5)^5\frac{1}{(1-x)^5}\\ &= x^{10}(1-5x^5+10x^{10}-10x^{15}\cdots)\\ &\quad\times(1+5x+\cdots+\binom{n+4}{n}x^n+\cdots) \end{aligned}$$

The coefficient of x^{16} is $\binom{6+4}{6} - 5\cdot 5 = \binom{10}{6} - 25 = 185$.