Solution to the written test for examination in MVE135
Random processes with applications, 18.08.2008

Problem 1. In pulse amplitude modulation (PAM), a PAM word consists of a sequence of pulses, where each pulse may take on a given number of amplitude levels. Suppose a PAM word is n pulses long and each pulse may take on m different levels. If each PAM word, four pulses long, is equally likely to occur and each pulse can have one of three levels $\{a, b, c\}$, what is the probability of a PAM word occurring with exactly two pulses of level c? 3p

Solution
There are $3^4 = 81$ different PAM words. Of these, $\binom{4}{2} \times 2^2 = 24$ words have exactly two pulses of level c.

$$P(2\ ones) = \frac{24}{81} = \frac{8}{27} \simeq 0.296$$

Problem 2. X and Y are independent Gaussian random variables with expectation 0 and variance 1.

(a) Evaluate $P(X < Y)$. 3p

For a fixed constant θ, consider the random variables

$$U = X\cos\theta - Y\sin\theta, \quad V = X\sin\theta + Y\cos\theta.$$

(b) Why are U and V jointly Gaussian? 3p

(c) What is the distribution of $U^2 + V^2$? 3p

Solution

(a) Since X and Y are independent and equally distributed, and also continuos, we have

$$P(X < Y) = P(Y < X), \quad P(X < Y) + P(Y < X) = 1.$$

Hence $P(X < Y = \frac{1}{2}$.

(b) The vector $[U, V]^T$ is Gaussian, because it is obtained from the Gaussian vector $[X, Y]^T$ by a non-degenerate linear transformation.

(c) Since X and Y are independent Gaussian with mean xero and variance one, $U^2 + V^2$ is a *Chi*-squared random variable with two degrees of freedom.

Problem 3. The number X of electrons counted by a receiver in an optical communication system is a Poisson random variable with rate λ, when a signal is present, and with rate $\lambda_0 < \lambda$, when a signal is absent. Suppose that a signal is present with probability p.

(a) Compute the expected value of the electrons counted. 3p

(b) Suppose the receiver uses the following decision rule:

If $X > t$, decide signal present, otherwise, decide signal absent.

Find the error probability of the above decision rule. 3p

Solution

(a)

$$E[X] = E[X|signal\ present]p + E[X|signal\ absent](1-p) = \lambda p + \lambda_0(1-p)$$

(b)

$$\begin{aligned} P_{error} &= P(error|signal\ present)p + P(error|signal\ absent)(1-p) \\ &= P(X \leq t|signal\ present)p + P(X > t|signal\ absent)(1-p) \\ &= \sum_{k=0}^{t} \frac{\lambda^k}{k!} e^{-\lambda} p + \sum_{t+1}^{\infty} \frac{\lambda_0^k}{k!} e^{-\lambda_0}(1-p) \end{aligned}$$

Problem 4. A WSS random process with a PSD $S_X(f) = \frac{1}{1+f^2}$ is input to a filter. The filter is to be designed so that the output process is white, i.t., has a constant PSD. Find the transfer function of the whitening filter. Be sure that the filter is causal. 4p

Solution
Denote the output process by W. We have $S_W(f) = c > 0$ and

$$|H(f)|^2 = c(1+f^2) = (1+jf)\sqrt{c} \times (1-jf)\sqrt{c}.$$

Thus

$$H(f) = (1+jf)\sqrt{c}.$$

Problem 5. Let $Y_n = X_n + \beta X_{n-1}$, where $\{X_n\}$ is a zero-mean, first-order autoregressive process with autocorrelation

$$R_X(k) = \sigma^2 \alpha^{|k|}, \quad |\alpha| < 1.$$

Compute the power spectral density functions of X and Y. 4p

Solution

$$\begin{aligned} S_X(f) &= \sigma^2 \sum_{-\infty}^{\infty} e^{-j2\pi fk} \alpha^{|k|} = \sigma^2 \Big[\sum_{-\infty}^{0} e^{-j2\pi fk} \alpha^{-k} + \sum_{0}^{\infty} e^{-j2\pi fk} \alpha^k - 1 \Big] \\ &= \sigma^2 \Big[\frac{1}{1-\alpha e^{j2\pi f}} + \frac{1}{1-\alpha e^{-j2\pi f}} - 1 \Big] = \sigma^2 \frac{1+\alpha^2}{1+\alpha^2 - 2\alpha \cos 2\pi f} \end{aligned}$$

$$
\begin{aligned}
R_Y(\tau) &= E\Big[(X_{n+\tau} + \beta X_{n-1+\tau})(X_n + \beta X_{n-1})\Big] \\
&= (1+\beta^2)R_X(\tau) + \beta\Big[R_X(\tau+1) + R_X(\tau-1)\Big]
\end{aligned}
$$

$$
S_Y(f) = (1+\beta^2)S_X(f) + \beta\Big[e^{j2\pi f} + e^{-j2\pi f}\Big]S_X(f) = (1+\beta^2 + 2\cos 2\pi f)S_X(f)
$$

Problem 6. The spectrum of a stationary stochastic process is to be estimated from the following data:

$$
x[n] = \{0,6,\ -0,7,\ 0,2,\ 0,3\}.
$$

Due to the small sample support, a simple AR(1)-model is exploited:

$$
x[n] + a_1 x[n-1] = e[n].
$$

Determine estimates of the AR-parameter a_1 and the white noise variance σ_e^2. Based on these, give a parametric estimate of the spectrum $P_X(e^{j\omega})$. 4p

Solution

The Yule-Walker method gives the estimate

$$
\hat{a}_1 = -\hat{r}_x[0]^{-1}\hat{r}_x[1]
$$

With the given data, the sample autocorrelation function is calculated as

$$
\hat{r}_x[0] = \frac{1}{4}(0.6^2 + 0.7^2 + 0.2^2 + 0.3^2) = 0.245
$$

and

$$
\hat{r}_x[1] = \frac{1}{4}(0.6 \times (-0.7) + (-0.7) \times 0.2 + 0.2 \times 0.3) = -0.125
$$

Thus, we get

$$
\hat{a}_1 = \frac{0.125}{0.245} \simeq 0.51.
$$

The noise variance estimate follows as

$$
\hat{\sigma}^2 = \hat{r}_x[0] + \hat{a}_1\hat{r}_x[1] \simeq 0.18
$$