Written test for the examination
"Random Processes with Applications", 2009-10-22, 14:00 - 18:00
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Allowed material: The handbook *Beta*, *Collection of formulas for MVE135*, calculators approved by Chalmers.

There are 30 total points in the examination. One needs 12 points for grade 3 (to pass), 18 points for grade 4, and 24 points for grade 5.

Problem 1. Suppose Z_1, Z_2, and Z_3 are independent Gaussian random variables with expectation zero and variance one. What is the distribution of $Z_4 = 3/5\, Z_1 + 4/5\, Z_2$? Write the joint PDF of Z_1 and Z_4. Show that $P\{|Z_1| \leq |Z_3|\} = 1/2$. 6p

Solution.

Since Z_1 and Z_2 are independent and Gaussian, Z_4 is Gaussian, too, with $E[Z_4] = 0$ and $Var(Z_4) = \frac{9}{25} + \frac{16}{25} = 1$.

Z_1 and Z_4 are jointly Gaussian with correlation coefficient $\rho = 3/5$. The joint PDF is

$$f(z_1, z_4) = \frac{1}{2\pi \times 4/5} \exp\left\{ -\frac{1}{2 \times 16/25}\left[z_1^2 - 2 \times 3/5\, z_1 z_4 + z_4^2\right]\right\}$$

$|Z_1|$ and $|Z_3|$ are continuous, independent, and equally distributed, thus

$$P\{|Z_1| < |Z_3|\} = P\{|Z_3| < |Z_1|\}.$$

Since

$$P\{|Z_1| < |Z_3|\} + P\{|Z_3| < |Z_1|\} = 1$$

we must have

$$P\{|Z_1| < |Z_3|\} = \frac{1}{2}.$$

Problem 2. The input to a communication channel is a random variable X with equiprobable values -1 and 1. The output of the channel is the random variable $Y = X + N$, where the noise random variable N is independent of X and has Gaussian distribution with mean zero and variance one. Find the probability density function of the output. Suppose you observe a negative output and have to decide whether the input was 1 or -1. What would your choice be? 6p

Solution.

$$F_Y(y) = P\{Y \leq y\} = \frac{1}{2}P\{X + N \leq y | X = -1\} + \frac{1}{2}P\{X + N \leq y | X = 1\}$$
$$= \frac{1}{2}\left[P\{N \leq y + 1\} + P\{N \leq y - 1\}\right] = \frac{1}{2}[\Phi(y+1) + \Phi(y-1)]$$

Hence

$$f_Y(y) = \frac{1}{2\sqrt{2\pi}}\left[e^{-\frac{(y+1)^2}{2}} + e^{-\frac{(y-1)^2}{2}}\right].$$

$$P\{X = 1 \,|\, Y < 0\} = \frac{P\{X = 1,\ Y < 0\}}{P\{Y < 0\}} = \frac{1/2\,\Phi(-1)}{F_Y(0)}$$
$$P\{X = -1 \,|\, Y < 0\} = \frac{P\{X = -1,\ Y < 0\}}{P\{Y < 0\}} = \frac{1/2\,\Phi(1)}{F_Y(0)}$$

When the output is negative, input -1 is more probable than 1 and we decide on -1.

Problem 3. Messages arrive in a multiplexer according to a Poisson process of rate λ messages per second.

(a) Suppose it is observed that exactly one message has arrived in the time interval $[0,\ t_0]$. Find the probability density function of the arrival time of the message. (3)

(b) Assume $\lambda = 10$. Use the CLT to estimate the probability that more then 640 messages arrive in one minute. (3)

Solution.

(a) Let T_1 be the arrival time of the first message. We have

$$P\{T_1 \le t \,|\, N(t_0) = 1\} = \frac{P\{T_1 \le t,\ N(t_0) = 1\}}{P\{N(t_0) = 1\}}$$
$$= \frac{P\{N(t) - N(0) = 1\}P\{N(t_0) - N(t) = 0\}}{P\{N(t_0) = 1\}}$$
$$= \frac{\lambda t\, e^{-\lambda t} \cdot e^{-\lambda(t_0 - t)}}{\lambda t_0 e^{-\lambda t_0}} = \frac{t}{t_0}, \quad 0 \le t \le t_0.$$

Then

$$f_{T_1|N(t_0)}(t|N(t_0) = 1) = \frac{1}{t_0}, \quad 0 \le t \le t_0.$$

The conditional distribution is uniform in $[0,\ t_0]$.

(b) Let $T_1, T_2, \ldots$ are the interarrival times of the process. The interarrival times are IID exponenrially random variables with $E[T_i] = 0.1, \quad Var(T_i) = 0.01$. Let S_k be the arrival time of the k-th message. We have $S_{641} = T_1 + T_2 + \ldots + T_{641}$. By the CLT

$$P\{S_{641} \le 60\} = P\left\{\frac{S_{641} - 0.1 \cdot 641}{\sqrt{641 \cdot 0.01}} \le \frac{60 - 0.1 \cdot 641}{\sqrt{641 \cdot 0.01}}\right\}$$
$$\approx \Phi(-1.62) = Q(1.62) = 0.055$$

Problem 4. Let $Y_n = X_n + \beta X_{n-1}$, where $\{X_n\}$ is a zero-mean, first-order autoregressive process with autocorrelation

$$R_X(k) = \sigma^2 \alpha^{|k|}, \quad |\alpha| < 1.$$

Compute the power spectral density of Y. Find a value of β for which $\{Y_n\}$ is the white noise process. 6p

Solution. From

$$\sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{-j2\pi fk} = \underbrace{\sum_{-\infty}^{0} \alpha^{-k} e^{-j2\pi fk}}_{-k=k_1: \sum_0^\infty \alpha^{k_1} e^{j2\pi fk_1}} -1 + \sum_{0}^{\infty} \alpha^{k} e^{-j2\pi fk}$$

$$= \frac{1}{1-\alpha e^{j2\pi f}} - 1 + \frac{1}{1-\alpha e^{-j2\pi f}} = \frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(2\pi f)}$$

we obtain

$$S_X(f) = \frac{\sigma^2(1-\alpha^2)}{1+\alpha^2-2\alpha\cos(2\pi f)}.$$

The unit impulse response of the system is $h_n = \delta_n + \beta\delta_{n-1}$, thus the transfer function is

$$H(f) = 1+\beta e^{-j2\pi f} \quad \text{and then} \quad |H(f)|^2 = 1+\beta^2+2\beta\cos(2\pi f).$$

Thus

$$S_Y(f) = |H(f)|^2 S_X(f) = \sigma^2(1-\alpha^2)\frac{1+\beta^2+2\beta\cos(2\pi f)}{1+\alpha^2-2\alpha\cos(2\pi f)}$$

Obviousely, when $\beta = -\alpha$ we have $S_Y(f) = \sigma^2(1-\alpha^2)$ and Y_n is then the white noise process. Another such value is $\beta = -1/\alpha$.

Problem 5. Y_n is a WSS process defined as

$$Y_n = \frac{1}{2}Y_{n-1} + X_n,$$

where X_n is the white-noise process of average power 1. Compute the autocorrelation function of Y_n. Find the unite impulse response of the filter producing the best linear predictor of Y_n from Y_{n-2} and Y_{n-3} and compute the mean-square estimation error. 6p

Solution. Multiplying by Y_{n-k} and taking expectation from both sides we obtain

$$R_Y(0) = \frac{1}{2}R_Y(1) + 1$$
$$R_Y(k) = \frac{1}{2}R_Y(k-1), \; k \geq 1.$$

The first two equations

$$R_Y(0) = \frac{1}{2}R_Y(1) + 1$$
$$R_Y(1) = \frac{1}{2}R_Y(0)$$

give $R_Y(0) = \frac{4}{3}$. Thus

$$R_Y(k) = \frac{4}{3}\left(\frac{1}{2}\right)^{|k|}, \quad k = 0, \pm 1, \ldots.$$

Let $\hat{Y}_n = z_2 Y_{n-2} + z_3 Y_{n-3}$ be the best linear predictor. The orthogonality condition

$$Y_n - \hat{Y}_n \perp Y_{n-2} \quad \text{and} \quad Y_n - \hat{Y}_n \perp Y_{n-3}$$

gives

$$z_2 R_Y(0) + z_3 R_Y(1) = R_Y(2)$$
$$z_2 R_Y(1) + z_3 R_Y(0) = R_Y(3)$$

or

$$4z_2 + 2z_3 = 1$$
$$2z_2 + 4z_3 = 1/2.$$

The solution is $z_2 = \frac{1}{4}, \quad z_3 = 0.$

$E[e_n^2] = E[(Y_n - \hat{Y}_n)e_n] = E[Y_n(Y_n - \hat{Y}_n)] = R_Y(0) - z_2 R_Y(2) = \frac{4}{3} - \frac{1}{4} \times \frac{1}{3} = 1.25$