Written test for the examination
"Random Processes with Applications", 2010-01-11, 08:30 - 12:30
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Allowed material: The handbook *Beta*, *Collection of formulas for MVE135*, calculators approved by Chalmers.

There are 30 total points in the examination. One needs 14 points for grade 3 (to pass), 18 points for grade 4, and 24 points for grade 5.

Problem 1. A power supply has five intermittent loads connected to it and each load, when in operation, draws a power of 10 W. Each load is in operation only one-quarter of the time and operates independently of all other loads. Find the mean value and the variance of the power required by the loads. If the power supply can provide only 40 W, what is the probability that it will be overloaded?

Solution

X = the # of loads in operation. We have $X \sim \text{Bin}(5, 0.25)$
The required power is $Y = 10X$.

$$E[Y] = 10E[X] = 10 * 5 * 0.25 = \boxed{12.5w}$$
$$\text{Var}(Y) = 100 \ \text{Var}(X) = 100 * 5 * 0.25 * 0.75 = \boxed{93.75w^2}$$
$$P\{Y > 40\} = P\{X > 4\} = P\{X = 5\} = (0.25)^5 = \boxed{9.77 * 10^{-4}}$$

Problem 2. A common method for detecting a signal in a presence of noise is to establish a threshold value and compare the value of any observation with this threshold. If the threshold is exceeded, it is decided that a signal is present. Sometimes, of course, noise alone will exceed the threshold and this is known as a "false alarm". Usually, it is desired to make the probability of a false alarm very small. At the same time, we would like any observation that does contain a signal plus the noise to exceed the threshold with a large probability. This is the probability of detection and it should be as close to 1 as possible. Suppose we have Gaussian noise with zero mean and variance 1 and we set a threshold level of 5 V. Compute the probability of false alarm. If a signal of value 8 V is observed in the presence of noise, what is the probability of detection? Compute the conditional mean value of the noise that exceeds the threshold when only noise is present. 6p

Solution

$\mathcal{N}$ - the noise
$\mathcal{N} \sim \mathcal{N}(0, 1)$
$P\{\text{false alarm}\} = P\{N > 5\} = Q(5) = \boxed{2.87 * 10^{-7}}$

$P\{\text{detection}\} = P\{N+8>5\} = Q(-3) = 1 - Q(3) = 1 - 1.35 * 10^{-3} = \boxed{0.9986}$

$$\begin{aligned} f_N(x|N>5) &= \frac{f_N(x)}{Q(5)}, \quad x>5 \\ E[N|N>5] &= \frac{1}{Q(5)} \int_5^\infty \frac{1}{\sqrt{2\pi}} x e^{-x^2/2} dx \\ &= \frac{1}{Q(5)\sqrt{2\pi}} \Big[-e^{-x^2/2} \Big|_5^\infty \Big] = \boxed{\frac{e^{-5^2/2}}{Q(5)\sqrt{2\pi}}} = 5.18 \end{aligned}$$

Problem 3. Consider a random process $\{X(t),\, t \geq 0\}$ that assumes values ± 1. Suppose that $X(0) = \pm 1$ with probability 1/2 and suppose that $X(t)$ changes polarity with each occurrence of an event in a Poisson process $\{N(t),\, t \geq 0\}$ of rate α. What is the distribution of $X(t)$? For $0 < t_1 < t_2 < t_3$ compute

$$P\{X(t_3) - X(t_2) = 2, X(t_2) - X(t_1) = 2\}.$$

Does the process have independent increments? (6)

Solution.

$$P\{X(t)=1\} = P\{X(t)=1|X(0)=1\}\frac{1}{2} + P\{X(t)=1|X(0)=-1\}\frac{1}{2}$$
$$\frac{1}{2}[P\{N(t) = \textit{even number}\} + P\{N(t) = \textit{odd number}\}] = \frac{1}{2}$$

Next we have

$$P\{X(t_3) - X(t_2) = 2, X(t_2) - X(t_1) = 2\} = P\{X(t_3) = 1,\, X(t_2) = -1, X(t_2) = 1, X(t_1) = -1\} = 0$$

The increments of the process are not independent, since

$$P\{X(t_3) - X(t_2) = 2\} = P\{X(t_3) = 1,\, X(t_2) = -1\} = \frac{1}{2} P\{N(t_3) - N(t_2) = \textit{odd number}\} \neq 0$$

and similarly

$$P\{X(t_2) - X(t_1) = 2\} \neq 0.$$

Problem 4. A zero-mean, WSS process $X(t)$ with autocorrelation function

$$R_X(\tau) = \begin{cases} 1 - |\tau|, & \text{if } \ |\tau| < 1 \\ 0, & \text{otherwise} \end{cases}$$

is to be processed by a filter designed so that the system output $Y(t)$ has autocorrelation function

$$R_Y(\tau) = \frac{\sin \pi\tau}{\pi\tau}.$$

Find a formula for $H(f)$, the transfer function of the filter. 6p

Solution. We have from the transform table

$$S_X(f) = \left[\frac{\sin \pi f}{\pi f}\right]^2, \quad S_Y(f) = rect(f)$$

From $S_Y(f) = |H(f)|^2 S_X(f)$ we obtain

$$H(f) = \begin{cases} \frac{\pi f}{\sin(\pi f)}, & |f| \leq 1/2 \\ 0 & |f| > 1/2 \end{cases}$$

Problem 5. The process $\{X_n\}$ is WSS with autocorrelation function

$$R_X(k) = 9(1/3)^{|k|}, \quad k = 0, \pm 1, \pm 2, ...$$

Find the optimum linear filter for estimating X_n from the observations X_{n-2} and X_{n+1} and compute the mean-square estimation error. (6)

Solution.

$$R_X(k) = 9(1/3)^{|k|}, \quad k = 0, \pm 1, \ldots$$

Consider a linear estimator

$$\hat{X}_n = h_2 X_{n-2} + h_{-1} X_{n+1}$$

By the orthogonality principle, $\hat{X}_n$ is optimal if

$$X_n - \hat{X}_n \perp X_{n-2}, \, X_n - \hat{X}_n \perp X_{n+1}.$$

This gives

$$\begin{bmatrix} R_X(0) & R_X(3) \\ R_X(3) & R_X(0) \end{bmatrix} \begin{bmatrix} h_2 \\ h_{-1} \end{bmatrix} = \begin{bmatrix} R_X(2) \\ R_X(1) \end{bmatrix}$$

and

$$\begin{bmatrix} h_2 \\ h_{-1} \end{bmatrix} = \begin{bmatrix} 9/91 \\ 30/91 \end{bmatrix} = \begin{bmatrix} 0.0989 \\ 0.3297 \end{bmatrix}$$

$$e_n^2 = R_X(0) - h_2 R_X(2) - h_{-1} R_X(1) = 7.912$$