Examination for the course Foundations of Probability Theory.
Monday, 14 January 2008, 08.30-13.30 in the V house.
Examiner: Torgny Lindvall. Telephone connect. 3574 or mobile 0705-987486.
Teacher available at the examination site around 10.00 and 11.45.
Facilities: Dictionaries, from and into English.
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A completely solved problem gives 5 credit points.

We suppose that events and random variables are defined on a probability space that we call $(\Omega, \mathcal{F}, \mathbf{P})$.

1. We have a random variable $X \sim N(0,1)$. Determine the probability density function of X^2.

2. We make repeated independent identical trials until a certain outcome with probability p occurs; this means that the number of trials has a geometric(p) distribution. Determine the probability generating function of such a random variable, we call it X, and use that to find $\mathbf{E}[X]$.

3. Let $B_1, B_2, \ldots$ be a sequence of events such that $B_1 \supseteq B_2 \supseteq B_3 \ldots$. Prove that $\mathbf{P}(B_n) \to \mathbf{P}(B)$ as $n \to \infty$ where $B = \cap_1^\infty B_n$ and use that result to show that the distribution function of a random variable is right continuous.

4. Give an example of a sequence of random variables $X_1, X_2, \ldots$ such that $X_n \to 0$ a.s. but $\mathbf{E}[X_n] \to \infty$ as $n \to \infty$.

5. Let $X_1, X_2, \ldots$ be independent 0-1 random variables, with $\mathbf{P}(X_n = 1) = n^{-\alpha}$ for $n = 1, 2, \ldots$, where $\alpha > 0$. What values can $\mathbf{P}(\sum_1^\infty X_n < \infty)$ take, and for what values of α are the different values taken?

6. We have two coins. Coin I is symmetric (shows Head or Tail with equal probability $=\frac{1}{2}$ at a toss), but coin II is skew: it shows Head with probability 0.7. It is appealing to intuition that if we choose one of the coins at random, and at a repeated number of tosses it shows Head a distinctly larger number of times than it shows Tail, then we become more and more convinced that we have chosen coin II. Prove that
$\mathbf{P}(\text{coin II was chosen} \mid \text{there are } \geq 0.6 \cdot n \text{ Heads among the first } n \text{ tosses}) \to 1$ as $n \to \infty$, under the obvious necessary condition that we choose coin II with a probability > 0.

FOR SHORT SOLUTIONS: PLEASE TURN PAGE!

Short solutions to Foundations of Probability Theory 14 January 2008.
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1. Cf. Williams [W], **Fc**, p.56. We have that $f_X(x) = \varphi(x) = (2\pi)^{-\frac{1}{2}} \cdot \exp(-x^2/2)$ for $x \in \mathbf{R}$. So $Y = X^2$ has, due to symmetry, pdf $y^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \cdot \varphi(y^{\frac{1}{2}}) = y^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \exp(-y/2)$.

2. Cf. [W], **Db**, p.53, and **C**, p.144. We have $\mathbf{P}(X = k) = q^{k-1}p$, for $k = 1, 2, \ldots$, where $q = 1 - p$. So we get the pgf $g_X(s) = \sum_1^\infty ps \cdot (qs)^{k-1} = ps/(1 - qs)$. It is now easy to derive $\mathbf{E}[X] = g_X'(1) = 1/p$.

3. Cf. [W], **A**, p.42, and **D**, p.43. We have $\mathbf{P}(B_n) = \mathbf{P}(B) + \sum_n^\infty \mathbf{P}(B_k \backslash B_{k+1})$ for all $n = 1, 2, \ldots$. Now the sum $\to 0$ as $n \to \infty$ since $\sum_1^\infty \mathbf{P}(B_k \backslash B_{k+1}) \leq 1$, and we have proved $\mathbf{P}(B_n) \to \mathbf{P}(B)$ as $n \to \infty$.
Let F be the distribution function of a random variable X, and let $x \in \mathbf{R}$. We have $\lim_{y \searrow x}(F(y) - F(x)) = \lim_{n\to\infty}(F(x + 1/n) - F(x)) = \lim_{n\to\infty} \mathbf{P}(X \in (x, x + 1/n]) = 0$, because $\cap_{n=1}^\infty \{X \in (x, x + 1/n]\} = \phi$.

4. Let $(\Omega, \mathcal{F}, \mathbf{P}) = ([0, 1], \mathcal{B}_{[0,1]}, \mathbf{P})$, where $\mathbf{P}$ is the uniform distribution on $[0, 1]$, and let $X_n(\omega) = n^2 \cdot I_{[0,1/n]}(\omega)$. Then $X_n(\omega) \to 0$ for all $\omega > 0$, but $\mathbf{E}[X_n] = n$ which of course $\to \infty$ as $n \to \infty$.

5. Cf. [W], the Borel-Cantelli Lemmata, **K**, p.45, and **E**, p.98. Since the event $\{\sum_1^\infty X_n = \infty\}$ equals the event $\{X_n = 1 \text{ i.o.}\}$, we find, using the B-C lemmata and the independence, that $\mathbf{P}(\sum_1^\infty X_n < \infty)$ equals 0 or 1, and equals 1 if and only if $\sum_1^\infty \mathbf{P}(X_n = 1) < \infty$, which is the case if and only if $\alpha > 1$. For $\alpha \leq 1$, we have $\sum_1^\infty n^{-\alpha} = \infty$, so then $\mathbf{P}(\sum_1^\infty X_n < \infty) = 0$.

6. Cf. [W], WLLN, **J**, p.107, and Bayes' Theorem, **Ca**, p.75. Let A_n be the event $\{\geq 0.6 \cdot n \text{ Heads among the first } n \text{ tosses}\}$. The WLLN yields that $\mathbf{P}(A_n|\text{coin II}) \to 1$ and $\mathbf{P}(A_n|\text{coin I}) \to 0$ as $n \to \infty$. Bayes' Theorem gives $\mathbf{P}(\text{coin II}|A_n) = \mathbf{P}(A_n|\text{coin II})p/(\mathbf{P}(A_n|\text{coin II})p + \mathbf{P}(A_n|\text{coin I})(1-p))$ which certainly tends to 1 as $n \to \infty$.