Examination for the course Foundations of Probability Theory.
Saturday, 17 January 2009, 08.30-13.30 in the V house.
Examiner: Torgny Lindvall. Telephone connect. 3574 or mobile 0705-987486.
Teacher available at the examination site around 10.00 and 11.45.
Facilities: Dictionaries, from and into English.
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A completely solved problem gives 5 credit points.

We suppose that events and random variables are defined on a probability space that we call $(\Omega, \mathcal{F}, \mathbf{P})$.

1. For a finite sequence of events $E_1, E_2, \ldots, E_n$, prove Boole's Inequality which states that $\mathbf{P}(\cup_1^n E_i) \leq \sum_1^n \mathbf{P}(E_i)$.

2. We throw a symmetric dice twice. The numbers of dots shown are independent and uniformly distributed on $\{1, 2, \ldots, 6\}$; we call these random variables X and Y. Make a detailed calculation of $\mathbf{E}[X|X + Y = 9]$.

3. Let $X_1, X_2, \ldots, X_n$ be IID 0-1 variables with $\mathbf{P}(X_i = 1) = p$, and let $Y = \sum_1^n X_i$. Then Y has a $Bin(n, p)$-distribution. Find the probability generating function of Y, and use that to determine $\mathbf{E}[Y]$.

4. The $Exp(\lambda)$-distribution, where $\lambda > 0$, has density function $\lambda \cdot e^{-\lambda x}, x > 0$. Now let U be a random variable that is uniformly distributed on $(0, 1)$. Prove that there exists a function $g : (0, 1) \to (0, \infty)$ such that $g(U)$ is $Exp(\lambda)$-distributed.

5. Let $Z_0, Z_1, Z_2, \ldots$ be a Galton-Watson branching process with $Z_0 = 1$. Prove that $\mathbf{E}[Z_n] = \mu^n$ for $n = 0, 1, 2, \ldots$, where μ is the reproduction mean, i.e., the expected number of children of an individual.

6. The random variables $X_1, X_2, X_3, \ldots$ are uniformly bounded: there exists a K such that $|X_n(\omega)| \leq K$ for all n and all $\omega \in \Omega$ We also have that $X_n \to X$ in probability. Show that $\mathbf{E}[|X_n - X|] \to 0$ as $n \to \infty$.

FOR SHORT SOLUTIONS: PLEASE TURN PAGE!

Short solutions to Foundations of Probability Theory 17 Jan. 2009.
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1. Cf. Williams, 2.2.B, p. 39. For any events F and G, we have $\mathbf{P}(F \cup G) = \mathbf{P}(F)+\mathbf{P}(G)-\mathbf{P}(F\cap G)$, so $\mathbf{P}(F\cup G) \le \mathbf{P}(F)+\mathbf{P}(G)$. Repeated use of that inequality yields: $\mathbf{P}(\cup_1^n E_i) \le \mathbf{P}(E_1) + \mathbf{P}(\cup_2^n E_i) \le \sum_1^2 \mathbf{P}(E_i) + \mathbf{P}(\cup_3^n E_i) \le \ldots \le \sum_1^n \mathbf{P}(E_i)$.

2. Cf. W-s, 9.1,A-B, p. 385f. Let A be the event $\{X+Y = 9\}$, it has probability $4/36 = 1/9$, and $\mathbf{P}(\{X = i\} \cap A) = 1/36$ for $3 \le i \le 6$, it is 0 otherwise. So $\mathbf{E}[X|A] = (\sum_3^6 i/36)/(1/9) = 4.5$.

3. Cf. W-s, 5.2,A-D, p. 143f, especially Exercise Da. With $q = 1 - p$, we find that the pgf $g_{X_i}(s)$ for an X_i equals $q + ps$, so the "Independence means multiply" rule gives $g_Y(s) = (q + ps)^n$. Since $\mathbf{E}[Y] = g_Y'(1)$ and $g_Y'(s) = n(q + ps)^{n-1}p$, we get $\mathbf{E}[Y] = np$.

4. Cf. W-s, 3.2,B, p. 50f. The $Exp(\lambda)$-distribution has distribution function $1 - e^{-\lambda x}, x > 0$, its inverse function on $(0, 1)$ equals $u \to -log(1 - u)/\lambda$. Since $1 - U$ has the same distribution as U, we may let $g(u) = -log(u)/\lambda$.

5. Cf. W-s, 9.1,K, p. 394. From the relation (K1), we have that $Z_{n+1} = X_1^{(n+1)} + X_2^{(n+1)} + \ldots + X_{Z_n}^{(n+1)}$ for $n \ge 0$, where $X_j^{(n+1)}$ is the number of children of individual j in generation n. Since $Z_0 = 1$, we certainly have $\mathbf{E}[Z_0] = 1 = \mu^0$. Now suppose we have proved that $\mathbf{E}[Z_j] = \mu^j$ for $j \le n$. Then by conditioning on Z_n we get $\mathbf{E}[Z_{n+1}] = \sum_1^\infty \mathbf{E}[Z_{n+1}|Z_n = j]\mathbf{P}(Z_n = j) = \sum_1^\infty j\mu\mathbf{P}(Z_n = j) = \mu \sum_1^\infty j\mathbf{P}(Z_n = j) = \mu\mathbf{E}[Z_n] = \mu \cdot \mu^n = \mu^{n+1}$. We have completed a prove by induction.
   One may also use the relation $g_{n+1} = g \circ g_n$, cf. W-s, Kb, p. 397; we pick the notation from there. Then we obtain the asked for expectation quicker by derivation: $\mathbf{E}[Z_{n+1}] = g_{n+1}'(1)$. But the relation $g_{n+1} = g \circ g_n$ is proved with a conditioning arguments of the type we used above!

6. Cf. W-s, 3.5,L, p. 65.