Examination for the course Foundations of Probability Theory,
Friday, 18 December 2009, 08.30-13.30 in the V house.
Examiner: Torgny Lindvall. Telephone connect. 3574 or mobile 0705-987486.
Teacher available at the examination site around 10.00 and 11.45.
Facilities: Dictionaries, from and into English.
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A completely solved problem gives 5 credit points.

We suppose that events and random variables are defined on a probability space that we call $(\Omega, \mathcal{F}, \mathbf{P})$.

1. Use probability generating functions to prove that if the random variables X and Y are independent and Poisson distributed, then $X+Y$ is Poisson distributed too.

2. The random variable X is Cauchy distributed with the density function $f_X(x) = 1/\pi(1+x^2)$, $x \in \mathbf{R}$. Determine the density function of $1/X$.

3. The random variables $X_1, X_2, \dots$ are uncorrelated, have all the same expectation, μ say, and their variances are uniformly bounded, so we have $\sup_n \mathbf{Var}[X_n] < \infty$. Formulate and prove the weak law of large numbers for $S_n, n = 1, 2, \dots$ where $S_n = \sum_1^n X_i$.

4. For the random variables X and $X_1, X_2, \dots$ we know that $|X_n| \leq 1$ for all n, and that $X_n \to X$ in probability as $n \to \infty$. Prove that $|X| \leq 1$ a.s.

5. The random variable X is N(0,1)-distributed. Determine the expectations $\mathbf{E}[X^{2k}]$ for $k = 1, 2, \dots$.
   Hint: recall that the MGF of X is $M_X(s) = e^{\frac{1}{2}s^2}$.

6. The random variables $X_1, X_2, \dots$ are exponentially distributed with the same intensity λ. Prove that $sup_n(X_n/log(n)) < \infty$ a.s.
   Hint: find a constant K such that $\mathbf{P}(X_n/log(n) > K\ i.o.) = 0$.

FOR SHORT SOLUTIONS: TURN PAGE!

Short solutions to Foundations of Probability Theory 18 Dec. 2009.
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1. Cf. Williams, p.144. Let X and Y have parameters λ and μ respectively. We find that $g_X(s) = \sum_{k=0}^{\infty} e^{-\lambda}\lambda^k s^k/k! = e^{\lambda(s-1)}$, and $g_Y(s) = e^{\mu(s-1)}$. "Independence means multiply" gives that $g_{X+Y}(s) = g_X(s) \cdot g_Y(s)$, which implies: $g_{X+Y}(s) = e^{(\lambda+\mu)(s-1)}$, so $X+Y$ is Poisson distributed with parameter $\lambda+\mu$, due to the uniqueness theorem.

2. Cf. W-s, p.55. For $x > 0$, we obtain that $\mathbf{P}(1/X \leq x) = \mathbf{P}(X \geq 1/x) = 1 - F_X(1/x)$, so $f_{1/X}(x) = -F'_X(1/x)(-x^2)$, which turns out to be $f_X(x)$! With an analogous analysis of $\mathbf{P}(1/X \leq x)$ for $x < 0$ we obtain that $1/X$ has the same distribution as X.

3. Cf. W-s, 4.3 H-J, and 3.5 K. Since the variables are uncorrelated, we get $\mathbf{Var}[S_n] = \sum_1^n \mathbf{Var}[X_i]$, so we have $\mathbf{Var}[S_n/n] \leq n \cdot \sup_i \mathbf{Var}[X_i]/n^2 = \sup_i \mathbf{Var}[X_i]/n$. Since this tends to 0 as $n \to \infty$, the WLLN for the sequence $S_n, n = 1, 2, \ldots$ follows along the ideas used for the IID case on p.107.

4. Assume that $\mathbf{P}(|X| > 1) > 0$. Then at least one of the two probabilities $\mathbf{P}(X > 1)$ or $\mathbf{P}(X < -1)$ has to be strictly positive; we may let it be the first one. Due to the Monotone-Convergence Properties of probability measures, cf. W-s, p.43, we have that $\mathbf{P}(X > 1+\epsilon) > 0$ for some $\epsilon > 0$. But the probability $\mathbf{P}(|X_n - X| > \epsilon)$ tends to 0 as $n \to \infty$ since $|X_n| \leq 1$ a.s., and we have reached a contradiction.

5. Cf. W-s, 5.5. C,G. We have that $M_X(s) = e^{\frac{1}{2}s^2}$, and the Maclaurin expansion of the exponential function gives $M_X(s) = \sum_0{}^{\infty}(\frac{1}{2}s^2)/k!$. But we also have that $M_X(s) = \mathbf{E}[e^{sX}] = \sum_{-0}^{\infty}\mathbf{E}[(sX)^k/k!]$. Comparing the coefficients for s^{2k}, $k = 1, 2, \ldots$, we obtain $\mathbf{E}[X^{2k}] = (2k)(2k-1)\cdots 1/2^k k(k-1)\cdots 1 = (2k-1)(2k-3)\cdots 1$, which sometimes is denoted by $(2k-1)!!$. We made use of the fact that if two power series are equal, then they have the same coefficients.

6. Let K be so large that $\lambda \cdot K = 2$. We find that $\mathbf{P}(X_n/log(n) > K) = \mathbf{P}(X_n > K \cdot log(n)) = e^{-\lambda \cdot K \cdot log(n)} = e^{-2 \cdot log(n)} = n^{-2}$. But $\sum_1^{\infty} n^{-2} < \infty$, so Borel-Cantelli's 1:st lemma yields that $\mathbf{P}(X_n/log(n) > K\ i.o.) = 0$, and we are done.