Examination for the course Foundations of Probability Theory,
Saturday, 16 January 2010, 08.30-13.30 in the V house.
Examiner: Torgny Lindvall. Telephone connect. 3574 or mobile 0705-987486.
Teacher available at the examination site around 10.00 and 11.45.
Facilities: Dictionaries, from and into English.
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A completely solved problem gives 5 credit points.

We suppose that events and random variables are defined on a probability space that we call $(\Omega, \mathcal{F}, \mathbf{P})$.

1. The random variable Y is uniformly distributed on [0,1]. Determine the density function, expectation and variance of $\sqrt{Y}$.

2. The random variable X has density function $e^{-x^2/2}/\sqrt{2\pi}$, $x \in R$, i.e., it is normally distributed. Find the moment generating function of X, and use that to determine $\mathbf{E}[X]$ and $\mathbf{Var}[X]$.

3. We say that a random variable X is discrete if there exists a countable set $A = \{a_1, a_2, \ldots\}$ such that $\mathbf{P}(X \in A) = 1$. Show the existence of a discrete variable with a strictly increasing distribution function.
   Hint: you have to find a countable set which is dense.

4. The random variables X and $X_1, X_2, \ldots$ have all absolute values bounded by a constant K. Prove the following Bounded-Convergence Theorem: if $X_n \to X$ in probability as $n \to \infty$, then $\mathbf{E}[X_n] \to \mathbf{E}[X]$ as $n \to \infty$.

5. Give an example of a sequence $X_1, X_2, \ldots$ of non-negative random variables such that $\mathbf{E}[X_n] \to \infty$ as $n \to \infty$, but $X_n \to 0$ a.s. as $n \to \infty$.

6. A symmetric dice is thrown N times, where N is Poisson distributed with parameter λ. Also, N is independent of the sequence of the numbers of spots shown. Let S_N denote the number of times the dice shows six spots. Determine the distribution of S_N.

FOR SHORT SOLUTIONS: PLEASE TURN PAGE!

Short solutions to Foundations of Probability Theory 16 January 2010.
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1. We let $\sqrt{Y}$ be denoted by X, and we obtain that $F_X(x) = \mathbf{P}(X \leq x) = \mathbf{P}(Y \leq x^2) = x^2$ for $x \in [0,1]$, so $f_X(x) = 2x$. We get $\mathbf{E}[X] = \int_0^1 x \cdot 2x\,dx = \mathbf{2/3}$, $\mathbf{E}[X^2] = \int_0^1 x^2 \cdot 2x\,dx = 1/2$, which gives $\mathbf{Var}[X] = \mathbf{E}[X^2] - \mathbf{E}[X]^2 = \mathbf{1/18}$.

2. Cf. Williams, 5.3 C, p.147, and 5.3 G, p.153. We find that $M_X(s) = e^{s^2/2}, \mathbf{E}[X] = M_X'(0) = 0$, and $\mathbf{Var}[X] = \mathbf{E}[X^2] = M_X''(0) = 1$.

3. Let A be the set of rational numbers; it is countable and dense. With a_i denoting the i:th rational number, $i = 1, 2, \ldots$, and with X such that $\mathbf{P}(X = a_i) = 2^{-i}$, we have produced a random variable with a strictly increasing distribution function. Indeed, for any $x, y,\ x < y$, we find that $F_X(y) - F_X(x) = \mathbf{P}(x < X \leq y) \geq \mathbf{P}(\{a\}) > 0$ where a is any rational number $\in (x, y]$.

4. Cf. W-s, 3.4 L, p.65. Notice that convergence in probability is exactly what W-s uses in his "*Heuristic proof*" !

5. Let $X_1, X_2, \ldots$ be random variables such that $\mathbf{P}(X_n = n^3) = n^{-2}$, and $\mathbf{P}(X_n = 0) = 1 - n^{-2}$. Then $\mathbf{E}[X_n] = n$, but due to the first Borel-Cantelli Lemma, $\mathbf{P}(X_n > 0 \text{ i.o.}) = 0$, so $X_n \to 0$ a.s. as $n \to \infty$. Notice that no condition on the X_n:s, such as independence, is needed!

6. Cf. W-s, 9.1 J, p.393. We use the statement of Exercise Ja and find that the probability generating generating function of S_N, g_{S_N}, satisfies $g_{S_N}(s) = g_N(g_{X_1}(s))$, where $X_i = 1$ if the dice shows six spots at the i:th throw, and $X_i = 0$ otherwise, for $i = 1, 2, \ldots$. With $1/6$ denoted by p for convenience, we have $g_N(s) = \exp(\lambda(s-1))$, and $g_{X_1}(s) = 1 - p + ps$, so we obtain $g_{S_N}(s) = \exp(\lambda(1 - p + ps - 1)) = \exp(\lambda p(s-1))$. Using the uniqueness theorem for pgf:s, we find that S_N is Poisson distributed with parameter $\lambda/6$.
It is also possible to find that using direct calculations after conditioning w.r.t. to N, but that is less elegant.