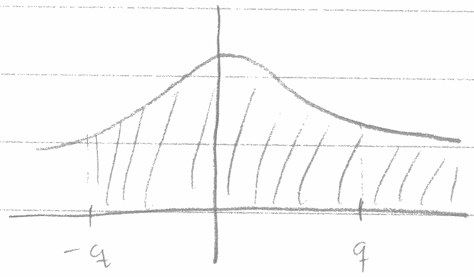
Chapter 8

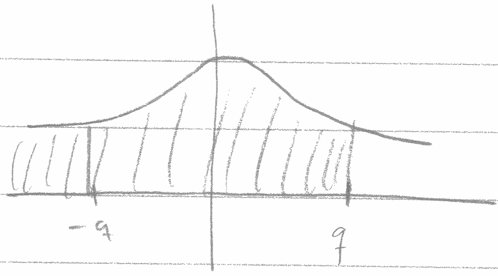
(33) $X_1, X_2, \dots, X_n \sim \text{iid } N(\mu, \sigma^2)$ where μ and σ are unknown
Choose c so $(-\infty, \bar{x}+c)$ is a 95% C.I. for μ

c should be chosen so that $P(-\infty < \mu \leq \bar{x}+c) = 0.95$, so
$P(-\infty < \mu \leq \bar{x}+c) = P(\mu \leq \bar{x}+c) = P(\mu - \bar{x} < c) = P(\bar{x}-\mu \geq -c) =$
$P\left(\frac{\bar{x}-\mu}{S_{\bar{x}}} \geq \frac{-c}{S_{\bar{x}}}\right) = P(T \geq -q) = 0.95$ where $T = \frac{\bar{x}-\mu}{S_{\bar{x}}} = \frac{\sqrt{n}(\bar{x}-\mu)}{s}$

and $-q = \frac{-c}{S_{\bar{x}}} = \frac{-\sqrt{n}\,c}{s}$



equal to



so we want $q \to t_{(n-1)}(0.95)$
$\Rightarrow c = q\frac{s}{\sqrt{n}}$ and $q = t_{(n-1)}(0.95)$

Rayleigh distribution
Let X, Y be independent variables with standard bivariate normal distribution

$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2}{2} - \frac{y^2}{2}}$$

In polar coordinates: $R = \sqrt{x^2+y^2}$
$\Theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow \Theta \sim U[0, 2\pi]$ and $R \sim \text{Rayleigh}(1)$ $\quad f_R(r) = re^{-\frac{r^2}{2}}$
$\text{Rayleigh}(\theta)$ $\quad f_R(r) = \frac{r}{\theta^2} e^{-x^2/2\theta^2}$

(50) $X_1, X_2, \ldots, X_n \sim iid \sim \text{Rayleigh}(\theta)$

$$f(x|\theta) = \frac{x}{\theta^2} \exp\{-x^2/2\theta^2\} \qquad x \geq 0$$

a) MOM estimate for θ

$$E[X] = \int_0^\infty x \, \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} dx = \int_0^\infty \frac{x^2}{\theta^2} e^{-\frac{x^2}{2\theta^2}} dx = \frac{\sqrt{2\pi}}{\theta} \int_0^\infty x^2 \frac{1}{\sqrt{2\pi}\,\theta^2} e^{-\frac{x^2}{2\theta^2}} dx$$

$$= \frac{\sqrt{2\pi}}{2\theta} E[\tilde{X}^2] \quad \text{where } \tilde{X} \sim N(0, \theta^2)$$

$$\theta^2 = Var(X) = E[\tilde{X}^2] - E^2[\tilde{X}] \implies E[\tilde{X}^2] = \theta^2$$

$$\implies E[X] = \frac{\sqrt{2\pi}}{2\theta}\theta^2 = \sqrt{\frac{\pi}{2}}\,\theta$$

$$\text{MOM: } E[X] = \bar{x} \iff \sqrt{\frac{\pi}{2}}\,\theta = \bar{x} \implies \hat{\theta} = \bar{x}\sqrt{\frac{2}{\pi}}$$

b) MLE for θ

$$L(\theta|x) = \prod_{i=1}^n \frac{x_i}{\theta_i} e^{-x_i^2/2\theta^2}$$

$$\ell(\theta|x) = \sum_{i=1}^n \log\left[\frac{x_i}{\theta^2} e^{-x_i^2/2\theta^2}\right] = \sum_{i=1}^n \left[\log(x_i) - 2\log(\theta) - \frac{x_i^2}{2\theta^2}\right]$$

$$\ell'(\theta|x) = \sum_{i=1}^n \left[-\frac{2}{\theta} + \frac{x_i^2}{\theta^3}\right] = -\frac{2n}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^n x_i^2$$

$$\ell'(\theta|x) = 0 \iff -\frac{2n}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^n x_i^2 = 0 \iff 2n = \frac{1}{\theta^2}\sum_{i=1}^n x_i^2$$

$$\iff \theta^2 = \frac{\sum x_i^2}{2n} \iff \hat{\theta} = \sqrt{\frac{\sum x_i^2}{2n}}$$

c) Find the asymptotic variance of the MLE

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \log f(\theta|x)\right] = -E\left[\frac{\partial^2}{\partial\theta^2} \log \frac{x}{\theta^2} e^{-x^2/2\theta^2}\right] = -E\left[\frac{\partial^2}{\partial\theta^2} \log x - 2\log\theta - \frac{x^2}{2\theta^2}\right]$$

$$= -E\left[\frac{\partial}{\partial\theta}\left(-\frac{2}{\theta} + \frac{1}{\theta^3}x^2\right)\right] = -E\left[\frac{2}{\theta^2} - \frac{1}{3\theta^4}x^2\right]$$

$$= -\frac{2}{\theta^2} + \frac{3}{\theta^4} E(x^2)$$

$$E(x^2) = \int_0^\infty x^2 \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} dx = \left| \begin{array}{l} y = x^2 \\ dy = 2x\,dx \end{array} \right| = \int_0^\infty \frac{y}{2\theta^2} e^{-\frac{y}{2\theta^2}} dy$$

$\int u\,dv = uv - \int v\,du$

$u = y \quad dv = \frac{1}{2\theta^2} e^{-\frac{1}{2\theta^2}y}$

$du = dy \quad v = -e^{-\frac{1}{2\theta^2}y}$

$$= -y e^{-\frac{1}{2\theta^2}y}\Big|_{y=0}^{\infty} + \int_0^\infty e^{-\frac{1}{2\theta^2}y} dy$$

$$= \int_0^\infty e^{-\frac{1}{2\theta^2}y} dy = -2\theta^2 e^{-\frac{1}{2\theta^2}y}\Big|_{y=0}^{\infty} = 2\theta^2$$

$$\Rightarrow I(\theta) = -\frac{2}{\theta^2} + \frac{3 \cdot 2\theta^2}{\theta^4} = \frac{-2}{\theta^2} + \frac{6}{\theta^2} = \frac{4}{\theta^2}$$

$$Var(\theta) \approx \frac{1}{nI(\theta)} = \frac{1}{n\,4/\theta^2} = \frac{\theta^2}{4n}$$

(73) Find a sufficient statistic for the Rayleigh density

$$f(x|\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-\frac{x_i^2}{2\theta^2}} = \frac{1}{\theta^{2n}} \prod_{i=1}^{n} x_i \, e^{-\frac{1}{2\theta^2}\Sigma x_i^2} = \frac{1}{\theta^{2n}} e^{-\frac{1}{2\theta^2}\Sigma x_i^2} \prod_{i=1}^{n} x_i$$

$$= g(T(x),\theta)\,h(x) \quad \text{where} \quad T(x) = \Sigma x_i^2 \, , \; g(t,\theta) = \frac{1}{\theta^{2n}} e^{-\frac{1}{2\theta}t}$$

and $h(x) = \prod_{i=1}^{n} x_i$

Sufficient statistic

A statistic $T(X_1, \dots, X_n)$ is said to be sufficient for the parameter θ if the conditional joint distribution of $X_1, \dots, X_n$ given $T=t$ does not depend on θ for any value of t.

Factorization theorem.

A statistic $T(X_1, \dots, X_n)$ is sufficient for the parameter $\theta \iff$ $f(x_1, \dots, x_n|\theta) = g[T(x_1, \dots, x_n), \theta]\, h(x_1, \dots, x_n)$ for some g and h

Observation: $\ell(\theta|X_1, \dots, X_n) = \log(g(T,\theta)) + \log(h)$

$$\ell'(\theta|X_1, \dots, X_n) = \frac{g'(T,\theta)T'}{g(T,\theta)}$$

so $\ell' = 0$ does not involve h

(71) Let $X_1, \dots, X_n$ be an iid sample from

$f(x|\theta) = \frac{\theta}{(1+x)^{\theta+1}}$ $\quad 0<\theta<\infty \quad 0\le x<\infty$ $\quad$ find a sufficient statistic for θ.

$f(X|\theta) = \prod_{i=1}^{n} \frac{\theta}{(1+x_i)^{\theta+1}} = \frac{\theta^n}{\prod_{i=1}^{n}(1+x_i)^{\theta+1}}$ $\quad$ If $\quad t = \prod_{i=1}^{n}(1+x_i)$

then f factors as $f(X|\theta) = g(t,\theta)h(X)$

where $g(t,\theta) = \frac{\theta^n}{t^{\theta+1}}$ and $h(X) = 1$ $\Rightarrow$ t is a sufficient statistic

Chapter 9

5) a) False

	True	False
Accept	$1-\alpha$	β Type II error
Reject	α Type I error	$1-\beta$

Significance level: α

α = probability of type I error

Power of the test: $1-\beta$

β = probability of type II error

b) False

According to Neyman-Pearson Lemma, for simple hypothesis H_0 and H_1 that rejects H_0 whenever the likelihood ratio is less than c and significance level α, any other test with less or equal significance level has the same or less power.

c) False

α is the probability of rejecting a true hypothesis

d) False

The power of the test is the probability of rejecting a false hypothesis. Here "falsely rejected" means rejecting when true: α

e) False

Not necessarily, because it can fall in the rejection region and the hypothesis can actually be false.

f) False

It's a matter of choosing the null hypothesis. In general the consequences of incorrectly rejecting one hypothesis could be as grave as those of incorrectly accepting. However, if we determine that the consequences of incorrectly rejecting one hypothesis are graver than those of incorrectly rejecting the other, then is customary to chose the former as the null hypothesis because the probability of falsely rejecting could be controlled by choosing alfa.

g) False
The null distribution is the probability distribution of the test statistic when the null hypothesis is true

h) True
It's based in the likelihood function, which is a function of the sample and therefore random

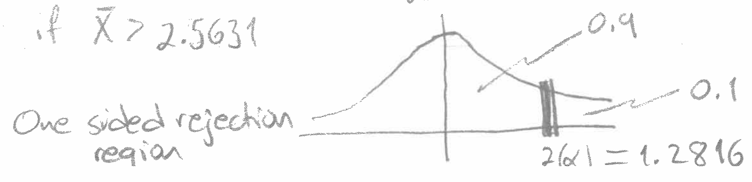
(9) $X_1, \dots, X_{25} \sim N(\mu, 100)$. Find the rejection region for a test at level $\alpha = 0.10$ of $H_0: \mu = 0$. What is the power of the test? Repeat for $\alpha = 0.01$
$H_A: \mu = 1.5$

Likelihood ratio: $$\frac{f_0(X)}{f_A(X)} = \frac{\frac{1}{\sqrt{200\pi}} \exp\left\{\frac{-1}{200} \sum_{i=1}^{25} X_i^2\right\}}{\frac{1}{\sqrt{200\pi}} \exp\left\{\frac{-1}{200} \sum_{i=1}^{25} (X_i - 1.5)^2\right\}} =$$

$$\frac{\exp\left\{\frac{-1}{200} \sum X_i^2\right\}}{\exp\left\{\frac{-1}{200} \sum X_i - 1.5)^2\right\}} = \exp\left\{\frac{1}{200}\left[\sum (X_i - 1.5)^2 - \sum X_i^2\right]\right\} =$$

$$\exp\left\{\frac{1}{200}\left[\sum X_i^2 - 3\sum X_i + \sum (1.5)^2 - \sum X_i^2\right]\right\} = \exp\left\{\frac{1}{200}\left[-3 \cdot 25\bar{X} + (1.5)^2 25\right]\right\}$$

The method rejects for small values of the likelihood ratio and small values correspond here to small values of $-3 \cdot 25\bar{X} + (1.5)^2 25$ which in turn corresponds to large values of $\bar{X}$. We reject thus for $\bar{X} > x_0$ for some x_0, and x_0 is chosen so that $P(\bar{X} > x_0) = \alpha$ if H_0 is true
Under H_0, the null distribution of $\bar{X}$ is $N(0, \frac{100}{25}) = N(0, 4)$ and
$P(\bar{X} > x_0) = P\left(\frac{\bar{X}}{2} > \frac{x_0}{2}\right) = P\left(Z > \frac{x_0}{2}\right)$ where $Z \sim N(0,1) \Rightarrow \frac{x_0}{2} = z(\alpha)$ $\xrightarrow{\alpha = 0.10}$
$x_0 = 2 \cdot 1.2816 = 2.5631$ then we reject if $\bar{X} > 2.5631$



The power of the test is the probability that the null hypothesis is rejected when it is false, that is

$1-\beta = P(\text{reject } H_0 \mid H_A) = P(\bar{X} > 2.5631 \mid H_A) = P\left(\frac{\bar{X}-1.5}{2} > \frac{2.5631-1.5}{2}\right) = P(Z > 0.5315)$

(where $Z \sim N(0,1)$) $= 1 - P(Z \leq 0.5315) = 1 - 0.7025 = 0.2975$

For $\alpha = 0.01$ the procedure is the same so $\frac{x_0}{2} = z(\alpha) \Rightarrow x_0 = 2 \cdot 2.3263 = 4.6526$

So we reject if $\bar{X} > 4.6526$. The power of the test is

$P(\bar{X} > 4.4526 \mid H_A) = P\left((\bar{X}-1.5)/2 > \frac{4.4526-1.5}{2}\right) = 1 - P(Z \leq 1.5763) = 0.0575$

(12) $X_1, X_2, \ldots, X_n \sim Exp(\theta)$: $f(x|\theta) = \theta e^{-\theta x}$. Derive a likelihood ratio test of H_0: $\theta = \theta_0$ vs H_A: $\theta \neq \theta_0$ and show that the rejection region is of the form $\{\bar{X} e^{-\theta_0 \bar{X}} \leq c\}$

likelihood ratio: $\frac{f_0(x)}{f_A(x)} = \frac{\prod_{i=1}^{n} \theta_0 e^{-\theta_0 x_i}}{\prod_{i=1}^{n} \theta e^{-\theta x_i}} = \frac{\theta_0^n e^{-\theta_0 \sum X_i}}{\theta^n e^{-\theta \sum X_i}}$

We use generalized likelihood ratio test because the alternative hypothesis is composite

$\max_{\theta \in \Omega} \theta^n e^{-\theta \sum X_i}$ is achieved for the mle estimator of θ which is $\frac{1}{\bar{X}}$

$$\Lambda = \frac{\theta_0^n e^{-\theta_0 \sum X_i}}{\left(\frac{1}{\bar{X}}\right)^n e^{-\frac{1}{\bar{X}} \sum X_i}} = \frac{\theta_0^n e^{-\theta_0 \sum X_i}}{\left(\frac{1}{\bar{X}}\right)^n e^{-n}} = \theta_0^n e^n \bar{X}^n e^{-\theta_0 n \bar{X}} = (\theta_0 e)^n (\bar{X} e^{-\theta_0 \bar{X}})^n$$

and we reject for small values of Λ which correspond to small values of $\bar{x} e^{-\theta_0 \bar{x}}$, that is, when $\bar{x} e^{-\theta_0 \bar{x}} \leq c$

(23) Yes. Such an interval has the form $(\bar{X} - \sigma_{\bar{x}} z(\alpha/2), \bar{X} + \sigma_{\bar{x}} z(\alpha/2))$ where α is the confidence level and we know that it has the property that $P\left[\frac{|\bar{X}-\mu|}{\sigma_{\bar{x}}} \le z(\alpha/2)\right] = 1-\alpha \iff P\left[|\bar{X}-\mu| > \sigma_{\bar{x}} z(\alpha/2)\right] = \alpha$ which is

*(with composite alternative hypothesis, page 340)

the condition for a test to reject at level α. In this case $(-2, 3)$ is constructed so the probability that the real value of μ being inside the interval is 0.99 therefore we reject a null hypothesis with values outside this interval with significance 0.01.

(29) A test based on test statistic T rejects if $T > t_0$ at level α. g is a monotone-increasing function and $S = g(T)$. Is the test that rejects if $S > g(t_0)$ a level α test?

Yes. We have that $P(T > t_0) = \alpha$ and, as g is monotone increasing

$\alpha = P(T > t_0) = P(g(T) > g(t_0)) = P(S > g(t_0)) \Rightarrow P(S > g(t_0)) = \alpha$

therefore the test rejects at level α.

(31) Values of the generalized likelihood ratio Λ needed to reject the null hypothesis at significance level $\alpha = 0.1$ for $n = 1, 5, 10$ and 20

As the null distribution of $-2\log\Lambda \to \chi^2_{(n)}$ [redacted] where $n = \dim \Omega - \dim \omega_0$

and $\Lambda = \dfrac{\max_{\theta \in \omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)}$ ω_0 a subset of all the possible values θ can take

$\Lambda < e^{-\frac{1}{2}\chi^2_{(n)}(1-\alpha)}$

Then we reject H_0 for big values of Λ, that is, if $-2\log\Lambda > \chi^2_{(n)}(1-\alpha)$

$n=1)$ we reject if $\Lambda < e^{-\frac{1}{2}\chi^2_{(1)}(0.90)} = 0.26 = 2.6 \times 10^{-1}$

$n=5)$ " " " $\Lambda < e^{-\frac{1}{2}\chi^2_{(5)}(0.90)} = 0.0098 = 9.8 \times 10^{-3}$

$n=10)$ " " " $\Lambda < e^{-\frac{1}{2}\chi^2_{(10)}(0.90)} = 3.37 \times 10^{-4}$

$n=20)$ " " " $\Lambda < e^{-\frac{1}{2}\chi^2_{(20)}(0.90)} = 6.77 \times 10^{-7}$

$\Rightarrow$ less likely to reject for larger samples

(37) We state H_0: Accidents are uniformly distibuted (doesn't work the other
H_1: Accidents are not uniformly distributed way around)

Under the null hypothesis, the expected frequencies of the number of deaths on each of the twelve month would be $\bar{x}$:

$$\bar{x} = \frac{1}{12}(1668 + 1407 + \dots + 1526) = 1409.\bar{6}$$

Month	Observed	Expected	$(O_i - E_i)^2 / E_i$
Jan	1668	$1409.\bar{6}$	47.3418
Feb	1407	"	0.005
Mar	1370	"	1.1162
Apr	1309	"	7.1988
May	1341	"	3.3448
Jun	1338	"	3.6435
Jul	1406	"	0.0095
Aug	1446	"	0.9365
Sep	1332	"	4.2791
Oct	1363	"	1.5444
Nov	1410	"	0.00008
Dec	1526	"	9.6005

$$\chi^2 = \sum (O_i - E_i)^2 / E_i = 79.0107$$

The number of degrees of freedom is $12 - 1 = \underline{11}$

$P(\chi^2 > 79.0107) = 2.29 \times 10^{-12}$. Which is enough evidence to reject H_0, therefore we conclude that the accidents are not uniformly distibuted.

Seasonal variance. Max in Dic-Jan, min in Mar-Jun. High incidence in Jul-Aug

(47) $X \sim \text{Poisson}(\lambda)$. Show that $Y = \sqrt{X}$ is variance-stabilizing. Note: $\mu = \lambda$, $\sigma^2 = \lambda$

If $Y = f(X)$, the method of propagation of error* shows that

$Var(Y) \approx \sigma^2(\mu)[f'(\mu)]^2 \qquad \sigma^2 = \sigma^2(\mu)$

Thus if f is chosen so that $\sigma^2(\mu)[f'(\mu)]$ is constant then the variance of Y will not depend on μ. And indeed

$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(\mu) = f'(\lambda) = \frac{1}{2\sqrt{\lambda}} \Rightarrow$$

$$Var(Y) \approx \lambda^2 \left(\frac{1}{2\sqrt{\lambda}}\right)^2 = \frac{1}{4}$$ therefore f is variance-stabilizing

* Linearization through Taylor series of $Y = g(X)$