Stochastic processes in finance

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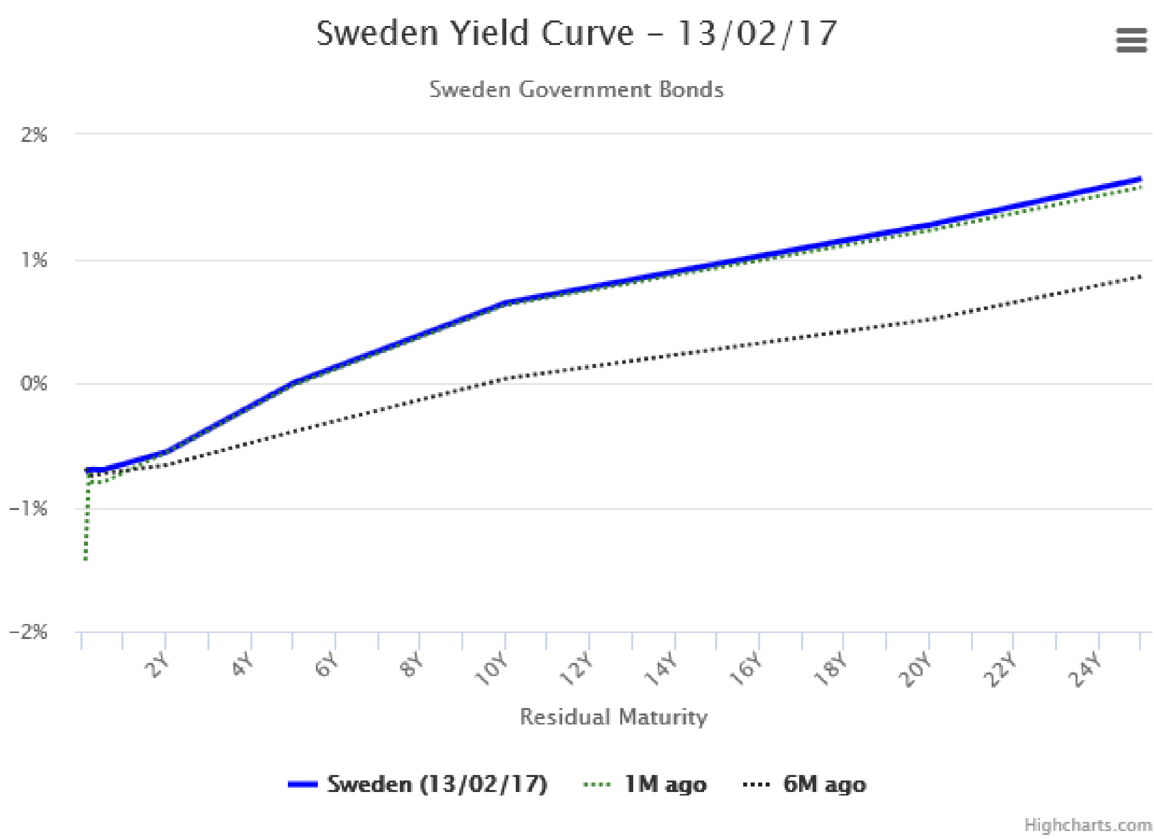


Contents

Chapter 1

The binomial options pricing model

The purpose of this section is to review the main features of the binomial pricing model for European style options. For a more detailed discussion on this topic, see Chapters 2-3 and Section 5.4 in [2].

1.1 Review of probability theory in finite spaces

We begin by recalling a few results on finite probability spaces. For more details on this subject, see Chapter 5 in [2].

Let $\Omega = \{\omega_1, \dots, \omega_m\}$ be a sample space containing m elements. Let $p = (p_1, \dots, p_m)$ such that

$$0 < p_i < 1, \text{ for all } i = 1, \dots, m, \quad \text{and} \quad \sum_{i=1}^{m} p_i = 1.$$

We define $p_i = \mathbb{P}(\{\omega_i\})$ to be the probability of the event $\{\omega_i\}$. If $A \subseteq \Omega$ is a non-empty event, we define the probability of A as

$$\mathbb{P}(A) = \sum_{i:\omega_i \in A} p_i = \sum_{\omega \in A} \mathbb{P}(\{\omega\}).$$

Moreover $\mathbb{P}(\emptyset) = 0$. The pair $(\Omega, \mathbb{P})$ is called a **finite probability space**. For example, given $p \in (0,1)$, the probability space

$$\Omega_N = \{H, T\}^N, \quad \mathbb{P}_p(\{\omega\}) = p^{N_H(\omega)}(1-p)^{N_T(\omega)}$$

is called the N**-coin toss probability space**. Here $N_H(\omega)$ is the number of heads in the toss $\omega \in \Omega_N$ and $N_T(\omega) = N - N_H(\omega)$ is the number of tails. In this probability space, tosses are independent and each toss has the same probability p to result in a head.

A **random variable** is a function $X : \Omega \to \mathbb{R}$. The **expectation** of X is denoted by $\mathbb{E}[X]$, and satisfies the properties in the following theorem.

Theorem 1. *Let X, Y be random variables, $g : \mathbb{R} \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$. The following holds:*

1. $\mathbb{E}[\alpha X + \beta Y] = \alpha\mathbb{E}[X] + \beta\mathbb{E}[Y]$ *(linearity).*
2. *If $X \geq 0$ and $\mathbb{E}[X] = 0$, then $X = 0$.*
3. *If $Y = g(X)$, i.e., if Y is X-measurable, then*

$$\mathbb{E}[g(X)] = \sum_{x \in \mathrm{Im}(X)} g(x) f_X(x), \tag{1.1}$$

where $\mathrm{Im}(X) = \{y \in \mathbb{R} : y = X(\omega) \text{ for some } \omega \in \Omega\}$ is the image of X.

4. *If X, Y are independent then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.*

The **conditional expectation** of X given Y is denoted by $\mathbb{E}[X|Y]$. It is a Y-measurable random variable and satisfies the following properties.

Theorem 2. *Let $X, Y, Z : \Omega \to \mathbb{R}$ be random variables and $\alpha, \beta \in \mathbb{R}$. Then*

1. $\mathbb{E}[\alpha X + \beta Y|Z] = \alpha\mathbb{E}[X|Z] + \beta\mathbb{E}[Y|Z]$ *(linearity).*
2. *If X is independent of Y, then $\mathbb{E}[X|Y] = \mathbb{E}[X]$.*
3. *If X is Y-measurable, then $\mathbb{E}[X|Y] = X$.*
4. $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.
5. *If X is Z-measurable, then $\mathbb{E}[XY|Z] = X\mathbb{E}[Y|Z]$.*
6. *If Z is Y-measurable then $\mathbb{E}[\mathbb{E}[X|Y]|Z] = \mathbb{E}[X|Z]$.*

These properties remain true if the conditional expectation is taken with respect to several random variables.

A **discrete stochastic process** is a (possibly finite) sequence $\{X_0, X_1, X_2, \dots\} = \{X_n\}_{n\in\mathbb{N}}$ of random variables. We refer to the index n in X_n as **time step**. If the discrete stochastic process is finite, i.e., if it runs only for a finite number $N \geq 1$ of time steps, we shall denote it by $\{X_n\}_{n=0,\dots,N}$ and call it a N**-period process**. At each time step, a discrete stochastic process on a finite probability space is a random variable with finitely many possible values. More precisely, for all $n = 0, 1, 2, \dots$, the value x_n of X_n satisfies $x_n \in \mathrm{Im}(X_n)$. We call x_n an **admissible state** of the stochastic process. Note that x_n is an admissible state if and only if $\mathbb{P}(X_n = x_n) > 0$.

Definition 1. *Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be two discrete stochastic processes on a finite* *probability space. The process $\{Y_n\}_{n\in\mathbb{N}}$ is said to be* **measurable** *with respect to $\{X_n\}_{n\in\mathbb{N}}$ if for all $n \in \mathbb{N}$ there exists a function $g_n : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $Y_n = g_n(X_0, X_2, \dots, X_n)$. If $Y_n = h_n(X_0, \dots, X_{n-1})$ for some function $h_n : \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, then $\{Y_n\}_{n\in\mathbb{N}}$ is said to be* **predictable** *from the process $\{X_n\}_{n\in\mathbb{N}}$.*

The interpretation of $\{Y_n\}_{n\in\mathbb{N}}$ being measurable (resp. predictable) with respect to $\{X_n\}_{n\in\mathbb{N}}$ is that the stochastic process $\{Y_n\}_{n\in\mathbb{N}}$ at the time step n is determined by looking at the process $\{X_n\}_{n\in\mathbb{N}}$ up to the time step n (resp. $n-1$). Of course predictable $\Rightarrow$ measurable but the opposite implication is in general not true.

A discrete stochastic process $\{X_n\}_{n\in\mathbb{N}}$ on the finite probability space $(\Omega, \mathbb{P})$ is called a **martingale** if

$$\mathbb{E}[X_{n+1}|X_1, X_2, \dots X_n] = X_n, \quad \text{for all } n \in \mathbb{N}. \tag{1.2}$$

The interpretation is the following: The variables $X_0, X_1, \dots X_n$ contains the information obtained by "looking" at the stochastic process up to the time step n. For a martingale process, this information is not enough to estimate whether, in the next step, the process will raise or fall. Martingales have constant expectation, i.e., $\mathbb{E}[X_n] = \mathbb{E}[X_0]$, for all $n \in \mathbb{N}$.

Definition 2. *A discrete stochastic process $\{X_n\}_{n\in\mathbb{N}}$ on the finite probability space $(\Omega, \mathbb{P})$ is called a* **Markov chain** *if it satisfies the* **Markov property***:*

$$\mathbb{P}(X_{n+1} = x_{n+1}|X_n = x_n) = \mathbb{P}(X_{n+1} = x_{n+1}|X_1 = x_1, X_2 = x_2, \dots, X_n = x_n), \tag{1.3}$$

for all $n \in \mathbb{N}$ and for all admissible states $x_0 \in \mathrm{Im}(X_0), \dots, x_{n+1} \in \mathrm{Im}(X_{n+1})$ such that $\mathbb{P}(X_0 = x_0, X_1 = x_1, \dots X_n = x_n)$ is positive[1]*. The left hand side of* (1.3) *is called* **transition probability** *from the state x_n to the state x_{n+1} and is denoted also as $\mathbb{P}(x_n \to x_{n+1})$. If $\mathbb{P}(x_n \to x_{n+1})$ is independent of $n \in \mathbb{N}$, the Markov process is said to be* **time homogeneous**.

The interpretation is the following: If $\{X_n\}_{n\in\mathbb{N}}$ is a Markov process, then the probability of transition from the state x_n to the state x_{n+1} (i.e., the left hand side of (1.3)) does not depend on the states occupied by the process before time n. Thus Markov processes are "memoryless": at each time step they "forget" what they did earlier.

Note that both the Markov property and the martingale property depend on the probability measure, i.e., a stochastic process can be a martingale and/or a Markov process in one probability $\mathbb{P}$ and neither of them in another probability $\mathbb{P}'$.

Exercise 1. *Prove the following.*

(a) If $\{Y_n\}_{n\in\mathbb{N}}$ is predictable from the martingale $\{X_n\}_{n\in\mathbb{N}}$, then $\{Y_n\}_{n\in\mathbb{N}}$ is also a martingale.

[1] That is to say, there must be a path of the stochastic process that connects the states $x_0 \dots, x_n$.

(b) If $\{Y_n\}_{n\in\mathbb{N}}$ is measurable with respect to the Markov chain $\{X_n\}_{n\in\mathbb{N}}$, then$\{Y_n\}_{n\in\mathbb{N}}$ is also a Markov chain

Example: Random Walk. Consider the following stochastic process $\{X_n\}_{n=1,\dots,N}$ defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$:

$$\omega = (\gamma_1, \dots, \gamma_N) \in \Omega_N, \quad X_n(\omega) = \begin{cases} 1 & \text{if } \gamma_n = H \\ -1 & \text{if } \gamma_n = T \end{cases}$$

Clearly, the random variables $X_1, \dots, X_N$ are independent and identically distributed (**i.i.d**), namely

$$\mathbb{P}_p(X_n = 1) = p, \quad \mathbb{P}_p(X_n = -1) = 1 - p, \quad \text{for all } n = 1, \dots, N.$$

Hence

$$\mathbb{E}[X_n] = 2p - 1, \quad \mathrm{Var}[X_n] = 1, \quad \text{for all } n = 1, \dots, N.$$

Now, for $n = 1, \dots, N$, let

$$M_0 = 0, \quad M_n = \sum_{i=1}^{n} X_i.$$

The stochastic process $\{M_n\}_{n=0,\dots,N}$ is measurable (but not predictable) with respect to the process $\{X_n\}_{n=1,\dots,N}$ and is called **(N-period) random walk**. It satisfies

$$\mathbb{E}[M_n] = n(1 - 2p), \quad \text{for all } n = 0, \dots, N.$$

Moreover, since it is the sum of independent random variables, the random walk has variance given by

$$\mathrm{Var}[M_0] = 0, \quad \mathrm{Var}[M_n] = \mathrm{Var}(X_1 + X_2 + \cdots + X_n) = n.$$

When $p = 1/2$, the random walk is said to be **symmetric**. In this case $\{M_n\}_{n=0,\dots,N}$ satisfies $\mathbb{E}[M_n] = 0$, $n = 0, \dots, N$. When $p \neq 1/2$, $\{M_n\}_{n=0,\dots,N}$ is called an **asymmetric** random walk, or a random walk with **drift**.

If $M_n = k$ then M_{n+1} is either $k + 1$ (with probability p), or $k - 1$ (with probability $1 - p$). Hence we can represent the paths of the random walk by using a binomial tree, as in the following example for $N = 3$:

$$\begin{tikzcd}
 & & & M_3 = 3 \\
 & & M_2 = 2 \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & M_1 = 1 \arrow[ur, "p"] \arrow[dr, "1-p"] & & M_3 = 1 \\
M_0 = 0 \arrow[ur, "p"] \arrow[dr, "1-p"] & & M_2 = 0 \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & M_1 = -1 \arrow[ur, "p"] \arrow[dr, "1-p"] & & M_3 = -1 \\
 & & M_2 = -2 \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & & & M_3 = -3
\end{tikzcd}$$

Note that the admissible states of the random walk at the step n are given by

$$\mathrm{Im}(M_n) = \{-n, -n+2, -n+4, \dots, n-2, n\} = \{-n+2j,\ j = 0, \dots, n\}.$$

Next we show that the *symmetric* random walk is a martingale. In fact, using the linearity of the conditional expectation we have

$$\begin{aligned}
\mathbb{E}[M_n|M_1, \dots, M_{n-1}] &= \mathbb{E}[M_{n-1} + X_n|M_1, \dots, M_{n-1}] \\
&= \mathbb{E}[M_{n-1}|M_1, \dots, M_{n-1}] + \mathbb{E}[X_n|M_1, \dots, M_{n-1}].
\end{aligned}$$

As M_{n-1} is measurable with respect to $M_1, \dots, M_{n-1}$, then $\mathbb{E}[M_{n-1}|M_1, \dots, M_{n-1}] = M_{n-1}$, see Theorem 2 (3). Moreover, as X_n is independent of $M_1, \dots, M_{n-1}$, Theorem 2 (2) gives $\mathbb{E}[X_n|M_1, \dots, M_{n-1}] = \mathbb{E}[X_n] = 0$. It follows that $\mathbb{E}[M_n|M_1, \dots, M_{n-1}] = M_{n-1}$, i.e., the symmetric random walk is a martingale. However the asymmetric random walk ($p \neq 1/2$) is *not* a martingale, as it follows by the fact that its expectation is not constant.

Exercise 2. *Let $\{M_n\}_{n=0,\dots,N}$ be an asymmetric random walk ($p \neq 1/2$). Show that the process $\{K_n\}_{n=0,\dots,N}$ defined by $K_n = M_n - (2p-1)n$ is a martingale. The quantity $(2p-1)n$ is called the* **drift** *of the asymmetric random walk.*

Next we show that random walks are Markov chains. Let $\{M_n\}_{n=0,\dots,N}$ be a random walk (not necessarily symmetric). Let $m_0 = 0$, $m_1 \in \{-1, 1\} = \mathrm{Im}(M_1)$, ..., $m_N \in \{-N, -N+2, \dots, N-2, N\} = \mathrm{Im}(M_N)$ be the admissible states at each time step. From the binomial tree of the process it is clear that there exists a path connecting $m_0, m_1, \dots, m_N$ if and only if $m_n = m_{n-1} \pm 1$, for all $n = 1, \dots, N$, and we have

$$\begin{aligned}\mathbb{P}(M_n = m_n | M_1 = m_1, \dots, M_{n-1} = m_{n-1}) &= \mathbb{P}(M_n = m_n | M_{n-1} = m_{n-1}) \\ &= \begin{cases} p & \text{if } m_n = m_{n-1} + 1 \\ 1 - p & \text{if } m_n = m_{n-1} - 1 \end{cases}\end{aligned}$$

Hence the random walk is an example of Markov chain. Moreover it is time homogeneous, because

$$\mathbb{P}(m_{n-1} \to m_n) = \mathbb{P}(M_n = m_n | M_{n-1} = m_{n-1}) = \begin{cases} p & \text{if } m_n = m_{n-1} + 1 \\ 1 - p & \text{if } m_n = m_{n-1} - 1 \end{cases}$$

is independent of the time step n.

Generalized random walk. A random walk is any discrete stochastic process $\{M_n\}_{n \in \mathbb{N}}$ which satisfies the following properties:

- $\mathrm{Im}(M_n) = \{-n, -n+2, -n+4, \dots, n-2, n\}$, for all $n \in \mathbb{N}$
- $\{M_n\}_{n \in \mathbb{N}}$ is a time-homogeneous Markov chain
- There exists $p \in (0,1)$ such that for $(m_{n-1}, m_n) \in \mathrm{Im}(M_{n-1}) \times \mathrm{Im}(M_n)$, the transition probability $\mathbb{P}(m_{n-1} \to m_n)$ is given by

$$\mathbb{P}(m_{n-1} \to m_n) = \begin{cases} p & \text{if } m_n = m_{n-1} + 1 \\ 1 - p & \text{if } m_n = m_{n-1} - 1 \\ 0 & \text{otherwise} \end{cases}$$

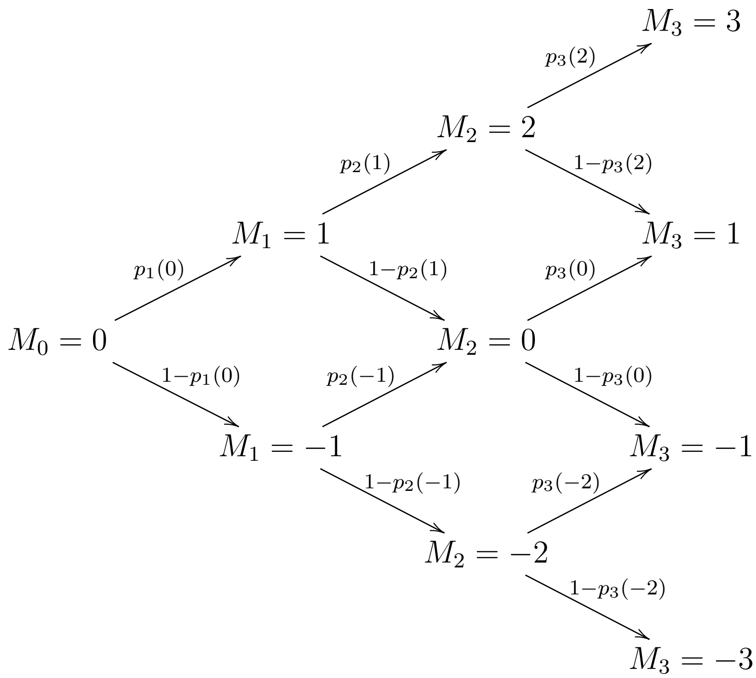
We may generalize this definition by relaxing the second and third properties as follows.

Definition 3. *A discrete stochastic process $\{M_n\}_{n \in \mathbb{N}}$ on a finite probability space is called a* **generalized random walk** *if it satisfies the following properties:*

1. $\mathrm{Im}(M_n) = \{-n, -n+2, -n+4, \dots, n-2, n\}$, *for all* $n \in \mathbb{N}$
2. $\{M_n\}_{n \in \mathbb{N}}$ *is a Markov chain*
3. *For all $n \in \mathbb{N}$ there exists $p_n : \mathrm{Im}(M_{n-1}) \to (0,1)$ such that*

$$\mathbb{P}(m_{n-1} \to m_n) = \begin{cases} p_n(m_{n-1}) & \textit{if } m_n = m_{n-1} + 1 \\ 1 - p_n(m_{n-1}) & \textit{if } m_n = m_{n-1} - 1 \\ 0 & \textit{otherwise} \end{cases}$$

The binomial tree of a generalized random walk will be written as in the following example:



Note that when $p_n \equiv p$ for all $n \in \mathbb{N}$, the generalized random walk becomes the random walk considered before.

Exercise 3. *Consider a 3-period generalized random walk with the following transition probabilities:*

$$p_1(0) = 3/4,\ p_2(1) = 3/5,\ p_2(-1) = 1/2,\ p_3(2) = 4/5,\ p_3(0) = 1/2,\ p_3(-2) = 1/5.$$

Compute $\mathbb{E}[M_n]$*, for* $n = 0, 1, 2, 3$*. Compute also* $\mathbb{P}(M_3 = 1 | M_1 = -1)$*.*

1.2 The binomial stock price

Given $0 < p < 1$, $S_0 > 0$ and $u > d$, the **binomial stock price** at time t is given by $S(0) = S_0$ and

$$S(t) = \begin{cases} S(t-1)e^u & \text{with probability } p \\ S(t-1)e^d & \text{with probability } 1-p \end{cases}, \qquad \text{for } t = 1, \dots, N. \tag{1.4}$$

If $S(t) = S(t-1)e^u$ we say that the stock price goes up at time t, while if $S(t) = S(t-1)e^d$ we say that it goes down at time t (although this terminology is strictly correct only when $u > 0$ and $d < 0$, which is often the case in the applications). The possible stock prices at time t belong to the set

$$\mathrm{Im}(S(t)) = \{S_0 e^{N_u(t)u + (t - N_u(t))d},\ N_u(t) = 0, \dots, t\},$$

where $N_u(t)$ is the number of times that the price goes up up to and including time t. For instance, for $N = 3$ the binomial tree is

$$
\begin{tikzcd}[column sep=small]
 & & & S(3) = S_0 e^{3u} \\
 & & S(2) = S_0 e^{2u} \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & S(1) = S_0 e^{u} \arrow[ur, "p"] \arrow[dr, "1-p"] & & S(3) = S_0 e^{2u+d} \\
S(0) = S_0 \arrow[ur, "p"] \arrow[dr, "1-p"] & & S(2) = S_0 e^{u+d} \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & S(1) = S_0 e^{d} \arrow[ur, "p"] \arrow[dr, "1-p"] & & S(3) = S_0 e^{u+2d} \\
 & & S(2) = S_0 e^{2d} \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & & & S(3) = S_0 e^{3d}
\end{tikzcd}
$$

The binomial stock price can be interpreted as a stochastic process defined on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$. To see this, consider the following i.i.d. random variables

$$X_t : \Omega_N \to \mathbb{R}, \quad X_t(\omega) = \left\{ \begin{array}{ll} 1, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } H \\ -1, & \text{if the } t^{th} \text{ toss in } \omega \text{ is } T \end{array} \right. , \quad t = 1, \dots, N. \tag{1.5}$$

We can rewrite (1.4) as $S(t) = S(t-1)\exp[(u+d)/2 + (u-d)X_t/2]$, which upon iteration leads to

$$S(t) = S_0 \exp\left[t \left(\frac{u+d}{2} \right) + \left(\frac{u-d}{2} \right) M_t \right], \quad M_t = X_1 + \cdots + X_t, \ t = 1, \dots, N.$$

Hence $S(t) : \Omega_N \to \mathbb{R}$ and therefore $\{S(t)\}_{t=0,\dots,N}$ is a (discrete) stochastic process on the N-coin toss probability space $(\Omega_N, \mathbb{P}_p)$. In this context, $\mathbb{P}_p$ is called **physical probability measure**, to distinguish it from the martingale (or risk-neutral) probability introduced below. Letting $M_0 = 0$, we have that $\{M_t\}_{t=0,\dots,N}$ is a random walk (which is asymmetric for $p \neq 1/2$) and therefore it is a Markov chain. As $\{S(t)\}_{t=0,\dots,N}$ is measurable with respect to $\{M_t\}_{t=0,\dots,N}$ then, by Exercise 1, the binomial stock price is a Markov chain. For each $\omega \in \Omega_N$, the vector

$$(S(0), S(1,\omega), \dots, S(N,\omega))$$

is called a **path** of the binomial stock price.

The value at time t of the **risk-free asset** is the deterministic function of time $B(t) = B_0 \exp(rt)$, where r is the (constant, continuously compounded) interest rate of the money market (**spot rate**) and B_0 is the initial value of the risk-free asset. Recall that $S^*(t) = e^{-rt}S(t)$ is called the **discounted price** of the stock (at time $t = 0$). In the following we denote by $\mathbb{E}_p$ the (possibly conditional) expectation in the probability space $(\Omega_N, \mathbb{P}_p)$.

Theorem 3. *If $r \notin (d, u)$, there is no probability measure $\mathbb{P}_p$ on the sample space Ω_N such that the discounted stock price process $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale. For $r \in (d, u)$, $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale with respect to the probability measure $\mathbb{P}_p$ if and only if $p = q$, where*

$$q = \frac{e^r - e^d}{e^u - e^d}.$$

Proof. By definition, $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale if and only if

$$\mathbb{E}_p[S^*(t)|S^*(0), \dots, S^*(t-1)] = S^*(t-1), \quad \text{for all } t = 1, \dots, N.$$

Taking the expectation conditional to $S^*(0), \dots, S^*(t-1)$ is clearly the same as taking the expectation conditional to $S(0), \dots, S(t-1)$, hence the above equation is equivalent to

$$\mathbb{E}_p[S(t)|S(0), \dots, S(t-1)] = e^r S(t-1), \quad \text{for all } t = 1, \dots, N, \tag{1.6}$$

where we canceled out a factor e^{-rt} in both sides of the equation. Moreover

$$\begin{aligned}\mathbb{E}_p[S(t)|S(0), \dots, S(t-1)] &= \mathbb{E}_p[\frac{S(t)}{S(t-1)} S(t-1)|S(0), \dots, S(t-1)] \\ &= S(t-1)\mathbb{E}_p[\frac{S(t)}{S(t-1)}|S(0), \dots, S(t-1)],\end{aligned}$$

where we used that $S(t-1)$ is measurable with respect to the conditioning variables and thus it can be taken out from the conditional expectation (see property 5 in Theorem 2). As

$$S(t)/S(t-1) = \begin{cases} e^u & \text{with prob. } p \\ e^d & \text{with prob. } 1-p \end{cases}$$

is independent of $S(0), \dots, S(t-1)$, then by Theorem 2(2) we have

$$\mathbb{E}_p[\frac{S(t)}{S(t-1)}|S(0), \dots, S(t-1)] = \mathbb{E}_p[\frac{S(t)}{S(t-1)}] = e^u p + e^d(1-p).$$

Hence (1.6) holds if and only if $e^u p + e^d(1-p) = e^r$. The latter has a solution $p \in (0, 1)$ if and only if $r \in (d, u)$ and the solution, when it exists, is unique and given by $p = q$. □

Due to Theorem 3, $\mathbb{P}_q$ is called **martingale probability measure**. Moreover, since martingales have constant expectation, then

$$\mathbb{E}_q[S(t)] = S_0 e^{rt}. \tag{1.7}$$

Thus in the martingale probability measure one expects the same return on the stock as on the risk-free asset. For this reason, $\mathbb{P}_q$ is also called **risk-neutral probability**.

1.3 Arbitrage-free markets

A **portfolio process** in a binomial market is a stochastic process $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ such that, for $t = 1, \dots, N$, $(h_S(t), h_B(t))$ corresponds to the portfolio position in the stock and the risk-free asset held in the interval $(t-1, t]$. As portfolio positions held at one instant of time only are meaningless, we use the convention $h_S(0) = h_S(1), h_B(0) = h_B(1)$, that is to say, $h_S(1), h_B(1)$ is the portfolio position in the *closed* interval $[0, 1]$. We always assume that the portfolio process is predictable, i.e., $h_S(t), h_B(t)$ are measurable with respect to $S(0), \dots, S(t-1)$. The **value of the portfolio process** is the stochastic process $\{V(t)\}_{t=0,\dots,N}$ given by

$$V(t) = h_B(t)B(t) + h_S(t)S(t), \quad t = 0, \dots, N. \tag{1.8}$$

A portfolio process $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ is said to be **self-financing** if

$$V(t-1) = h_B(t)B(t-1) + h_S(t)S(t-1), \quad t = 1, \dots, N, \tag{1.9}$$

while it is said to generate the **cash flow** $C(t-1)$ if

$$V(t-1) = h_B(t)B(t-1) + h_S(t)S(t-1) + C(t-1), \quad t = 1, \dots, N. \tag{1.10}$$

Recall that $C(t) > 0$ corresponds to cash withdrawn from the portfolio at time t while $C(t) < 0$ corresponds to cash added to the portfolio at time t. The self-financing property means that no cash is ever added or withdrawn from the portfolio.

Theorem 4. *Let $\{(h_S(t), h_B(t))\}_{t=0,\dots,N}$ be a self-financing predictable portfolio process with value $\{V(t)\}_{t=0,\dots,N}$. Then the discounted portfolio value $\{V^*(t)\}_{t=0,\dots,N}$ is a martingale in the risk-neutral probability measure. Moreover the following identity holds:*

$$V^*(t) = \mathbb{E}_q[V^*(N)|S(0), \dots, S(t)], \quad t = 0, \dots, N. \tag{1.11}$$

In particular[2]*,*

$$V(0) = \mathbb{E}_q[V^*(N)] = e^{-rN} \sum_{\omega \in \Omega_N} q^{N_H(\omega)}(1-q)^{N-N_H(\omega)} V(N, \omega).$$

Proof. The martingale claim is

$$\mathbb{E}_q[V^*(t)|V^*(0), \dots, V^*(t-1)] = V^*(t-1).$$

We now show that this follows by

$$\mathbb{E}_q[V^*(t)|S(0), \dots, S(t-1)] = V^*(t-1). \tag{1.12}$$

[2] The number of heads $N_H(\omega)$ corresponds here to the number of times that the stock price goes up in the path ω.

In fact, computing the expectation of (1.12) conditional to $V^*(0),\dots,V^*(t-1)$, we obtain

$$\begin{aligned}V^*(t-1) &= \mathbb{E}_q[V^*(t-1)|V^*(0),\dots,V^*(t-1)]\\ &= \mathbb{E}_q\big[\mathbb{E}_q[V^*(t)|S(0),\dots,S(t-1)]|V^*(0),\dots,V^*(t-1)\big]\\ &= \mathbb{E}_q[V^*(t)|V^*(0),\dots,V^*(t-1)],\end{aligned}$$

where we have used property 3 of Theorem 2 in the first step and property 6 in the last step. The latter is possible because $V^*(t)$ is measurable with respect to $S(0),\dots,S(t)$. Now we claim that (1.12) also implies the formula (1.11). We argue by backward induction. Letting $t=N$ in (1.12) we see that (1.11) holds at $t=N-1$. Assume now that (1.11) holds at time $t+1$, i.e.,

$$V^*(t+1) = \mathbb{E}_q[V^*(N)|S(0),\dots,S(t+1)].$$

Taking the expectation conditional to $S(0),\dots,S(t)$ we have, by (1.12),

$$\begin{aligned}V^*(t) &= \mathbb{E}_q[V^*(t+1)|S(0),\dots,S(t)] = \mathbb{E}_q\Big[\mathbb{E}_q[V^*(N)|S(0),\dots,S(t+1)]|S(0),\dots,S(t)\Big]\\ &= \mathbb{E}_q[V^*(N)|S(0),\dots,S(t)].\end{aligned}$$

Hence (1.11) holds at time t and so (1.12) $\Rightarrow$ (1.11), as claimed. Finally we prove (1.12). As $B(t)=B(t-1)e^r$, (1.9) gives

$$h_B(t)B(t) = e^r V(t-1) - h_S(t)S(t-1)e^r.$$

Replacing in (1.8) we find

$$V(t) = e^r V(t-1) + h_S(t)[S(t) - S(t-1)e^r].$$

Taking the expectation conditional to $S(0),\dots,S(t-1)$ we obtain

$$\begin{aligned}\mathbb{E}_q[V(t)|S(0),\dots S(t-1)] &= e^r\mathbb{E}_q[V(t-1)|S(0),\dots,S(t-1)]\\ &\quad + \mathbb{E}_q[h_S(t)(S(t)-S(t-1)e^r)|S(0),\dots,S(t-1)]. \qquad (1.13)\end{aligned}$$

As $V(t-1)$ and $h_S(t)$ are measurable with respect to the conditioning variables we have $\mathbb{E}_q[V(t-1)|S(0),\dots,S(t-1)] = V(t-1)$, as well as

$$\begin{aligned}&\mathbb{E}_q[h_S(t)(S(t)-S(t-1)e^r)|S(0),\dots,S(t-1)]\\ &= h_S(t)\mathbb{E}_q[S(t)-S(t-1)e^r|S(0),\dots,S(t-1)]\\ &= h_S(t)\Big(\mathbb{E}_q[S(t)|S(0),\dots,S(t-1)] - S(t-1)e^r\Big) = 0,\end{aligned}$$

where in the last step we used that $\{S^*(t)\}_{t=0,\dots,N}$ is a martingale in the risk-neutral probability. Going back to (1.13) we obtain

$$\mathbb{E}_q[V(t)|S(0),\dots S(t-1)] = e^r V(t-1),$$

which is the same as (1.12). $\square$

Recall that a portfolio process $\{(h_S(t), h_B(t)\}_{t=0,\dots,N}$ invested in the binomial market is called an **arbitrage portfolio process** if it is predictable and if its value $V(t)$ satisfies

1) $V(0) = 0$;
2) $V(N, \omega) \geq 0$, for all $\omega \in \Omega_N$;
3) There exists $\omega_* \in \Omega_N$ such that $V(N, \omega_*) > 0$.

A market model is said to be **arbitrage free** if it does not admit self-financing arbitrage portfolios. Now we can use the martingale property of $\{V^*(t)\}_{t=0,\dots,N}$ to give a simple proof of the absence of arbitrage in the binomial market.

Theorem 5. *Assume $d < r < u$, i.e., assume the existence of a risk-neutral probability measure for the binomial market. Then the binomial market is arbitrage free.*

Proof. Assume that $\{h_S(t), h_B(t)\}_{t=0,\dots,N}$ is a self-financing arbitrage portfolio process. Then $V(0) = V^*(0) = 0$ and since martingales have constant expectation then $\mathbb{E}_q[V^*(t)] = 0$, for all $t = 0, 1 \dots, N$. As $V(N) \geq 0$, then $V^*(N) \geq 0$ and Theorem 1(2) entails $V^*(N, \omega) = 0$ for any sample $\omega \in \Omega_N$. Hence $V(N, \omega) = 0$, for all $\omega \in \Omega_N$, contradicting the assumption that the portfolio is an arbitrage. $\square$

Remark 1. As shown in [2], the existence of a risk-neutral probability measure in not only sufficient but also necessary for the absence of arbitrage in the binomial market. More precisely, if $r \notin (d, u)$ one can construct self-financing arbitrage portfolios in the market. Hence the binomial market is arbitrage free if and only if it admits a risk-neutral probability measure. The latter result is valid for any discrete (or even continuum) market model and is known as the **first fundamental theorem of asset pricing**.

1.4 Risk neutral price of European derivatives

Now let $Y : \Omega_N \to \mathbb{R}$ be a random variable and consider the European-style derivative with pay-off Y at maturity time N. This means that the derivative can only be exercised at time $t = N$. For standard European derivatives Y is a deterministic function of $S(N)$, while for non-standard derivatives Y depends also on $S(0), \dots, S(N-1)$. Let $\Pi_Y(t)$ be the binomial fair price of the derivative a time t. By definition, $\Pi_Y(t)$ equals the value $V(t)$ of self-financing, hedging portfolios. In particular, $\Pi_Y(t)$ is a random variable and so $\{\Pi_Y(t)\}_{t=0,\dots,N}$ is a stochastic process. Using the hedging condition $V(N) = Y$ (which means $V(N, \omega) = Y(\omega)$, for all $\omega \in \Omega_N$)) and (1.11), we have the following formula for the fair price at time t of the financial derivative:

$$\Pi_Y(t) = e^{-r(N-t)}\mathbb{E}_q[Y|S(0), \dots, S(t)]. \tag{1.14}$$

Equation (1.14) is known as **risk-neutral pricing formula** and it is the cornerstone of options pricing theory. It holds not only for the binomial model but for any discrete—or even continuum —pricing model for financial derivatives. It is used for standard as well as non-standard European derivatives. In the special case $t = 0$, (1.14) reduces to

$$\Pi_Y(0) = e^{-rN}\mathbb{E}_q[Y]. \tag{1.15}$$

Example. Consider a 2-period binomial model with the following parameters

$$e^u = \frac{4}{3}, \quad e^d = \frac{2}{3}, \quad r = 0, \quad p \in (0,1).$$

Assume further that $S_0 = 36$. Consider the European derivative with pay-off

$$Y = (S(2) - 28)_+ - 2(S(2) - 32)_+ + (S(2) - 36)_+$$

and time of maturity $T = 2$. According to (1.15), the fair value of the derivative at $t = 0$ is

$$\Pi_Y(0) = e^{-2r}\mathbb{E}_q[Y] = \mathbb{E}_q[(S(2) - 28)_+] - 2\mathbb{E}_q[(S(2) - 32)_+] + \mathbb{E}_q[(S(2) - 36)_+].$$

By the market parameters we find $q = 1/2$. Hence the distribution of $S(2)$ in the risk-neutral probability measure is

$$\mathbb{P}_q(S(2) = s) = \begin{cases} 1/4 & \text{if } s = 16 \text{ of } s = 64 \\ 1/2 & \text{if } s = 32 \\ 0 & \text{otherwise} \end{cases}.$$

It follows that

$$\mathbb{E}_q[(S(2) - 28)_+] = 11, \quad \mathbb{E}_q[(S(2) - 32)_+] = 8, \quad \mathbb{E}_q[(S(2) - 36)_+] = 7,$$

hence $\Pi_Y(0) = 2$.

Exercise 4. *Use (1.14) to show that (i) the discounted binomial price of European derivatives is a martingale in the risk-neutral probability measure and (ii) the following recurrence formula holds:*

$$\Pi_Y(N) = Y, \quad \Pi_Y(t) = e^{-r}[q\Pi_Y^u(t+1) + (1-q)\Pi_Y^d(t+1)], \; t = 0, \dots, N-1, \tag{1.16}$$

where $\Pi_Y^u(t)$, resp. $\Pi_Y^d(t)$, is the price of the derivative assuming that the stock price goes up, resp. down, at time t. TIP: Use that

$$\Pi_Y^u(t+1) = e^{-r(N-(t+1))}\mathbb{E}_q[Y|S(0), \dots, S(t), S(t+1) = S(t)e^u]$$

and similarly for $\Pi_Y^d(t+1)$.

Remark 2. It can be shown that any European derivative in the binomial market can be hedged by a self-financing portfolio invested in the underlying stock and the risk-free asset, see [2]. For this reason the binomial market is called a **complete market**. In fact, the **second fundamental theorem of asset pricing** states that market completeness is equivalent to the uniqueness of the risk-neutral probability measure. An arbitrage free market is said to be **incomplete** if the risk-neutral measure is not unique. When the market is incomplete the price of European derivatives is not uniquely defined and moreover there exist European derivatives which cannot be hedged by self-financing portfolios. An example of incomplete market is the trinomial model discussed in the following exercise.

Exercise 5. *Show that the trinomial model, in which the stock price is given by*

$$S(t) = \begin{cases} S(t-1)e^u & \text{with prob. } p_u \\ S(t-1)e^m & \text{with prob. } p_m \\ S(t-1)e^d & \text{with prob. } p_d \end{cases} \quad d < m < u, \;\; p_u + p_m + p_d = 1,$$

is incomplete, i.e., the martingale probability measure is not unique. TIP: Follow the argument in the proof of Theorem 3.

1.5 Implementation of the binomial model

Recall that in real world applications the binomial model must be properly rescaled in time. Precisely, let $T > 0$ be the maturity of the European derivative and consider the uniform partition of the interval $[0, T]$ with size $h > 0$:

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad t_i - t_{i-1} = h, \quad \text{for all } i = 1, \dots, N.$$

Then the binomial stock price on the given partition is given by $S(0) = S_0$ and

$$S(t_i) = \begin{cases} S(t_{i-1})e^u, & \text{with probability } p, \\ S(t_{i-1})e^d, & \text{with probability } 1-p, \end{cases} \qquad i = 1, \dots, N,$$

while

$$B(t_i) = B_0 e^{rhi}.$$

The **instantaneous mean of log-return** and the **instantaneous variance** of the binomial stock price are defined respectively by

$$\alpha = \frac{1}{h}\mathbb{E}_p[\log S(t_i) - \log S(t_{i-1})] = \frac{1}{h}[pu + (1-p)d],$$
$$\sigma^2 = \frac{1}{h}\mathrm{Var}_p[\log S(t_i) - \log S(t_{i-1})] = \frac{(u-d)^2}{h}p(1-p).$$

The parameter σ itself is called **instantaneous volatility**. Note carefully that these parameters are constant in the standard binomial model and that they are computed with the physical probability (and *not* with the risk-neutral probability). Inverting the equations above we obtain

$$u = \alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}, \quad d = \alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}. \tag{1.17}$$

In the applications of the binomial model it is customary to give the parameters α, σ and then compute u, d using (1.17). The risk-neutral probability then becomes

$$q = \frac{e^{rh} - e^{\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}}{e^{\alpha h + \sigma\sqrt{\frac{1-p}{p}}\sqrt{h}} - e^{\alpha h - \sigma\sqrt{\frac{p}{1-p}}\sqrt{h}}}. \tag{1.18}$$

Moreover the recurrence formula (1.16) now reads

$$\Pi_Y(T) = Y, \quad \Pi_Y(t_i) = e^{-rh}[q\Pi_Y^u(t_{i+1}) + (1-q)\Pi_Y^d(t_{i+1})], \; i = 0, \dots, N-1. \tag{1.19}$$

The binomial model is trustworthy only for h very small compared to T (i.e., $N >> 1$). In the time-continuum limit $N \to \infty, h \to 0$ such that $Nh = T$, the binomial price of the stock converges in distribution to the **geometric Brownian motion** with parameters (α, σ), that is

$$S(t) = S_0 e^{\alpha t + \sigma W(t)}, \tag{1.20}$$

where $\{W(t)\}_{t\geq 0}$ is a Brownian motion (see Theorem 6.3 in [2]). In the same limit the binomial price of standard European derivatives converges to the **Black-Scholes price**, which is given by

$$\Pi_Y(t) = v(t, S(t)), \quad v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\left(x e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}y}\right) e^{-\frac{y^2}{2}}\, dy, \quad \tau = T - t, \tag{1.21}$$

where g is the pay-off function of the derivative, see [2, Section 6.2]. Note that neither the stock price $\{S(t)\}_{t\in[0,T]}$, nor the Black-Scholes price $\{\Pi_Y(t)\}_{t\in[0,T]}$ of the derivative in the time-continuum limit depend on the physical probability p. The value of p only affects the rate of convergence, the fastest one being obtained for $p = 1/2$. Hence one typically assumes $p = 1/2$ in the binomial model. Moreover the Black-Scholes price does not depend on the parameter α either, hence one also typically sets $\alpha = 0$ in the binomial approximation. Note however that the assumptions $p = 1/2$ and $\alpha = 0$ are harmless only if N is sufficiently large! Numerical codes (e.g. with Matlab) should easily and quickly handle $N \sim 10000$.

Exercise 6 (Matlab)**.** *Write a function*

$$\texttt{EuroZeroBin}(\texttt{g}, \texttt{T}, \texttt{s}, \texttt{alpha}, \texttt{sigma}, \texttt{r}, \texttt{p}, \texttt{N})$$

that computes the initial price of the standard European derivative with pay-off $Y = g(S(T))$ using (1.19)*. The variable s is the initial price S_0 of the stock. The function should also check that the market is arbitrage-free and if not it should stop and return a warning message. Show numerically that, provided N is sufficiently large, the price is weakly dependent on the parameters α, p. Show also that the fastest convergence to the Black-Scholes price as $N \to \infty$ is obtained for $p = 1/2$. Verify your code by checking the validity of the put-call parity.*

Chapter 2

The Asian option

The Asian option is a non-standard European derivative whose pay-off depends on the average movement of the stock price in a given time interval. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a given partition of the interval $[0, T]$ and let $K > 0$. The Asian call option, respectively put option, with strike K and maturity T is the European style derivative with pay-off

$$Y_{\mathrm{call}} = \left[\Big(\frac{1}{N+1}\sum_{j=0}^{N} S(t_j)\Big) - K\right]_+, \quad \text{resp.} \quad Y_{\mathrm{put}} = \left[K - \Big(\frac{1}{N+1}\sum_{j=0}^{N} S(t_j)\Big)\right]_+.$$

Let's say that we want to price this derivative in the binomial model. Assume $T = N$, $t_i = i$, so that

$$Y_{\mathrm{call}} = \left[\Big(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\Big) - K\right]_+, \quad \text{resp.} \quad Y_{\mathrm{put}} = \left[K - \Big(\frac{1}{N+1}\sum_{t=0}^{N} S(t)\Big)\right]_+$$

and $S(t)$ is given by (1.4). According to the risk-neutral pricing formula (1.15), the binomial price $\Pi_{\mathrm{AC}}(0)$ of the call and $\Pi_{\mathrm{AP}}(0)$ of the put at time $t = 0$ are given respectively by

$$\Pi_{\mathrm{AC}}(0) = e^{-rN}\mathbb{E}_q[Y_{\mathrm{call}}], \quad \Pi_{\mathrm{AP}}(0) = e^{-rN}\mathbb{E}_q[Y_{\mathrm{put}}],$$

where $\mathbb{E}_q$ is the expectation in the martingale probability measure.

Exercise 7. *Prove the following put-call parity identity:*

$$\Pi_{\mathrm{AC}}(0) - \Pi_{\mathrm{AP}}(0) = e^{-rN}\Big[\frac{S_0}{N+1}\frac{e^{r(N+1)} - 1}{e^r - 1} - K\Big].$$

HINT: For $\alpha \neq 1$, $\sum_{k=0}^{N} \alpha^k = \frac{1-\alpha^{N+1}}{1-\alpha}$.

Exercise 8 (Matlab)**.** *Write a Matlab function that computes the binomial price of Asian call/put options and verify numerically the put-call parity in the previous exercise. How large can you choose N when you run the code?*

As it is clear from the results of Exercise 8, the binomial model is unsuitable to compute the price of the Asian option. The number of paths required in the computation is too large even for moderately few steps ($N \approx 20$).

The most commonly used numerical method to compute the price of Asian options is the Monte Carlo method applied to the Black-Scholes price of the option. To describe this method we need first to define the Black-Scholes price of non-standard European derivatives, as the definition given by formula (1.21) is only valid for standard European derivatives. To this end we need to discuss a few more advanced concepts in probability.

2.1 Equivalent probabilities

When the sample space Ω is uncountable, there is no general procedure to construct a probability space, but only an abstract definition. In particular a probability measure $\mathbb{P}$ on events $A \subseteq \Omega$ is defined only axiomatically by requiring that $0 \leq \mathbb{P}(A) \leq 1$, $\mathbb{P}(\Omega) = 1$ and that, for any sequence of disjoint events $A_1, A_2, \dots$,

$$\mathbb{P}(A_1 \cup A_2 \cup \dots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots$$

Moreover it is not necessary—and almost never convenient—to assume that $\mathbb{P}$ is defined for all events $A \subset \Omega$. We denote by $\mathcal{F}$ the set of events (i.e., subsets of Ω) which have a well defined probability satisfying the properties above.

Example. Let $\Omega = \mathbb{R}$ and $\mathcal{F}$ be the set of subsets of Ω which can be written as the union of countably many intervals. Let $p : \mathbb{R} \to \mathbb{R}$ be a continuous non-negative function such that

$$\int_{\mathbb{R}} p(x)\, dx = 1.$$

Then $\mathbb{P} : \mathcal{F} \to [0, 1]$ given by

$$\mathbb{P}(A) = \int_A p(x)\, dx$$

defines a probability.

In fact in many applications, and in particular for those in financial mathematics, it suffices to define the probability only for events of the form $\{X \in I\}$, where $X : \Omega \to \mathbb{R}$ is a random variable and $I \subset \mathbb{R}$ is an interval. Moreover, in most cases, financial random variables admit a density and therefore

$$\mathbb{P}(X \in I) = \int_I f_X(x)\, dx,$$

where f_X denotes the density of X. In our case we assume that the probability space admits a Brownian motion, that is to say, a stochastic process $\{W(t)\}_{t \geq 0}$ which (almost surely) starts in zero, i.e., $W(0) = 0$, has continuous paths, independent increments and satisfies

$$W(t) - W(s) \in \mathcal{N}(0, t - s), \text{ for all } t > s \geq 0.$$

In particular $W(t) \in \mathcal{N}(0,t)$, that is

$$f_{W(t)}(x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}.$$

All stochastic processes that we consider are measurable with respect to the Brownian motion. If for example $X(t) = g(W(t))$, then

$$\mathbb{P}(X(t) \in I) = \mathbb{P}(W(t) \in g^{-1}(I)) = \int_{g^{-1}(I)} f_{W(t)}(x)\, dx.$$

Moreover

$$\mathbb{E}[X(t)] = \int_{\mathbb{R}} g(x) f_{W(t)}(x)\, dx.$$

An example of $W(t)$-measurable process is the geometric Brownian motion (1.20). We recall that the density of $S(t)$ is given by

$$f_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}} \frac{1}{x} \exp\left(-\frac{(\log x - \log S_0 - \alpha t)^2}{2\sigma^2 t}\right), \tag{2.1}$$

where $H(x)$ is the Heaviside function. The formula (2.1) can easily be proved using the density of $W(t)$, see [2, Theorem 6.2].

One further technical complication arising for uncountable sample spaces is the existence of non-trivial events with zero measure, e.g., the event $\{W(t) = a\}$ that the Brownian motion $W(t)$ takes a given value $a \in \mathbb{R}$. We shall need to consider the concept of equivalent probability measures.

Definition 4. *Two probability measure $\mathbb{P}$, $\widetilde{\mathbb{P}}$ on the events $A \in \mathcal{F}$ are said to be equivalent if $\mathbb{P}(A) = 0 \Leftrightarrow \widetilde{\mathbb{P}}(A) = 0$.*

Hence equivalent probability measures agree on which events are impossible. The following important theorem characterizes the relation between equivalent probability measures and is known as the Radon-Nikodým theorem.

Theorem 6 (Radon-Nikodým theorem)**.** *Let $\mathbb{P}, \widetilde{\mathbb{P}} : \mathcal{F} \to [0,1]$ be two probability measures and denote by $\mathbb{E}[\cdot]$, $\widetilde{\mathbb{E}}[\cdot]$ the expectation in these measures. Then $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent if and only if there exists a random variable $Z : \Omega \to \mathbb{R}$ such that $Z > 0$ (almost surely), $\mathbb{E}[Z] = 1$ and $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{I}_A]$.*

For example, assume $\Omega = \mathbb{R}$ and that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are defined by

$$\mathbb{P}(A) = \int_A p(x)\, dx, \quad \widetilde{\mathbb{P}}(A) = \int_A \widetilde{p}(x)\, dx$$

where $p(x)$, $\widetilde{p}(x)$ are two continuous non-negative functions such that

$$\int_{\mathbb{R}} p(x)\, dx = \int_{\mathbb{R}} \widetilde{p}(x) = 1.$$

Then, according to Theorem 6, $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent if and only if there exists a function $Z : \mathbb{R} \to \mathbb{R}$ such that $Z > 0$,

$$\int_{\mathbb{R}} Z(x)\, dx = 1$$

and $\widetilde{p}(x) = Z(x)p(x)$ (for almost all $x \in \mathbb{R}$).

Exercise 9. *Let $\{W(t)\}_{t\geq 0}$ be a Brownian motion in the probability measure $\mathbb{P}$. Given $\theta \in \mathbb{R}$ and $T > 0$ define*

$$Z = e^{-\theta W(T) - \frac{1}{2}\theta^2 T}.$$

Show that $\mathbb{P}_\theta(A) = \mathbb{E}[Z\mathbb{I}_A]$ defines a probability measure equivalent to $\mathbb{P}$. Remark: Note that $\mathbb{P}_\theta$ also depends on T but this is not reflected in our notation. TIP: Use Theorem 6.

Now we can state a fundamental theorem in probability theory with deep applications in financial mathematics, namely Girsanov's theorem[1].

Theorem 7. *Let $\{W(t)\}_{t\geq 0}$ be a Brownian motion in the probability measure $\mathbb{P}$. Given $\theta \in \mathbb{R}$ and $T > 0$, let $\mathbb{P}_\theta$ be the probability measure equivalent to $\mathbb{P}$ introduced in Exercise 9. Define the stochastic process $\{W_\theta(t)\}_{t\geq 0}$ by*[2]

$$W_\theta(t) = W(t) + \theta t. \tag{2.2}$$

Then $\{W_\theta(t)\}_{t\geq 0}$ is a Brownian motion in the probability measure $\mathbb{P}_\theta$.

Corollary 1. *For all $\theta \in \mathbb{R}$ and $T > 0$, the geometric Brownian motion* (1.20) *has the following density in the probability measure $\mathbb{P}_\theta$:*

$$f^{(\theta)}_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}} \frac{1}{x} \exp\left(-\frac{(\log x - \log S_0 - (\alpha - \theta\sigma)t)^2}{2\sigma^2 t} \right). \tag{2.3}$$

Proof. Since

$$S(t) = S_0 e^{\alpha t + \sigma W(t)} = S_0 e^{(\alpha - \theta\sigma)t + \sigma W_\theta(t)}$$

and $\{W_\theta(t)\}_{t\geq 0}$ is a Brownian motion in the probability measure $\mathbb{P}_\theta$, then the density $f^{(\theta)}_{S(t)}$ is the same as $f_{S(t)}$ with α replaced by $\alpha - \theta\sigma$. □

2.2 Risk-neutral pricing formula in Black-Scholes markets

Our next purpose it to derive the risk-neutral pricing formula for the time-continuum Black-Scholes model. Motivated by the discussion in the time-discrete binomial model, we first look

[1] Actually we consider only a special case of this theorem, which suffices for our purposes.

[2] The process $\{W_\theta(t)\}_{t\geq 0}$ is a Brownian motion with drift θt in the probability measure $\mathbb{P}$, but a pure Brownian motion without drift in the equivalent probability measure $\mathbb{P}_\theta$. In other words, the drift vanishes when we pass to the new probability $\mathbb{P}_\theta$.

for a probability measure which makes the discounted geometric Brownian motion $\{S(t)\}_{t\geq 0}$ a martingale (martingale probability measure). It is natural to approach this problem by searching for θ so that the probability $\mathbb{P}_\theta$ defined in Exercise 9 is risk-neutral. Let $\mathbb{E}_\theta[\cdot]$ denote the expectation in the measure $\mathbb{P}_\theta$. Recall that in the risk-neutral measure it must hold that $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$. This condition alone suffices to single out a unique possible value of θ.

Theorem 8. *The identity $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$ holds if and only if $\theta = q$, where*

$$q = \frac{\alpha - r}{\sigma} + \frac{\sigma}{2}. \tag{2.4}$$

Proof. Using the density (2.3) of $S(t)$ in the measure $\mathbb{P}_\theta$ we have

$$\mathbb{E}_\theta[S(t)] = \int_{\mathbb{R}} x f^{(\theta)}_{S(t)}(x)\,dx = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_0^\infty \exp\left(-\frac{(\log x - \log S_0 - (\alpha - \theta\sigma)t)^2}{2\sigma^2 t}\right) dx.$$

With the change of variable $y = \frac{\log x - \log S_0 - (\alpha - \theta\sigma)t}{\sigma\sqrt{t}}$ we obtain

$$\mathbb{E}_\theta[S(t)] = S_0 e^{(\alpha - \theta\sigma + \frac{\sigma^2}{2})t}$$

by which the result follows. □

Although the validity of $\mathbb{E}_\theta[S(t)] = S_0 e^{rt}$ is only necessary for the discounted geometric Brownian motion to be a martingale, one can prove the following:

Theorem 9. *The discounted value of the geometric Brownian motion stock price is a martingale in the probability measure $\mathbb{P}_\theta$ if and only if $\theta = q$, where q is given by* (2.4). *In particular the Black-Scholes market admits a unique risk-neutral probability measure and therefore this market is complete.*

Remark 3. Note that the risk-neutral probability and the physical probability are equivalent. This condition is imposed in any time-continuum model and is clearly well-motivated: one does not want to make the impossible possible when going from the physical world to the risk-neutral world. In the time-discrete case it is trivially satisfied since all probability measures in a finite probability space are equivalent.

Replacing $\alpha = q$ in (2.3) we obtain the density of the geometric Brownian motion in the risk-neutral probability:

$$f^{(q)}_{S(t)}(x) = \frac{H(x)}{\sqrt{2\pi\sigma^2 t}} \frac{1}{x} \exp\left(-\frac{(\log x - \log S_0 - (r - \frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right). \tag{2.5}$$

We can finally reach our goal, which is to write the Black-Scholes price at time $t = 0$ of standard European derivatives as the expectation of the discounted pay-off in the risk-neutral probability measure.

Theorem 10. *The formula (1.21) for the Black-Scholes price of the European derivative with pay-off $Y = g(S(T))$ can be rewritten at time $t = 0$ as*

$$\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y]. \tag{2.6}$$

Proof. Using the density (2.5) of $S(t)$ in the risk-neutral probability measure we have

$$e^{-rT}\mathbb{E}_q[Y] = e^{-rT}\mathbb{E}_q[g(S(T))] = e^{-rT}\int_{\mathbb{R}} g(x) f^{(q)}_{S(T)}(x)\,dx$$
$$= \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 t}}\int_0^\infty \frac{g(x)}{x}\exp\left(-\frac{(\log x - \log S_0 - (r-\frac{\sigma^2}{2})t)^2}{2\sigma^2 t}\right)dx.$$

With the change of variable $y = \frac{\log x - \log S_0 - (\alpha - \theta\sigma)t}{\sigma\sqrt{t}}$ we obtain

$$e^{-rT}\int_{\mathbb{R}} g(S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}})e^{-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}},$$

which is exactly (1.21) at time $t = 0$. $\square$

The formula (2.6) has been derived only for standard European derivatives, however it makes perfectly sense even for non-standard European derivatives. Thus we introduce the following

Definition 5. *The Black-Scholes price at time $t = 0$ of the European derivative with pay-off Y at maturity time T is given by $\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y]$.*

For example, the Asian call, resp. put, option in the time-continuum case is defined as the non-standard European derivative with pay-off

$$Y_{\text{call}} = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+, \quad \text{resp.} \quad Y_{\text{put}} = \left(K - \frac{1}{T}\int_0^T S(t)\,dt\right)_+,$$

where $K > 0$ is the strike price of the option. The Black-Scholes prices at time $t = 0$ of these options are given respectively by

$$\Pi_{\text{AC}}(0) = e^{-rT}\mathbb{E}_q[Y_{\text{call}}], \quad \Pi_{\text{AP}}(0) = e^{-rT}\mathbb{E}_q[Y_{\text{call}}]. \tag{2.7}$$

Exercise 10. *Derive the following put-call parity identity:*

$$\Pi_{\text{AC}}(0) - \Pi_{\text{AP}}(0) = e^{-rT}\left(\frac{e^{rT}-1}{rT}S_0 - K\right). \tag{2.8}$$

2.3 Monte Carlo analysis of the Asian option

The formula (2.7) for the Black-Scholes price of Asian options cannot be written in a simple explicit form as the Black-Scholes price (1.21) of standard European derivatives. Hence in order to price Asian options the need of more advanced numerical methods become essential. In this final section we describe the Monte Carlo method, which, due to its simplicity, is the most used by financial institutions.

The Monte Carlo method

The Monte Carlo method is, in its simplest form, a numerical method to compute the expectation of a random variable. Its mathematical validation is based on the **Law of Large Numbers**, which states the following: Suppose $\{X_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables with expectation $\mathbb{E}[X_i]=\mu$. Then the sample average of the first n components of the sequence, i.e.,

$$\overline{X}_n = \frac{1}{n}(X_1+X_2+\cdots+X_n),$$

converges (in probability) to μ.

The law of large numbers can be used to justify the fact that if we are given a large number of independent trials $X_1,\dots,X_n$ of the random variable X, then

$$\mathbb{E}[X]\approx \frac{1}{n}(X_1+X_2+\cdots+X_n).$$

Example. Let $X:\Omega_2\to\mathbb{R}$ be the random variable that takes value 1 if the two tosses are different and value -1 if they are equal. If the coin is fair we have of course $\mathbb{E}[X]=0$. Suppose that we perform the experiment "tossing the coin twice" 100 times. Then we shall obtain 100 trials $X_1,\dots X_{100}$ for the random variable X. If our 2-tosses were different, say, 55 times and equal 45 times, then our approximation for $\mathbb{E}[X]=0$ is (55-45)/100=0.1.

To measure how reliable is the approximation of $\mathbb{E}[X]$ given by the sample average, consider the standard deviation of the trials $X_1,\dots,X_n$:

$$s=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\overline{X}_n-X_i)^2}.$$

If we interpret $X_1,\dots,X_n$ as the first n components of a sequence $\{X_i\}_{i\geq 1}$ of i.i.d. random variables with $\mathbb{E}[X_i]=\mu$, then a simple application of the Central Limit Theorem proves that the random variable

$$\frac{\mu-\overline{X}_n}{s/\sqrt{n}}$$

converges in distribution to a standard normal random variable. We use this result to show that the true value μ of $\mathbb{E}[X]$ has about 95% probability to be in the interval

$$[\overline{X}_n-1.96\frac{s}{\sqrt{n}},\overline{X}_n-1.96\frac{s}{\sqrt{n}}].$$

Indeed, for n large,

$$\mathbb{P}\left(-1.96\leq\frac{\mu-\overline{X}_n}{s/\sqrt{n}}\leq 1.96\right)\approx\int_{-1.96}^{1.96}e^{-x^2/2}\,\frac{dx}{\sqrt{2\pi}}\approx 0.95.$$

Application to options pricing theory

Consider now the European derivative with pay-off Y at maturity T. We approximate the price at time $t = 0$ by

$$\Pi_Y(0) = e^{-rT}\mathbb{E}_q[Y] \approx e^{-rT}\frac{Y_1 + \cdots + Y_n}{n} \tag{2.9}$$

where $Y_1, \ldots, Y_n$ is a large number of independent pay-off trials. As the pay-off depends on the path of the stock price, the trials $Y_1, \ldots, Y_n$ can be created by first generating a sample of paths for the stock price. Letting $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $[0, T]$ with size $t_i - t_{i-1} = h$, we may construct a sample of n paths of the geometric Brownian motion on the given partition with the following simple Matlab function:

```
function Path=StockPath(s,sigma,r,T,N,n)
h=T/N;
W=randn(n,N);
q=ones(n,N);
Path=s*exp((r-sigma^2/2)*h.*cumsum(q')+sigma*sqrt(h)*cumsum(W'));
Path=[s*ones(1,n);Path];
```

Note carefully that the stock price is modeled as a geometric Brownian motion with mean of log return $\alpha = r - \sigma^2/2$, which means that computations are made in the "risk-neutral world", see (2.5). This is of course correct, since the expectation in (2.9) that we want to compute is in the risk-neutral probability measure.

In the case of the Asian call option with strike K and maturity T the pay-off is given by

$$Y = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+ \approx \left(\frac{1}{N}\sum_{i=1}^{N} S(t_i) - K\right)_+ .$$

The following function computes the approximate price of the Asian option using the Monte Carlo method:

```
function [price, conf95]=MonteCarlo_AC(s,sigma,r,K,T,N,n)
tic
stockPath=StockPath(s,sigma,r,T,N,n);
payOff=max(0,mean(stockPath)-K);
price=exp(-r*T)*mean(payOff);
conf95=1.96*std(payOff)/sqrt(n);
toc
```

The function also return the 95% confidence interval of the result. For example, by running the command

```
[price, conf95]=MonteCarlo_AC(100,0.5,0.05,100,1/2,100,1000000)
```

we get `price`=8.5799, `conf95`=0.0283, which means that the Black-Scholes price of the Asian option with the given parameters has 95% probability to be in the interval 8.5799 ± 0.0283.

The calculation took about 4 seconds. Note that the 95% confidence is $0.0565/8.5799*100 \approx 0.66\%$ of the price. The Monte Carlo method can be improved to achieve the same level of accuracy (or even better) with a much lower number of paths (and thus much quicker) than in our example. See [3].

Exercise 11 (Matlab)**.** *Use the Monte Carlo method to study numerically how the price of the Asian call depend on the parameters of the option. In particular:*

- *(a) Verify numerically the put-call parity* (2.8)
- *(b) Show that the Asian call is less sensitive to volatility than the standard call. Do you have an intuitive explanation for this?*
- *(c) Show that, as the number of sample paths grows, the Monte Carlo approximation of the Asian call price stabilizes to a fixed value (which would be the exact Black-Scholes price of the option)*
- *(d) Show that for large volatilities the Monte Carlo method becomes unstable (the confidence interval grows very fast)*
- *(d) Show that the Asian call is cheaper than the standard call with the same strike. Do you have an intuitive explanation for this?*

Exercise 12 (Matlab)**.** *Write a matlab code which applies the Monte Carlo method to compute the Black-Scholes price and the confidence interval of barrier options with American barrier.*

Exercise 13 (Matlab)**.** *Write a matlab code which applies the Monte Carlo method to compute the Black-Scholes price and the confidence interval of lookback options*

Exercise 14 (Matlab)**.** *For each option in the previous two exercises perform an parameter-sensitivity analysis similar to the one outlined in Exercise 11 for the Asian option.*

Chapter 3

Interest rate contracts

Interest rate derivatives, such as swaps, caps and floors, are financial instruments used by companies, banks, and national governments to hedge against the risk derived from the uncertainty of interest rate markets. These products comprise the largest component of the financial derivative market, their total notional amount having estimated to be several times larger than the total gross world product. In this section we study the so-called classical approach to the problem of pricing interest rate contracts. Other important topics, such as hedging, calibration and the HJM methodology, are not discussed. For further reading on interest rate contracts see the books [5, 7], which have been used to prepare most of the material presented in this section.

Before we embark on the study of interest rate contracts we need to discuss the important difference between discretely compounded and continuously compounded interest rates. Consider a risk-free asset in the money market with value $B(t)$ at time t. Recall that assets in the money market are equivalent to short term loans (< 1 year) issued by institutions that bear no risk of default (risk-free). The risk-free asset is said to have **discretely compounded interest rate** $R(t)$ in the time period $[t, t+h]$ if the value of the asset satisfies

$$B(t+h) = B(t)(1 + R(t)h). \tag{3.1}$$

Inverting (3.1) we have

$$R(t) = \frac{B(t+h) - B(t)}{hB(t)},$$

i.e., $R(t)$ is the annualized relative return of the risk-free asset in the interval $[t, t+h]$. Letting $h \to 0$ we obtain the **continuously compounded interest rate** $r(t)$ of the risk-free asset

$$R(t) \to r(t) = \frac{B'(t)}{B(t)} = \frac{d}{dt}\log B(t), \quad \text{as } h \to 0. \tag{3.2}$$

Integrating (3.2) on $[t, t+h]$ we find

$$B(t+h) = B(t)e^{\int_t^{t+h} r(s)\,ds}, \tag{3.3}$$

which is the time-continuum analog of (3.1). Hence discretely and continuously compounded rates differ merely by how frequently the interest rate is compounded. In the discrete case the interest rate is compounded over finite time intervals, while in the continuous case the interest rate is compounded instantaneously. When h is small, (3.3) gives approximately $B(t+h) \approx B(t)e^{r(t)h} \approx B(t)(1+r(t)h)$, hence over sufficiently short time intervals we have $R(t) \approx r(t)$. However, despite being small, this difference can have a substantial impact over large capitals and therefore any interest rate contract *must* specify how the interest rate is compounded.

Example. Suppose that you contract a loan of 2500000 Kr from a bank with a fixed interest rate of 1.7% for 1 year (with no need of paying back the loan during this time). If the interest rate is compounded annually, then your debt with the bank after 1 year is

$$2500000(1 + 0.017) \approx 2542500.$$

Thus the total interest to be paid on the loan is approximately 2721630-2500000=42500 Kr. Most likely the bank will require you to pay a fraction of this interest every month, hence you will pay 42500/12≈ 3541 Kr each month. On the other hand, if the interest rate is compounded continuously, then your debt after 1 year grows to

$$2500000 \exp(0.017) = 2542860$$

and thus the amount of interest to pay every month is (2542860-2500000)/12≈3572 Kr.

The continuously compounded rate $r(t)$ is the interest rate used in the previous sections (where it was assumed to be constant). In this section we adopt both discretely and continuously compounded rates, as these two methods of measuring interest rates are equally common in the market of interest rate contracts. In general we shall denote discretely compounded rates by capital letters (e.g., R) and the corresponding continuously compounded rate by the same letter but in small font (e.g., r).

3.1 Zero-coupon bonds

A **zero-coupon bond** (ZCB) with **face** (or **nominal**) value K and maturity $T > 0$ is a contract that promises to pay to its owner the amount K at time T in the future. Zero-coupon bonds, and related contracts described in the following section, are issued by national governments and private companies as a way to borrow money and fund their activities. In the following we assume that all ZCB's are issued by one given institution, so that all bonds differ merely by their face values and maturities. Once a debt is issued in the so-called **primary market**, it becomes a tradable asset in the **secondary** bond market. It is therefore natural to model the value at time t of the ZCB maturing at time $T > t$ as a random variable, which we denote by $B(t,T)$. Hence $\{B(t,T)\}_{t\in[0,T]}$ is a stochastic process. We assume throughout the discussion that the institution issuing the bond bears no risk of default, i.e., $B(t,T) > 0$, for all $t \in [0,T]$. Clearly $B(T,T) = K$ and, under normal market conditions, $B(t,T) < K$, for $t < T$, i.e., ZCB's are risk-free assets ensuring a positive return[1]. Without loss of generality we assume from now on that $K = 1$, as owning a ZCB with face value K is clearly equivalent to own K shares of a ZCB with face value 1. The maturity of bonds can reach up to 30 or more years. A **zero-coupon bond market** (ZCB market) is a market in which the objects of trading are ZCB's with different maturities. Our main goal is to introduce models for the prices of ZCB's observed in the market. We discuss both time-discrete and time-continuum models. For time-discrete models we let $T \in \{0, \dots, N\}$ and $t \in \{0, 1, \dots, T\}$, where in the applications each time step corresponds typically to a period of 3 or 6 months. Moreover N is sufficiently large so that all ZCB's expire before N periods in the future (e.g., $N \approx 100$). For time-continuum models we let $T \in [0, S]$ and $t \in [0, T]$ where, again, S is sufficiently large so that all ZCB's in the market expire before time S (e.g., $S \approx 50$ years).

Remark 4. The term "bond" is more specifically used for long term loans with maturity $T > 1$ year. Short term loans have different names (e.g., bills, repo, etc.) and they constitute the component of the loan market called **money market**. A bond which has less than one year left to maturity is also considered a money market asset.

3.2 Interest rates and yield of ZCB's

Forward rate

The difference in value of ZCB's with different maturities is expressed through the implied forward rate of the bond. To define this concept, suppose that at the present time t we open a portfolio that consists of -1 share of a ZCB with maturity $t < T$ and $B(t,T)/B(t,T+\delta)$ shares of a ZCB expiring at time $T + \delta$. This investment has zero value and entails that we pay 1 at time T and receive $B(t,T)/B(t,T+\delta)$ at time $T + \delta$. Hence our investment at the present time t is equivalent to an investment in the future time interval $[T, T+\delta]$ with (annualized) return given by

$$F_\delta(t,T) = \frac{1}{\delta}\left(B(t,T)/B(t,T+\delta) - 1\right) = \frac{B(t,T) - B(t,T+\delta)}{\delta B(t,T+\delta)}. \tag{3.4}$$

The quantity $F_\delta(t,T)$ is also called **discretely compounded forward rate** in the interval $[T, T+\delta]$ **locked at time** t (or forward LIBOR, as it is commonly applied to LIBOR interest rate contracts). The name is intended to emphasize that the investment return in the future interval $[T, T+\delta]$ is locked at the *present* time $t \leq T$, that is to say, we know today which interest rate has to be charged to borrow in the future time interval $[T, T+\delta]$ (if a different rate were locked today, then an arbitrage opportunity would arise). When $\delta \to 0^+$ we obtain

[1] Actually, zero-coupon (and other) national bonds in Sweden with maturity shorter than 5 years yield currently (2017) a negative return.

the **continuously compounded T-forward rate**

$$f(t,T) = \lim_{\delta\to 0^+} \frac{1}{\delta}\frac{B(t,T) - B(t,T+\delta)}{B(t,T+\delta)} = -\partial_T \log B(t,T), \tag{3.5}$$

which is the rate locked at time t to borrow at time T for an "infinitesimal" period of time.

From now on we set $\delta = 1$ in the discretely compounded forward rate and rewrite the latter as

$$F(t,T) = B(t,T)/B(t,T+1) - 1, \tag{3.6}$$

where

$$t = 0,\dots,T, \quad T = 1,\dots N-1.$$

Thus $F(t,T)$ is the rate locked at time t to borrow in the one period interval $[T, T+1]$, where in the applications one period means typically 3 or 6 months.

The curve $T \to F(t,T)$ (or $T \to f(t,T)$) is called **forward rate curve** of the ZCB market. The next theorem shows that the knowledge of $F(t,T)$ (or $f(t,T)$) for all maturities (i.e., the knowledge of the forward rate curve) determines the price $B(t,T)$ of all ZCB's in the market.

Theorem 11. *The following identity holds:*

$$B(t,T) = \frac{1}{\prod_{j=t}^{T-1}(1+F(t,j))}, \quad T \in \{1,\dots,N\},\ t \in \{0,\dots,T-1\}. \tag{3.7}$$

In the continuously compounded case we have

$$B(t,T) = \exp\left(-\int_t^T f(t,s)\,ds\right), \quad 0 \le t \le T \le S. \tag{3.8}$$

Proof. The formula (3.8) follows easily by integrating (3.5). We prove (3.7) by induction on T. Replacing $B(t,t) = 1$ in (3.6), we obtain $F(t,t) = 1/B(t,t+1) - 1$, hence $B(t,t+1) = 1/(1+F(t,t))$, which is (3.7) for $T = t+1$. Now assume that (3.7) holds for $T = t+k$, i.e.,

$$B(t,t+k) = \frac{1}{\prod_{j=t}^{t+k-1}(1+F(t,j))}. \tag{3.9}$$

Replacing $T = t+k$ in (3.6) we obtain

$$F(t,t+k) = B(t,t+k)/B(t,t+k+1) - 1,$$

hence

$$B(t,t+k+1) = \frac{B(t,t+k)}{1+F(t,t+k)} = \frac{1}{\prod_{j=t}^{t+k-1}(1+F(t,j))}\frac{1}{1+F(t,t+k)} = \frac{1}{\prod_{j=t}^{t+k}(1+F(t,j))}.$$

It follows that (3.9) holds for all $k \ge 0$. Letting $k = T-t$ proves (3.7), for all $T = 1,2,\dots$ and $t = 0,1,\dots,T-1$. $\square$

Remark 5. One consequence of the previous theorem is that a theoretical price of ZCB's at all maturities can be derived by modeling the forward rate curve of the ZCB market. This approach to the problem of ZCB's pricing is called HJM (Heath, Jarrow, Morton) approach and is described for instance in [5]. In this text we consider only the so-called "classical approach" to bonds pricing, which is described in Section 3.5 below.

Spot rate

The quantity

$$R(t) = F(t,t), \quad t = 0,\dots,N-1$$

is called **discretely compounded spot rate** of the ZCB market at time t and represents the interest rate locked at time t to borrow in the interval $[t, t+1]$. It follows by (3.6) that

$$B(t,t+1) = \frac{1}{1+R(t)}, \tag{3.10}$$

hence the spot rate at time t determines the price at time t of the ZCB expiring at time $t+1$. As this is a short term loan, then $R(t)$ coincides with the discretely compounded interest rate of the money market. Similarly

$$r(t) = f(t,t), \quad t \in [0,S]$$

is the **continuously compounded spot rate**, i.e.., the continuously compounded interest rate of the money market.

Remark 6. In the previous sections $r(t)$ was assumed to be constant, which is reasonable for markets of short maturity contracts, such as stock options. In the case of ZCB markets, where assets mature after a very long time, we need to take into account that borrowing in the one period $[t_1, t_1+1]$ will in general entail a different interest rate than for borrowing in the one period $[t_2, t_2+1]$.

Note also that $R(t)$ and $r(t)$ do not depend on the maturity of the ZCB's and so they do not contain enough information to compute $B(t,T)$ for all maturities. However we shall see in Section 3.5 that a model on the spot rate combined with the so-called risk-neutral hypothesis, leads to a model for the price of ZCB's. This approach to the problem of ZCB's pricing is sometimes referred to as the classical approach (as opposed to the HJM approach which is based on the forward rate curve, see Remark 5).

As negative ZCB prices are clearly meaningless, we assume throughout the remainder of this text that

$$R(t) > -1, \quad t = 0,\dots,N-1.$$

Example. Suppose that today (November 1st, 2017) an investor wants to sign a contract to borrow 1000000 Kr on May 1st, 2018 for a period of 6 months. There are essentially two ways in which this loan can be issued. The first way is to fix the interest rate today as the forward rate $F(t,T)$, where $t =$November 1st, 2017 and $T =$May 1st, 2018. Note that $F(t,T)$ is known at time t, as it depends on the present day price of ZCB's (which can be found on-line or in the financial press). The second way to issue the loan is at the spot rate $R(T)$. However the interest rate $R(T)$ is not known at time t, hence in this case the investor must wait until the first of May 2018 to know which interest rate will be charged to the loan. Of course, this second method entails a risk for both the borrower. This risk can be hedged by interest rate derivatives, such as interest rate swaps, caps, floors, etc., which are described in Section 3.6.

Discount process

The spot rate can be used to define the **discount process**:

$$D(0) = 1, \quad D(t) = \frac{1}{\prod_{j=0}^{t-1}(1+R(j))}, \text{ for } t = 1, 2, \dots \quad \text{(discretely compounded)} \tag{3.11}$$

$$d(t) = \exp\left(-\int_0^t r(s)\,ds\right) \quad \text{(continuously compounded)} \tag{3.12}$$

If t is the present time and $X(\tau)$ is the value of an asset at some given future time $\tau > t$, then the quantity

$$\frac{D(\tau)}{D(t)}X(\tau) = \frac{X(\tau)}{\prod_{j=t}^{\tau-1}(1+R(j))}, \quad \text{or}$$

$$\frac{d(\tau)}{d(t)}X(\tau) = \exp\left(-\int_t^\tau r(s)\,ds\right)X(\tau)$$

is called the present (at time t) **discounted value** of the asset and represents the future value of the asset expressed in terms of the present value of money. When $t = 0$, we denote the discounted value as $D(\tau)X(\tau) = X^*(\tau)$ (or $d(\tau)X(\tau) = X^*(\tau)$). Note that in the time-continuum case, when $r(t) = r$ is constant, we have $d(t) = e^{-rt}$, hence $X^*(\tau) = e^{-r\tau}X(\tau)$ is precisely the discounted value used for stock prices in the previous sections. Similarly, if $R(t) = R$, then $D(t) = (1+R)^{-t}$.

Example. The value of a ZCB a maturity equals its face value, i.e., $B(T,T) = 1$. The value of this payment at time $t < T$ is $D(T)/D(t)$. Thus we can interpret $D(T)/D(t)$ as the present (at time t) value of 1 dollar payed in the future time T. When the spot rate is constant, we obtain the simple formulas

$$\frac{D(T)}{D(t)} = \frac{1}{(1+R)^{T-t}}, \quad \frac{d(T)}{d(t)} = e^{-r(T-t)}. \tag{3.13}$$

Yield to maturity

We conclude this section by presenting the fundamental concept of ZCB's yield to maturity. The discretely compounded **yield (to maturity)** $Y(t,T)$ at time t of the ZCB with maturity

T is the *constant* discretely compounded forward rate which entails the value $B(t,T)$ of the ZCB. Hence the yield $Y(t,T)$ of a ZCB is obtained by replacing $F(t,j) = Y(t,T)$ for all $j = t,\dots,T-1$ in the formula (3.7). We obtain

$$B(t,T) = \frac{1}{(1+Y(t,T))^{T-t}}, \quad \text{i.e.,} \quad Y(t,T) = \left(\frac{1}{B(t,T)}\right)^{\frac{1}{T-t}} - 1. \tag{3.14}$$

To put it in other words: Selling a ZCB for the price $B(t,T)$ at time t (i.e., borrowing $B(t,T)$ at time t) is equivalent to lock the constant forward rate $Y(t,T)$ until maturity. Note also that the first equation in (3.14) expresses $B(t,T)$ as the discounted value at time t of the future payment $= 1$ at maturity assuming that the spot rate is constant and equal to $Y(t,T)$ in the interval $[t,T]$ (see (3.13))

The formula for the continuously compounded yield is obtained similarly by replacing $f(t,v) = y(t,T)$ in (3.8), i.e.,

$$B(t,T) = e^{-y(t,T)(T-t)}, \quad \text{i.e.,} \quad y(t,T) = -\frac{\log B(t,T)}{T-t} \tag{3.15}$$

and the same interpretation of the yield as a constant spot rate to discount the future payment $= 1$ of the ZCB also holds in the time-continuum case.

Exercise 15 (?)**.** *By the second equation in* (3.15)*,* $y(t,T)$ *equals the annualized log-return of the ZCB in the interval* $[t,T]$*. Is* $y(t,T)$ *really a good measurement for the return of a ZCB (as it is often claimed in the financial literature)?*

ZCB 's are listed in the market in terms of their yield rather than in terms of their price $B(t,T)$. The curve $T \to Y(t,T)$ (or $T \to y(t,T)$) is called the **yield curve** of the ZCB market. Figure 3.1 shows an example of yield curve for governmental Swedish bonds.

Exercise 16. *Yield curves observed in the market are classified based on their shape (e.g., steep, flat, inverted, etc.). Find out on the Internet what the different shapes mean from an economical point of view.*

3.3 Coupon bonds

Let $0 < t_1 < t_2 < \dots < t_M = T$ be a partition of the interval $[0,T]$. A **coupon bond** with maturity T, face value 1 and coupons $c_1, c_2, \dots, c_M \in [0,1)$ is a contract that promises to pay the amount c_k at time t_k and the amount $1+c_M$ at maturity $T = t_M$. Note that some c_k may be zero, which means that no coupon is actually paid at that time. We set $c = (c_1,\dots,c_M)$ and denote by $B_c(t,T)$ the value at time t of the bond paying the coupons $c_1,\dots,c_M$ and maturing at time T. Now, let $t \in [0,T]$ and $k(t) \in \{1,\dots,M\}$ be the smallest index such that $t_{k(t)} > t$, that is to say, $t_{k(t)}$ is the first time after t at which a coupon is paid. Holding the coupon bond at time t is clearly equivalent to holding a portfolio containing $c_{k(t)}$ shares

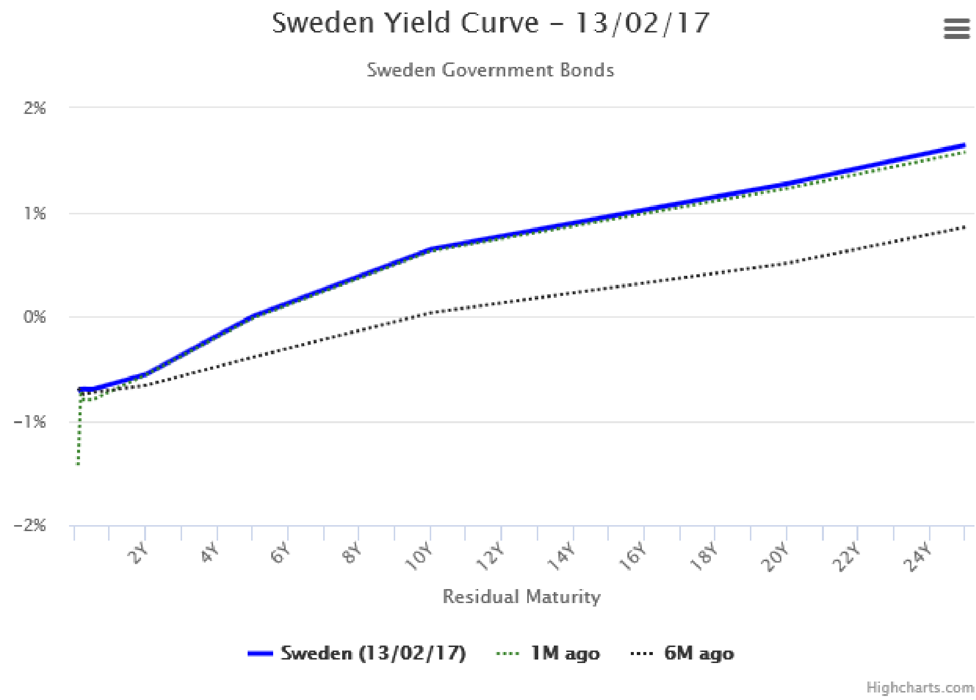


Figure 3.1: Yield curve for Swedish bonds. Note that the yield is negative for maturities shorter than 5 years. Bonds with maturity larger than 2 years have coupon and thus their yield is computed using (3.20) (instead of (3.15)).

of the ZCB expiring at time $t_{k(t)}$, $c_{k(t)+1}$ shares of the ZCB expiring at time $t_{k(t)+1}$, and so on, hence

$$B_c(t,T) = \sum_{j=k(t)}^{M-1} c_j B(t,t_j) + (1+c_M)B(t,T), \tag{3.16}$$

the sum being zero when $k(t) = M$.

Remark 7. Note that the value of the coupon bond at time t does not depend on the coupons paid at or before t. This of course makes sense, as purchasing a bond at time t does not give the buyer any right concerning previous coupon payments.

The yield of a coupon bond is defined as follows. Using (3.14), we can write the prices of the ZCB's in the sum (3.16) as $B(t,t_j) = (1+Y(t,t_j))^{-(t_j-t)}$. This leads to define the discretely compounded yield $Y_c(t,T)$ of a coupon bond with coupons $(c_1,\dots,c_M)$ and maturity T implicitly by the equation

$$B_c(t,T) = \sum_{j=k(t)}^{M-1} \frac{c_j}{(1+Y_c(t,T))^{(t_j-t)}} + \frac{(1+c_M)}{(1+Y_c(t,T))^{T-t}}. \tag{3.17}$$

Note that *the yield of the coupon bond is the constant discretely compounded spot rate used to discount the total future payments of the coupon bond.*

Example. Consider a 3 year maturity coupon bond with face value 1 which pays 1% coupon semiannually. Suppose that the bond is listed with an yield of 2%. What is the value of the bond at time zero? The coupon dates are

$$(t_1, t_2, t_3, t_4, t_5, t_6) = (1/2, 1, 3/2, 2, 5/2, 3),$$

and $c_1 = c_2 = \cdots = c_6 = 0.02$. Hence

$$\begin{aligned} B_c(0,T) &= \frac{0.02}{(1+0.01)^{1/2}} + \frac{0.02}{(1+0.01)^{1}} + \frac{0.02}{(1+0.01)^{3/2}} + \frac{0.02}{(1+0.01)^{2}} + \frac{0.02}{(1+0.01)^{5/2}} + \frac{1+0.02}{(1+0.01)^{3}} \\ &= 1.08852. \end{aligned}$$

Remark 8. Most commonly, as in the example above, the coupons are equal. Letting $c_j = c$, for all $j = 1, \dots, M$, the formula (3.18) simplifies to

$$B_c(t,T) = c \sum_{j=k(t)}^{M-1} \frac{1}{(1+Y_c(t,T))^{(t_j-t)}} + \frac{(1+c)}{1+Y_c(t,T)}. \tag{3.18}$$

In the example above, the yield was given and $B_c(0,T)$ was computed. However one is most commonly faced with the opposite problem, i.e., computing the yield of the coupon bond with given initial value $B_c(0,T)$. We can easily solve this problem numerically inverting (3.18). For instance, assume that $T = M$ years and that the coupons are paid annually, that is $t_1 = 1$, $t_2 = 2, \dots, t_M = M$. Then $x = (1+Y_c(0,T))^{-1}$ solves $p(x) = 0$, where p is the M-order polynomial given by

$$p(x) = c_1 x + c_2 x^2 + \cdots + (1+c_M)x^M - B_c(0,T). \tag{3.19}$$

The roots of this polynomial can easily be computed numerically, e.g., with the command `roots[p]` in matlab, see Exercise 17 below.

Similarly, the continuously compounded yield of a coupon bond is the quantity $y(t,T)$ defined implicitly by the equation

$$B_c(t,T) = \sum_{j=k(t)}^{M-1} c_j e^{-y_c(t,T)(t_j-t)} + (1+c_M)e^{-y_c(t,T)(T-t)}. \tag{3.20}$$

Exercise 17 (Matlab)**.** *Write a matlab function*

```
yield(B, Coupon, FirstCoupDate, CoupFreq, M)
```

that computes the continuously compounded yield of a coupon bond. Here, `B` *is the current (i.e., at time $t = 0$) price of the coupon,* `Coupon` $\in [0,1)$ *is the (constant) coupon,*

$\texttt{FirstCoupDate}$ *is the first future date at which the coupon is paid,* $\texttt{CoupFreq}$ *is the frequency of coupon payments and* $\texttt{M}$ *is the number of times that the coupons are paid from today ($t = 0$) up to and including maturity. For example*[2]

$$\texttt{yield}(1.01, 0.02, 46/252, 1, 4)$$

computes the yield of a 2% coupon bond which today is valued 1.01, pays the first coupon in 46 days and 3 more coupons every one year until maturity.

TIP: Note that the time left to maturity (i.e., T when $t = 0$) is given by

$$T = \texttt{FirstCoupDate} + (\texttt{M} - 1)\texttt{CoupFreq}.$$

Moreover the formula (3.20) becomes, with the data in the exercise,

$$\begin{aligned} B_c(0, T) = \texttt{Coupon} \sum_{j=1}^{M-1} e^{-y_c(0,T)(\texttt{FirstCoupDate}+(\texttt{j}-1)\texttt{CoupFreq})} \\ + (1 + \texttt{Coupon})e^{-y_c(0,T)(\texttt{FirstCoupDate}+(\texttt{M}-1)\texttt{CoupFreq})} \end{aligned}$$

Now, since times are rational numbers (fraction of years), then there exists $a, b, c, d \in \mathbb{N}$, $b, d \neq 0$, such that

$$\texttt{FirstCoupDate} = \frac{a}{b}, \quad \texttt{CoupFreq} = \frac{c}{d}.$$

Use this representation to write $B_c(0, T)$ as a polynom of grade $ad + bc(M - 1)$.

3.4 Arbitrage-free ZCB markets

Recall that an arbitrage is a risk-less investment with zero initial value and with positive probability to result in a profit. An arbitrage-free market is a market in which arbitrage investments are not possible. Before we discuss this concept in more details, let us show how the absence of arbitrage fixes the price of ZCB's in a (hypothetical) deterministic market.

Theorem 12. *Assume that the ZCB prices $B(t, T)$ are deterministic functions of t, for all $0 \le t \le T$. Then the equality*

$$B(t, T) = B(t, S)B(S, T), \quad t \le S \le T \tag{3.21}$$

must hold in order for the ZCB market to be arbitrage-free. Moreover

$$B(t, T) = \frac{D(T)}{D(t)}, \tag{3.22}$$

in the time-discrete case and similarly $B(t, T) = d(T)/d(t)$ in the time-continuum case.

[2] Remember that time in finance is measured in fraction of years and 1 year = 252 days (unless otherwise stated in the contract).

Proof. Suppose $B(t,T) > B(t,S)B(S,T)$. At time t we open a portfolio that consists of $-B(S,T)B(t,S)/B(t,T)$ shares of the ZCB maturing at time T and $B(S,T)$ shares of the ZCB maturing at time S. The value $V(t)$ of this portfolio is zero:

$$V(t) = -\frac{B(S,T)B(t,S)}{B(t,T)}\, B(t,T) + B(S,T)\, B(t,S) = 0.$$

At time S we receive $B(S,T)$, which we use to buy one share of the ZCB maturing at time T. Hence at time T we receive 1 and pay $-B(S,T)B(t,S)/B(t,T)$, thereby ensuring the positive return $1-B(S,T)B(t,S)/B(t,T)$. This portfolio is therefore an arbitrage. Similarly one can find an arbitrage opportunity when $B(t,T) < B(t,S)B(S,T)$. To prove (3.22), we first notice that it holds for $T = t+1$, as $B(t,t+1) = 1/(1+R(t)) = D(t+1)/D(t)$. Letting $S = t+1$ and $T = t+2$ in (3.21) proves (3.22) for $T = t+2$:

$$B(t,t+2) = B(t,t+1)B(t+1,t+2) = \frac{1}{1+R(t)}\frac{1}{1+R(t+1)} = \frac{D(t+2)}{D(t)},$$

and by iteration one proves (3.22) for all $T > t$. □

Next we discuss the concept of self-financing portfolio process invested in a ZCB market. We restrict the discussion to time-discrete markets[3].

Consider a ZCB market on the time set $t \in \{0,\dots,N\}$ and let $h_T(t)$ be the portfolio position held in the time period $[t,t+1)$ on the ZCB with maturity $T = 1,\dots,N$. Recall that for each t,T fixed, $h_T(t)$ and $B(t,T)$ are to be considered random variables, which means that they are defined on some (finite) probability space $(\Omega,\mathbb{P})$. We assume that the position $h_T(t)$ is **predictable**, i.e., it depends only on the price of the bonds up to time t. Note that $h_T(t)$ is only defined for $t \le T$, as the ZCB with maturity T ceases to exist after time T. Moreover, as changing the portfolio position at maturity is clearly meaningless, we use the convention

$$h_T(T) = h_T(T-1), \quad T = 1,\dots,N. \tag{3.23}$$

The number of shares $h_{t+1}(t)$ invested at time t in the ZCB maturing at time $t+1$ defines our position on the money market. The portion of the portfolio value invested in the money market is called **money market account** and its value at time t is given by

$$M(t) = h_{t+1}(t)B(t,t+1).$$

This component of the portfolio pays[4] $h_{t+1}(t)$ at the next time step (because $B(t+1,t+1) = 1$; recall that $h_{t+1}(t+1) = h_{t+1}(t)$, see (3.23), i.e., we do not change the position at maturity). We assume that the money market account is self-financing, that is to say the position

[3]The general definition of self-financing portfolio in a time-continuum ZCB market requires stochastic calculus and is therefore beyond the purpose of this text.

[4]We are assuming here that the portfolio has a long position on the money market. A similar argument works for a short position.

$h_{t+2}(t+1)$ in the money market at time $t+1$ is financed by the cash $h_{t+1}(t)$ deposited in the money market account at time $t+1$. Hence

$$M(t+1) = h_{t+2}(t+1)B(t+1, t+2) = h_{t+1}(t). \tag{3.24}$$

Theorem 13. *The value of the money market account at time t is given by*

$$M(t) = \frac{M(0)}{D(t)}. \tag{3.25}$$

Proof. At time $t = 0$, (3.25) is a trivial identity. Moreover $M(0) = h_1(0)B(0,1) = h_1(0)/(1+R(0))$, where we used (3.10). Hence $h_1(0) = M(0)(1 + R(0))$. Setting $t = 0$ in (3.24) gives $M(1) = h_1(0) = M(0)(1 + R(0)) = M(0)/D(1)$. Hence the claim holds at time 1 and by induction it is easily proved to hold at all times. □

It follows that, after re-balancing the money market account, the value of a portfolio process invested in the ZCB market is

$$V(t) = \frac{M(0)}{D(t)} + \sum_{T=t+2}^{N} h_T(t)B(t,T), \quad t = 0, \dots, N-2, \tag{3.26a}$$

while $V(t) = M(t) = M(0)/D(t)$, for $t = N-1, N$. In fact at time $t = N-1$ only the ZCB expiring at time N is left in the market and the position on this bond is financed by the money market account. To avoid this last trivial period we assume (following [5]) that all positions are liquidated at time $N-1$, i.e.,

$$V(N-1) = V(N) = 0. \tag{3.26b}$$

So far only the money market account is self-financing. To ensure that the full portfolio process is self-financing, we now impose that the change of position in the ZCB's does not generate a cash flow. To transform this condition into a mathematical formula, assume that we keep the positions $h_T(t)$ up to and including time $t+1$. Then the value of our portfolio at time $t+1$ would be

$$V^-(t+1) = M(t+1) + \sum_{T=t+2}^{N} h_T(t)B(t+1,T).$$

However having changed the positions at time $t+1$ to the new positions $h_T(t+1)$, then the value of our portfolio is

$$V(t+1) = M(t+1) + \sum_{T=t+3}^{N} h_T(t+1)B(t+1,T).$$

The self-financing condition is $V^-(t+1) = V(t+1)$, for all $t = 0, \dots, N-3$.

Definition 6. *A portfolio process* $\{h_T(t), t = 0, \dots, T,\ T = 1, \dots, N\}$ *invested in a ZCB market is said to be* **self-financing** *if its value at time* $t = 0, \dots, N$ *is given by* (3.26) *and*

$$\sum_{T=t+1}^{N} h_T(t-1)B(t,T) = \sum_{T=t+2}^{N} h_T(t)B(t,T), \quad t = 1, \dots, N-2. \tag{3.27}$$

In particular, for $t = 1, \dots, N-2$ *the value* (3.26a) *of self-financing portfolio processes can be rewritten as*

$$V(t) = \frac{M(0)}{D(t)} + \sum_{T=t+1}^{N} h_T(t-1)B(t,T), \quad t = 1, \dots, N-2. \tag{3.28}$$

Example for $N = 3$. We consider an example with three periods where we assume that the spot rates are deterministic (that is to say, we study a past state of the market). To be more specific, let

$$R(0) = 0.05, \quad R(1) = 0.03, \quad R(2) = 0.07.$$

Then the discount process is[5] $D(0) = 1$ and

$$D(1) = \frac{1}{1+R(0)} = 0.9524, \quad D(2) = \frac{D(1)}{1+R(1)} = 0.9246, \quad D(3) = \frac{D(2)}{1+R(2)} = 0.8642.$$

Using (3.22) we derive the following (deterministic) prices for the ZCB's in the market:

$$\begin{aligned}
&B(0,1) = D(1) = 0.9524,\ B(0,2) = D(2) = 0.9246,\ B(0,3) = D(3) = 0.8642,\\
&B(1,2) = D(2)/D(1) = 0.9708,\ B(1,3) = D(3)/D(1) = 0.9074,\\
&B(2,3) = D(3)/D(2) = 0.9347.
\end{aligned}$$

Suppose that at time $t = 0$ we buy 100 shares of the bond with maturity 1 ($h_1(0) = 100$), we buy 100 shares of the bond with maturity 2 ($h_2(0) = 100$) and sell 50 shares of the bond with maturity 3 ($h_3(0) = -50$). The initial value of our portfolio is

$$V(0) = 100 * 0.9524 + 100 * 0.9246 - 50 * 0.8642 = 144.49.$$

The value of the money market account at time 0 is $M(0) = 100 * 0.9524 = 95.24$. At time 1 the money market account has grown to $M(1) = M(0)(1 + R(0)) = 100$. Setting $t = 1$ in (3.27) we obtain that $h_3(1)$, i.e., the number of shares at time 1 on the bond with maturity 3, satisfies $h_3(1)B(1,3) = h_2(0)B(1,2) + h_3(0)B(1,3)$, i.e.,

$$h_3(1) = \frac{h_2(0)B(1,2) + h_3(0)B(1,3)}{B(1,3)} = 56.987.$$

Hence the portfolio value at time 1 is

$$V(1) = M(1) + h_3(1)B(1,3) = 100 + 56.987 * 0.9074 = 151.71.$$

At time $t = 2$ the portfolio is liquidated, hence $V(2) = V(3) = 0$.

[5] We truncate the computations at the fourth decimal digit. By abuse of notation we shall however keep using the equality sign "=" instead of the approximatively equal sign "≈".

Definition 7. *A portfolio process* $\{h_T(t), t = 0, \dots, T,\ T = 1, \dots, N\}$ *invested in a ZCB market and with value* $\{V(t), t = 0, \dots, N\}$ *is called an* **arbitrage** *if it is predictable and if there exists* $m \in \{1, \dots, N\}$ *such that* $\mathbb{P}(V(0) = 0) = 0$, $\mathbb{P}(V(m) \geq 0) = 1$ *and* $\mathbb{P}(V(m) > 0) > 0$. *A ZCB market is said to be* **arbitrage-free** *if every self-financing portfolio process invested in this market is not an arbitrage.*

Remark 9. Since $V(N-1) = V(N) = 0$, then m can be restricted to be less or equal to $N-2$ in the definition of arbitrage-free ZCB market.

Now, in a deterministic market we have shown in Theorem 12 that ZCB's must be priced using (3.22). The next theorem proves the reverse logical implication, namely that pricing ZCB's by formula (3.22) entails that the deterministic ZCB market is arbitrage free.

Theorem 14. *In a deterministic ZCB market where ZCB's are priced according to* (3.22) *there holds*

$$V(t) = \frac{V(0)}{D(t)}, \quad t = 0, \dots, N-2, \tag{3.29}$$

for all self-financing (predictable[6]*) portfolio processes invested in the ZCB market. Moreover the ZCB market is arbitrage-free.*

Proof. Using (3.22) in (3.26) we write the value at time t of the portfolio process as

$$V(t) = \frac{M(0)}{D(t)} + \sum_{T=t+2}^{N} h_T(t) \frac{D(T)}{D(t)}.$$

Hence $V^*(t) = D(t)V(t)$ satisfies

$$V^*(t) = M(0) + \sum_{T=t+2}^{N} h_T(t) D(T).$$

The self-financing property (3.27) gives

$$\sum_{T=t+2}^{N} h_T(t) \frac{D(T)}{D(t)} = \sum_{T=t+1}^{N} h_T(t-1) \frac{D(T)}{D(t)}, \quad \text{i.e.} \quad \sum_{T=t+2}^{N} h_T(t) D(T) = \sum_{T=t+1}^{N} h_T(t-1) D(T).$$

Hence

$$V^*(t) = M(0) + \sum_{T=t+1}^{N} h_T(t-1) D(T) = V^*(t-1).$$

Iterating the previous identity we obtain $V^*(t) = V^*(0) = V(0)$, which is (3.29). Using (3.29) we have $V(0) = 0 \Leftrightarrow V(t) = 0$, for all $t = 0, \dots, N-2$ and so the ZCB market is arbitrage-free. □

Remark 10. The generalization of Theorem 14 to random markets is Theorem 15 below.

[6] Of course, in a deterministic market all portfolios are predictable.

3.5 The classical approach to ZCB's pricing

In this section we describe the so-called **classical approach** to ZCB's pricing in time-discrete markets. This approach uses the discretely compounded spot rate process $\{R(t), t = 0, \dots, N-1\}$ as the unique source of randomness in the market, that is to say, it is assumed that all other stochastic processes in the market, such as ZCB prices, portfolio positions, etc., are measurable with respect to $\{R(t), t = 0, \dots, N-1\}$. Hence the knowledge of the spot rate up to time t suffices to uniquely fix the state of the market at time t.

Now, recall that a deterministic market is arbitrage-free if and only if ZCB's are priced according to $B(t,T) = \frac{D(T)}{D(t)}$, see Theorems 12, 14. The assumption that the spot rate process contains all the information about the state of the market motivates the following definition:

Definition 8. *Let $\{R(t), t = 0, \dots, N-1\}$, $R(t) > -1$, be a stochastic process modeling the spot interest rate of the ZCB market, where we assume that $R(0) = R_0$ is a deterministic constant. Then*

$$B(t,T) = \mathbb{E}[\frac{D(T)}{D(t)}|R(0), \dots, R(t)], \quad t = 0, \dots, T \tag{3.30}$$

is called the **risk-neutral price** *of the ZCB with maturity $T \le N$. At time $t = 0$ we have in particular*

$$B(0,T) = \mathbb{E}[D(T)] = \mathbb{E}[(1 + R(0)^{-1})(1 + R(1))^{-1} \dots (1 + R(T-1))^{-1}]. \tag{3.31}$$

Similarly, in the time-continuum case we define

$$B(0,T) = \mathbb{E}[d(T)] = \mathbb{E}[e^{-\int_0^T r(s)\,ds}]. \tag{3.32}$$

Remark 11. We may see the analogy of (3.30) with the risk-neutral pricing formula used in options pricing theory, as it defines the price of the ZCB at time t as the conditional expectation of the discounted pay-off (=1) to which the ZCB entitles at time T. Note however that this analogy with the risk-neutral pricing formula for options is only formal, as we have no way to claim here that the probability measure is a martingale probability! Put it differently we assume here *a priori* that the physical probability is risk-neutral (risk-neutral hypothesis). In order to confirm, or disprove, that this is the case we need to add a risky asset in the market, e.g., a stock. An example of such augmented market is given in Section 4.1.

Remark 12. Note that $B(t,T)$ is measurable with respect to $R(0), \dots, R(t)$, as it is defined by a conditional expectation in these variables. Moreover, since $D(t)$ is measurable with respect to $R(0), \dots R(t)$, then we can rewrite (3.30) as

$$B(t,T) = D(t)^{-1}\mathbb{E}[D(T)|R(0), \dots, R(t)], \text{ i.e., } B^*(t,T) = \mathbb{E}[D(T)|R(0), \dots, R(t)]. \tag{3.33}$$

Next we prove that the risk-neutral pricing formula (3.30) implies that the ZCB market is arbitrage-free.

Theorem 15. *Let $\{h_T(t), t = 0, \dots, T,\ T = 1, \dots, N\}$ be a self-financing portfolio process invested in a ZCB market. We assume that the portfolio position $h_T(t)$ is measurable with respect to $R(0), \dots, R(t)$, for all $t = 0, \dots, T$ and $T = 1, \dots, N$. Let $V(t)$ be the value of the portfolio at time t. Then the following holds:*

(i) The discounted portfolio value $\{V^*(t), t = 0, \dots, N-2\}$ is a martingale;

(ii) The discounted risk-neutral price $\{B^*(t,T), t = 0, \dots, T\}$ of the ZCB is a martingale;

(iii) The portfolio process is not an arbitrage.

Proof. [7] (i) The claim is

$$\mathbb{E}[V^*(t)|V^*(0), \dots, V^*(t-1)] = V^*(t-1), \quad t = 1, \dots, N-2. \tag{3.34}$$

The proof is by extending the argument used for the deterministic analogous statement in Theorem 14. Namely, using (3.33) in (3.28) we obtain

$$V^*(t) = M(0) + \sum_{T=t+1}^{N} h_T(t-1)\mathbb{E}[D(T)|R(0), \dots, R(t)].$$

Taking the expectation conditional to $R(0), \dots, R(t-1)$ and using that $h_T(t-1)$ is measurable with respect to these variables, we have

$$\mathbb{E}[V^*(t)|R(0), \dots, R(t-1)] = M(0) + \sum_{T=t+1}^{N} h_T(t-1)\mathbb{E}[D(T)|R(0), \dots, R(t-1)].$$

By (3.33) and (3.26a), the right hand side of the latter equation is precisely $V^*(t-1)$, hence

$$\mathbb{E}[V^*(t)|R(0), \dots, R(t-1)] = V^*(t-1).$$

Now (3.34) follows by taking the expectation conditional to $V^*(0), \dots V^*(t-1)$ and using that $V^*(t)$ is measurable with respect to $R(0), \dots, R(t)$. (ii) The claim is

$$\mathbb{E}[B^*(t,T)|B^*(0,T), \dots, B^*(t-1,T)] = B^*(t-1,T), \quad t = 1, \dots, T. \tag{3.35}$$

As $B^*(t,T) = D(t)B(t,T)$ is measurable with respect to $R(0), \dots, R(t)$, (3.35) follows by

$$\mathbb{E}[B^*(t,T)|R(0), \dots, R(t-1)] = B^*(t-1,T), \quad t = 1, \dots, T. \tag{3.36}$$

Using (3.33), the left hand side of (3.36) is

$$\mathbb{E}[\mathbb{E}[D(T)|R(0), \dots, R(t)]|R(0), \dots, R(t-1)] = \mathbb{E}[D(T)|R(0), \dots, R(t-1)],$$

and so, using again (3.33), the proof of (3.36) is completed. The proof of (iii) follows as usual. Namely, since martingales have constant expectation, then $\mathbb{E}[V^*(t)] = \mathbb{E}[V^*(0)]$ for all $t = 0, \dots, N-2$. If the portfolio is an arbitrage then $V^*(0) = 0$ and $V^*(m) \geq 0$ for some $m = 0, \dots, N-2$ and with probability 1. But then $\mathbb{E}[V^*(m)] = 0 \Rightarrow V^*(m) = 0$ with probability 1, and thus the portfolio is not an arbitrage. □

[7] In the proof we use the properties of the conditional expectation given in Theorem 3.

Hence in order to price ZCB's in the classical approach we must now fix a stochastic model for the spot rate process. There are dozens of such models used in the financial industry, see [1] for a review. Most of these models are actually formulated in the time-continuum theory, although for their calibration they need to be approximated by discrete models. We shall study only one example, known as the **Ho-Lee model** [4], which was introduced in the late 80's and is still widely used by financial institutions.

The Ho-Lee model (discrete case)

Let $\{M_t\}_{t=0,\dots,N}$ be a (possibly asymmetric) N-period random walk with transition probability

$$\mathbb{P}(m_{n-1} \to m_n) = \begin{cases} p & \text{if } m_n = m_{n-1} + 1 \\ 1 - p & \text{if } m_n = m_{n-1} - 1 \\ 0 & \text{otherwise} \end{cases}$$

Let also $\sigma > 0$ be a constant and $\theta(t)$ be a *deterministic* function of $t = 0, \dots, N-1$ such that $\theta(0) = 0$. The Ho-Lee model is given by[8]

$$R(t) = R(0) + \theta(t) + \sigma M_t, \quad t = 0, \dots, N-1. \tag{3.37}$$

Moreover, since the possible values of the random variables M_t are

$$M_t \in \{-t, -t+2, -t+4, \dots, t-2, t\} = \{-t + 2j,\ j = 0, \dots, t\}, \quad t = 0, \dots, N,$$

then the image of $R(t)$ is the set

$$R(t) \in \{R(0) + \theta(t) + (2j - t)\sigma,\ j = 0, \dots, t\}, \quad t = 0, \dots, N-1.$$

Remark 13. $R(0)$ is the current (at time $t = 0$) spot rate and is therefore known. The function $\theta(t)$ and the constant $\sigma > 0$ are chosen by calibrating the model, i.e., by imposing that the outcomes of the model best fit the market data. Of course this can only be done by relying on past states of the market. It then requires a "leap of faith" to believe that the model still works in the future (if not, the model needs to be re-calibrated).

Remark 14. The spot rate in the Ho-Lee model may reach negative values, unless $\theta(t)$ and σ are constrained to satisfy $R(0) + \theta(t) > \sigma t$, for all $t = 0, \dots, N-1$. While this condition may be imposed a priori on the model, it could be (and often is) in disagreement with the result of the calibration. Moreover negative interest rates can actually appear in the market (e.g., Swedish bonds with maturity less than 5 years have currently (2017) a negative yield). Note also that $R(0) + \theta(t) > \sigma t - 1$ must hold in order for $R(t) > -1$ to be verified at all times.

Exercise 18. *Compute* $\mathbb{E}[R(t)]$, $\mathrm{Var}[R(t)]$ *and the distribution of* $R(t)$.

[8] In a N-period model, the spot rate $R(N)$ is meaningless, hence we need to define $R(t)$ only for $t = 0, \dots, N-1$.

For instance, for $N = 3$ we obtain the following binomial tree for the spot rate:

$$
\begin{tikzcd}
 & & R(2) = R(0) + \theta(2) + 2\sigma \\
 & R(1) = R(0) + \theta(1) + \sigma \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
R(0) = R(0) \arrow[ur, "p"] \arrow[dr, "1-p"'] & & R(2) = R(0) + \theta(2) \\
 & R(1) = R(0) + \theta(1) - \sigma \arrow[ur, "p"] \arrow[dr, "1-p"] & \\
 & & R(2) = R(0) + \theta(2) - 2\sigma
\end{tikzcd}
$$

Assuming for instance $p = 1/2$, $R(0) = 0.03$ and

$$\theta(1) = 0.02, \quad \theta(2) = 0.01, \quad \sigma = 0.02,$$

the above tree for the spot rate becomes:

$$
\begin{tikzcd}
 & & R(2) = 0.08 \\
 & R(1) = 0.07 \arrow[ur, "1/2"] \arrow[dr, "1/2"] & \\
R(0) = 0.03 \arrow[ur, "1/2"] \arrow[dr, "1/2"'] & & R(2) = 0.04 \\
 & R(1) = 0.03 \arrow[ur, "1/2"] \arrow[dr, "1/2"] & \\
 & & R(2) = 0
\end{tikzcd}
$$

The discount process $\{D(t), t = 0, 1, 2, 3\}$ is given as follows:

$$D(0) = 1, \quad D(1) = \frac{1}{1 + R(0)} = 0.9709, \quad \text{with prob. } 1,$$

$$D(2) = \frac{D(1)}{1 + R(1)} = \begin{cases} \frac{0.9709}{1+0.07} = 0.9074, & \text{with prob. } 1/2 \\ \frac{0.9709}{1+0.03} = 0.9426 & \text{with prob. } 1/2 \end{cases},$$

$$D(3) = \frac{D(2)}{1 + R(2)} = \begin{cases} \frac{0.9074}{1+0.08} = 0.8402, & \text{with prob. } 1/4 \\ \frac{0.9074}{1+0.04} = 0.8725 & \text{with prob. } 1/4 \\ \frac{0.9426}{1+0.04} = 0.9063 & \text{with prob. } 1/4 \\ \frac{0.9426}{1+0} = 0.9426 & \text{with prob. } 1/4 \end{cases}$$

Let us now compute the risk-neutral ZCB prices. At time $t = 0$ we have $B(0,T) = \mathbb{E}[D(T)]$. For $T = 1$ this gives

$$B(0,1) = \frac{1}{1 + R(0)} = D(1) = 0.9709.$$

The initial price of the bond expiring at time $T = 2$ is

$$B(0,2) = \mathbb{E}[D(2)] = \frac{1}{2}0.9074 + \frac{1}{2}0.9426 = 0.9250$$

and for the ZCB maturing at time $T = 3$ we have

$$B(0,3) = \mathbb{E}[D(3)] = \frac{1}{4}0.8402 + \frac{1}{4}0.8725 + \frac{1}{4}0.9063 + \frac{1}{4}0.9426 = 0.8904.$$

Using (3.14) we can compute the yields of the ZCB's a time $t = 0$:

$$Y(0,1) = R(0) = 0.03, \quad Y(0,2) = \frac{1}{B(0,2)^{1/2}} - 1 = 0.0397, \quad Y(0,3) = \frac{1}{B(0,3)^{1/3}} - 1 = 0.0394$$

and thus plot the yield curve $T \to Y(0,T)$. The model can be calibrated by choosing the parameters so that the predicted yield curve best fits the observed yield curve.

Exercise 19. *In the example just considered, compute the possible values of the ZCB's at all times.*

Remark 15. As it is clear from (3.31), in the classical approach to ZCB pricing the bond is a *non-standard* European derivative on the sport rate. Consequently the numerical implementation of the discrete Ho-Lee model is feasible only for up to $N \sim 20$ periods; see the analogous remark following Exercise 8 for the Asian option in the binomial model. As shown below, this deficiency of the Ho-Lee model is resolved by considering the time-continuum version of the model.

Exercise 20 (Matlab)**.** *Write a Matlab function that computes the initial price of ZCB's using the time-discrete Ho-Lee model. The function $\theta(t)$, the constants $\sigma, R(0)$ and the probability $p \in (0,1)$ should appear as arguments of the Matlab function. Plot $B(0,T)$ in terms of $p \in (0,1)$, for all $T = 1, \dots, N$ (the function should work properly for $N \sim 20$). Draw the yield curve and the forward rate curve at time $t = 0$. Experiment different choices of $\theta(t)$ and σ and study how this choice affects the shape of the yield curve. Can you reproduce all the profiles found in Exercise 16?*

The Ho-Lee model (time-continuum case)

A simple application of the Central Limit Theorem shows that, in the time-continuum limit, the discretely compounded Ho-Lee spot rate converges to a process $\{r(t)\}_{t\in[0,T]}$ of the form

$$r(t) = r(0) + \theta(t) + \sigma W(t), \quad \theta(0) = 0, \tag{3.38}$$

where $\{W(t)\}_{t\geq 0}$ is a Brownian motion, $\sigma > 0$ is constant and $\theta(t)$ is a deterministic function of time. Next we use (3.32) to compute the initial price of the ZCB with face value 1 and maturity T.

Theorem 16. *When the continuously compounded spot rate is modeled by the process* (3.38)*, the initial price of the ZCB with face value 1 and maturity $T > 0$ is given by*

$$B(0,T) = \exp\left(-r(0)T - \int_0^T \theta(s)\,ds + \sigma^2\frac{T^3}{6}\right) \tag{3.39}$$

Proof. Replacing (3.38) into (3.32) we obtain

$$B(0,T) = e^{-\int_0^T \theta(s)\,ds}\mathbb{E}[e^{-\sigma\int_0^T W(s)\,ds}]$$

It can be shown that

$$\int_0^T W(s)\,ds \in \mathcal{N}\Big(0, \frac{T^3}{3}\Big).$$

Hence

$$B(0,T) = \frac{e^{-\int_0^T \theta(s)\,ds}}{\sqrt{2\pi\frac{T^3}{3}}}\int_{\mathbb{R}} e^{-\sigma x - x^2/(2T^3/3)}\,dx.$$

Computing the integral leads to (3.39). □

Remark 16. Using the fact that $\{r(t)\}_{t\in[0,T]}$ is a Markov process, it can be shown that

$$B(t,T) = \exp\left(-r(t)(T-t) - \int_t^T \theta(s)\,ds + \sigma^2\frac{(T-t)^3}{6}\right), \quad t \in [0,T]. \tag{3.40}$$

Remark 17. The existence of a closed formula for the price of ZCB's is of course a very convenient feature of the time-continuum Ho-Lee model, as it makes the computation of yields a very simple numerical problem (see next exercise). Several other interest rate models (e..g., the Vasicek model and the CIR model) used in the financial industry lead to a closed formula solution for ZCB's, which is one of the reason why they are used in the first place!

Exercise 21 (Matlab)**.** *Write a Matlab function that computes the yield of coupon bonds using the time-continuum Ho-Lee model. Plot the yield curve. Experiment different choices of $\theta(t)$ and σ and study how this choice affects the shape of the yield curve. Can you reproduce all the profiles found in Exercise 16? TIP: You can use the function in Exercise 17 to compute the yield.*

3.6 Other interest rate derivatives

In this section we present briefly two more examples of interest rate derivatives (other than ZCB's). The theoretical price of these contracts is discussed in the time-discrete case only.

Interest rate swaps

An **interest rate swap** can be seen as a coupon bond with variable (random) coupons, which can be positive or negative. More precisely, assume that $t = 0$ is the present time and let $R(t) = F(t,t)$ be the spot rate at time t, where $t = 1, \dots, N-1$. Recall that $R(t)$ represents the interest rate to borrow in the short period $[t, t+1]$ and it is known at time t, but not at the present time 0 (however the t-forward rate $F(0,t)$ is known at time 0 and represents the interest rate locked at time 0 to borrow in the same short period $[t, t+1]$). An interest rate swap stipulated at time 0 is a contract between two parties which at each time $t = 1, \dots, N-1$, entails the exchange of cash $L(R(t) - \rho)$, where ρ is a fixed interest rate and $L > 0$ is the **notional amount** converting units of interest rates into units of currency. Without loss of generality, we assume $L = 1$ in the following. The party receiving this cash flow when it is positive is called the receiver, while the opposite party is called the payer. Hence the receiver has a long position on the spot rate, while the payer has a short position on the spot rate. Letting $c = (c_1, \dots, c_{N-1})$, where

$$c_t = R(t) - \rho, \ \text{for } t = 1, \dots, N-1,$$

we define the risk-neutral value at time $t = 0, \dots, N-2$ of the interest rate swap as[9]

$$IRS(t) = \sum_{k=t+1}^{N-1} \mathbb{E}[c_k D(k) | R(0), \dots, R(t)] \tag{3.41}$$

and in particular at time 0,

$$IRS(0) = \sum_{t=1}^{N-1} \mathbb{E}[(R(t) - \rho) D(t)], \tag{3.42}$$

where $D(t)$ is the discount process. Hence the value at time t of the interest rate swap is the conditional expectation of the total discounted cash flow in the future. Now, in a fair swap, neither the payer nor the receiver should have an advantaged position on the contract, and thus none of them should a pay a premium to the other. In other words, *the fair value of interest rate swaps is zero*[10]. The value of the interest rate ρ which makes the risk-neutral price of the interest rate swap equal to 0 at the present time $t = 0$ is called the **swap rate**. It represents the "fair value" of the interest rate ρ which has to be agreed by the payer and the receiver of the interest rate swap.

Theorem 17. *The swap rate is given by*

$$\rho_{\text{swap}} = \frac{\sum_{t=1}^{N-1} F(0, t-1) B(0,t)}{\sum_{t=1}^{N-1} B(0,t)}. \tag{3.43}$$

[9]We assume that the interest rate swap exists up to time $N-2$. Note also that the interest rate swap at time $N-1$ is not defined, as $R(N)$ is meaningless in a N-period model.

[10]This is a general property of forward contracts, see Section 4.2.

Proof. By (3.42) we have

$$IRS(0) = \sum_{t=1}^{N-1} \mathbb{E}[R(t)D(t)] - \rho \sum_{t=1}^{N-1} \mathbb{E}[D(t)].$$

Using $\mathbb{E}[D(t)] = B(0,t)$ we obtain $IRS(0) = 0$ if and only if $\rho = \rho_{\text{swap}}$, where

$$\rho_{\text{swap}} = \frac{\sum_{t=1}^{N-1} \mathbb{E}[R(t)D(t)]}{\sum_{t=1}^{N-1} B(0,t)}.$$

Moreover

$$\begin{aligned}\mathbb{E}[R(t)D(t)] &= \mathbb{E}[(1+R(t))D(t)] - \mathbb{E}[D(t)] = \mathbb{E}[D(t-1)] - \mathbb{E}[D(t)] \\ &= B(0,t-1) - B(0,t) = F(0,t-1)B(0,t),\end{aligned}$$

where we used the definition (3.6) of forward rate. This concludes the proof. □

Exercise 22 (Matlab)**.** *Write a Matlab function that computes the swap rate using the Ho-Lee model.*

Caps and Floors

An **interest rate cap** is a contract that caps (i.e., put a maximum limit on) the spot rate. More precisely, an interest rate cap with strike rate ρ and notional amount $L = 1$ pays to its owner the amount $(R(t) - \rho)_+$ at each time $t = 1, \dots, N-1$. Hence the spot rate for the owner of the interest rate cap is no higher than ρ: any excess to the strike rate is paid by the seller of the interest rate cap. Similarly, an interest rate floor put a minimum on the spot rate and pays to its owner the amount $(\rho - R(t))_+$ at every time $t = 1, \dots, N-1$. The risk-neutral price of the interest rate cap/floor is defined by

$$Cap(t) = \sum_{k=t+1}^{N-1} \mathbb{E}[(R(k) - \rho)_+ D(k) | R(0), \dots, R(t)],$$

$$Floor(t) = \sum_{k=t+1}^{N-1} \mathbb{E}[(\rho - R(k))_+ D(k) | R(0), \dots, R(t)],$$

where $t = 0, \dots, N-2$. As $(R(k) - \rho)_+ - (\rho - R(k))_+ = (R(k) - \rho)$, the **cap-floor parity** identity holds:

$$Cap(t) - Floor(t) = IRS(t).$$

In particular if the strike rate coincides with the swap rate then the cap and the floor have the same initial price.

Exercise 23 (Matlab)**.** *Write a Matlab function that computes the initial price of caps and floors using the Ho-Lee model. Verify numerically the cap-floor parity identity.*

Chapter 4

Generalized binomial model

4.1 Binomial markets with stochastic interest rate

In Section 1 we have discussed the binomial options pricing model under the assumption that the risk-free asset has constant interest rate. In this section we consider a binomial market in which the interest rate of the risk-free asset is a stochastic process. For the sake of concreteness we model the interest rate by the (time-discrete) Ho-Lee model. This working example should be enough to grasp the general theory.

Let $\{M_t\}_{t=0,\dots,N}$ be a *generalized* random walk and consider a binomial market consisting of a stock with price

$$S(t) = S_0 \exp\left[t\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right)M_t\right], \quad t = 0, 1, 2, \dots N, \tag{4.1a}$$

together with a risk-free asset with interest rate:

$$R(t) = R(0) + \theta(t) + \sigma M_t, \quad t = 0, 1, \dots, N-1, \text{ where } \theta(0) = 0 \text{ and } \sigma > 0. \tag{4.1b}$$

The first problem to be addressed for the model (4.1) is its completeness, i.e., the existence a unique martingale probability measure. Recall that in the standard binomial model with constant interest rate, $\{M_t\}_{t=0,\dots,N}$ is an homogeneous Markov chain with transition probabilities

$$\mathbb{P}(m_{t-1} \to m_t) = \mathbb{P}(M_t = m_t | M_{t-1} = m_{t-1}) = \begin{cases} p & \text{if } m_t = m_{t-1}+1 \\ 1-p & \text{if } m_t = m_{t-1}-1 \end{cases},$$

where $m_t \in \mathrm{Im}(M_t) = \{-t, -t+2, \dots, t-2, t\}$, and in this case there exists a unique probability measure (i.e., a unique value of p), such that the discounted value of the stock is a martingale. It turns out that in the case of a stochastic interest rate we need to enlarge the class of admissible probability measures in order to find a martingale probability. We do so by assuming that $\{M_t\}_{t=0,\ldots,N}$ is a generalized Markov chain with transition probabilities

$$\mathbb{P}(m_{t-1} \to m_t) = \begin{cases} p_t(m_{t-1}) & \text{if } m_t = m_{t-1} + 1 \\ 1 - p_t(m_{t-1}) & \text{if } m_t = m_{t-1} - 1 \end{cases} \tag{4.2}$$

for some functions $p_t : \mathrm{Im}(M_{t-1}) \to (0, 1)$. We want to show that there exists a unique sequence of functions $p_1, \ldots, p_N$ such that

$$\mathbb{E}[S^*(t)|S^*(0), \ldots, S^*(t-1)] = S^*(t-1), \quad t = 1, 2, \ldots, N, \tag{4.3}$$

where we recall that

$$S^*(t) = D(t)S(t) = \frac{S(t)}{(1 + R(0))(1 + R(1)) \ldots (1 + R(t-1))}. \tag{4.4}$$

As the process $\{S(t), R(t)\}_{t=0,\ldots,N}$ carries the the same information as $\{M_t\}_{t=0,\ldots,N}$, then (4.3) is equivalent to

$$\mathbb{E}[S^*(t)|M_0, \ldots, M_{t-1}] = S^*(t-1), \quad t = 1, 2, \ldots, N. \tag{4.5}$$

Theorem 18. *The market* (4.1) *admits a martingale probability measure if and only if the parameters* $R(0), \sigma, \theta(t)$ *are such that*

$$e^d < 1 + R(0) + \theta(t) - \sigma t \text{ and } 1 + R(0) + \theta(t) + \sigma t < e^u, \text{ for all } t = 0, 1, \ldots, N-1. \tag{4.6}$$

Moreover, when it exists, the martingale probability measure is unique and it is given by (4.2) *with* $p_t = q_t$*, where* $q_t : \mathrm{Im}(M_{t-1}) \to (0, 1)$*,*

$$q_t(x) = \frac{1 + R(0) + \theta(t-1) + \sigma x - e^d}{e^u - e^d}, \quad t = 1, \ldots, N. \tag{4.7}$$

Thus, under the conditions (4.6)*, the market is complete.*

Proof. As $\{R(t)\}_{t=0,\ldots,N}$ is measurable with respect to $\{M_t\}_{t=0,\ldots,N}$, the the discount process can be taken out from the conditional expectation in the left hand side of (4.5), hence

$$\mathbb{E}[S^*(t)|M_0, \ldots, M_{t-1}] = \frac{\mathbb{E}[S(t)|M_0, \ldots, M_{t-1}]}{(1 + R(0)) \ldots (1 + R(t-1))} = \frac{\mathbb{E}[S(t)|M_{t-1}]}{(1 + R(0)) \ldots (1 + R(t-1))},$$

where for the second equality we use that $S(t)$ is measurable with respect to M_t and that $\{M_t\}_{t=0,\ldots,N}$ is a Markov process. Writing $S(t) = \frac{S(t)}{S(t-1)} S(t-1)$ and using that $S(t-1)$ is M_{t-1}-measurable we obtain

$$\mathbb{E}[S^*(t)|M_0, \ldots, M_{t-1}] = \frac{S(t-1)}{(1 + R(0)) \ldots (1 + R(t-1))} \mathbb{E}[\frac{S(t)}{S(t-1)}|M_{t-1}].$$

Next we use

$$\frac{S(t-1)}{(1 + R(0)) \ldots (1 + R(t-1))} = \frac{S^*(t-1)}{1 + R(t-1)}, \quad \frac{S(t)}{S(t-1)} = e^{\frac{u+d}{2}} e^{\frac{u-d}{2}(M_t - M_{t-1})}.$$

According to (4.2), the increments of the process $\{M_t\}_{t=0,\dots,N}$ satisfy

$$\mathbb{P}(M_t - M_{t-1} = 1 | M_{t-1} = m_{t-1}) = \mathbb{P}(M_t = m_{t-1} + 1) | M_{t-1} = m_{t-1} = p_t(m_{t-1}),$$
$$\mathbb{P}(M_t - M_{t-1} = -1 | M_{t-1} = m_{t-1}) = \mathbb{P}(M_t = m_{t-1} - 1) | M_{t-1} = m_{t-1} = 1 - p_t(m_{t-1}).$$

Hence

$$\mathbb{E}[S^*(t)|M_0, \dots, M_{t-1}] = \frac{S^*(t-1)}{1 + R(t-1)} \mathbb{E}[e^{\frac{u+d}{2}} e^{\frac{u-d}{2}(M_t - M_{t-1})} | M_{t-1}]$$
$$= S^*(t-1) \frac{e^{\frac{u+d}{2}}}{1 + R(0) + \theta(t-1) + \sigma m_{t-1}} \left(e^{\frac{u-d}{2}} p_t(m_{t-1}) + e^{-\frac{u-d}{2}} (1 - p_t(m_{t-1})) \right).$$

Thus in order for $p_t(m_{t-1})$ to be a martingale probability it must hold that

$$\frac{e^{\frac{u+d}{2}}}{1 + R(0) + \theta(t-1) + \sigma m_{t-1}} \left(e^{\frac{u-d}{2}} p_t(m_{t-1}) + e^{-\frac{u-d}{2}} (1 - p_t(m_{t-1})) \right) = 1.$$

Solving the latter equation we find that $p_t(x) = q_t(x)$ is given by (4.7). Moreover $0 < q_t(x) < 1$ for all $x \in \mathrm{Im}(M_{t-1})$ and $t = 1, \dots, N$ if and only if (4.6) is satisfied, which concludes the proof of the theorem. $\square$

Remark 18. It is clear that the transition martingale probabilities are constant if and only if $\sigma \equiv 0$ and $\theta(t) = 0$, for all $t = 1, \dots, N-1$, i.e., if and only if the spot rate is a deterministic constant, in which case Theorem 18 reduces to Theorem 3.

Thanks to the completeness of the market (4.1) we can now define a unique arbitrage-free price for European derivatives in this market.

Definition 9. *Consider a European derivative with maturity $T = N$ and pay-off Y which is measurable with respect to $\{M_t\}_{t=0,\dots,N}$ (e.g., $Y = g(S(N))$ for a standard European derivative on the stock). Assume that the market* (4.1) *is complete, i.e., the market parameters satisfy* (4.6)*. Then the risk-neutral price of the derivative is given by*

$$\Pi_Y(t) = D(t)^{-1} \widetilde{\mathbb{E}}[D(T)Y | M_0, \dots, M_t], \quad t = 0, \dots, T, \tag{4.8}$$

where $\widetilde{\mathbb{E}}$ denotes the (conditional) expectation in the martingale probability measure. In particular, $\Pi_Y(0) = \widetilde{\mathbb{E}}[D(T)Y]$ and $\Pi_Y(T) = Y$.

Remark 19. Note that the pay-off of the derivative may also depend on the interest rate process, which is the case for the interest rate contracts discussed in the previous sections. However, having now added a stock to the market, we *must* choose the martingale probability to price these contracts! Otherwise an arbitrage opportunity would arise by trading on ZCB's *and* the stock. For instance, letting $Y = 1$ in (4.8), the price of the ZCB with maturity T in the market (4.1) is given by

$$B(t, T) = D(t)^{-1} \widetilde{\mathbb{E}}[D(T) | M_0, \dots, M_t], \tag{4.9}$$

for $t = 0, \dots, T$, where the conditional expectation is taken with respect to the martingale probability (and not in the physical probability as in (3.30)).

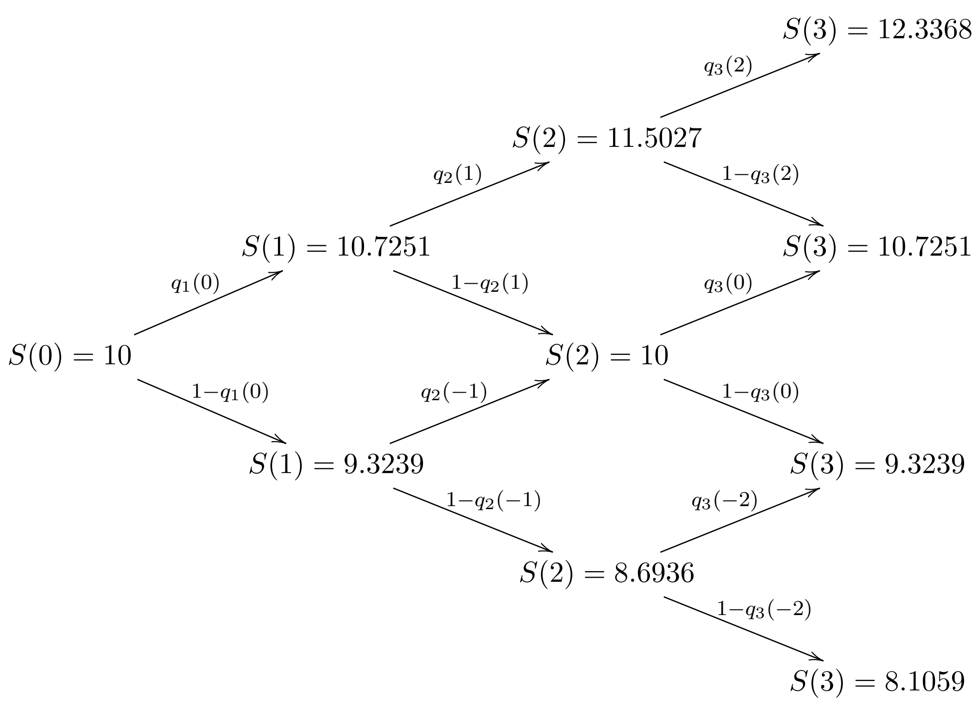
Example for $N = 3$. Consider a binomial stock price with $N = 3$, $u = -d = 0.07$, $S_0 = 10$ and a Ho-Lee model for the interest rate with parameters

$$R(0) = 0.03, \quad \theta(1) = 0.02, \quad \theta(2) = 0.01, \quad \sigma = 0.01.$$

The martingale transition probabilities are

$$\begin{aligned}
q_1(0) &= \frac{1 + R(0) - e^d}{e^u - e^d} = 0.6966 \\
q_2(1) &= \frac{1 + R(0) + \theta(1) + \sigma - e^d}{e^u - e^d} = 0.9107 \\
q_2(-1) &= \frac{1 + R(0) + \theta(1) - \sigma - e^d}{e^u - e^d} = 0.7680 \\
q_3(2) &= \frac{1 + R(0) + \theta(2) + 2\sigma - e^{,}3d}{e^u - e^d} = 0.9107 \\
q_3(0) &= \frac{1 + R(0) + \theta(2) - e^d}{e^u - e^d} = 0.7680 \\
q_3(-2) &= \frac{1 + R(0) + \theta(2) - 2\sigma - e^d}{e^u - e^d} = 0.6252
\end{aligned}$$

As $q_t(x) \in (0, 1)$, the market is complete. The binomial tree for the stock price in the risk-neutral probability measure is as follows



The binomial tree for the interest rate is

$$
\begin{tikzcd}
 & & R(2) = 0.06 \\
 & R(1) = 0.06 \arrow[ur, "q_2(1)"] \arrow[dr, "1-q_2(1)"] & \\
R(0) = 0.03 \arrow[ur, "q_1(0)"] \arrow[dr, "1-q_1(0)"] & & R(2) = 0.04 \\
 & R(1) = 0.04 \arrow[ur, "q_2(-1)"] \arrow[dr, "1-q_2(-1)"] & \\
 & & R(2) = 0.02
\end{tikzcd}
$$

We don't need to compute $R(3)$. The discount process is

$$D(0) = 1, \quad D(1) = \frac{1}{1+R(0)} = 0.9709, \quad \text{with prob. } 1,$$

$$D(2) = \frac{D(1)}{1+R(1)} = \begin{cases} \frac{0.9709}{1+0.06} = 0.9159, & \text{with prob. } q_1(0) \\ \frac{0.9709}{1+0.04} = 0.9336 & \text{with prob. } 1-q_1(0) \end{cases},$$

$$D(3) = \frac{D(2)}{1+R(2)} = \begin{cases} \frac{0.9159}{1+0.06} = 0.8641, & \text{with prob. } q_1(0)q_2(1) \\ \frac{0.9159}{1+0.04} = 0.8807 & \text{with prob. } q_1(0)(1-q_2(1)) \\ \frac{0.9336}{1+0.04} = 0.8977 & \text{with prob. } (1-q_1(0))q_2(-1) \\ \frac{0.9336}{1+0.02} = 0.9153 & \text{with prob. } (1-q_1(0))(1-q_2(-1)) \end{cases}$$

Now assume that we want to compute the initial price of a call option on the stock with strike $K = 10$ and maturity $T = 3$. According to Definition 9, this price is given by

$$\Pi(0) = \widetilde{\mathbb{E}}[D(3)(S(3)-10)_+],$$

where the expectation is in the martingale probability $q_t(x)$. To compute this expectation we need the joint distribution of the random variables $D(3), S(3)$. Using our results above we find that this joint distribution is given as in the following table:

$D(3)\downarrow$, $S(3)\rightarrow$	12.3368	10.7251	9.3239	8.1059
0.8641	$q_1(0)q_2(1)q_3(2)$	$q_1(0)q_2(1)(1-q_3(2))$	0	0
0.8807	0	$q_1(0)(1-q_2(1))q_3(0)$	$q_1(0)(1-q_2(1))(1-q_3(0))$	0
0.8977	0	$(1-q_1(0))q_2(-1)q_3(0)$	$(1-q_1(0))q_2(-1)(1-q_3(0))$	0
0.9153	0	0	$(1-q_1(0))(1-q_2(-1))q_3(-2)$	$(1-q_1(0))(1-q_2(-1))(1-q_3(-2))$

We conclude that

$$\begin{aligned}\Pi(0) = 0.8641[&(12.3368-10)q_1(0)q_2(1)q_3(2) + (10.7251-10)q_1(0)q_2(1)(1-q_3(2))] \\ &+ 0.8807(10.7251-10)q_1(0)(1-q_2(1))q_3(0) + 0.8977(10.7251-10)(1-q_1(0))q_2(-1)q_3(0) \\ &= 1.3491.\end{aligned}$$

Exercise 24. *In the example just considered compute the possible prices of the call at times* $t = 1, 2$.

For future purpose we also compute the initial price of the ZCB expiring at time $T = 3$. According to (9), the value at time $t = 0$ it is given by $B(0,T) = \widetilde{\mathbb{E}}[D(T)]$. Hence

$$B(0,3) = \widetilde{\mathbb{E}}[D(3)] = 0.8641q_1(0)q_2(1) + 0.8807q_1(0)(1 - q_2(1)) + 0.8977(1 - q_1(0))q_2(-1) + 0.9153(1 - q_1(0))(1 - q_2(-1)) = 0.8766 \tag{4.10}$$

4.2 Forward contracts

A **forward contract** with **delivery price** K and maturity (or delivery) time T on an asset $\mathcal{U}$ is a European type of financial derivative stipulated by two parties in which one promises to the other to sell (and possibly deliver) the asset $\mathcal{U}$ at time T in exchange for the cash K. As opposed to option contracts, both parties in a forward contract are *obliged* to fulfill their part of the agreement. In particular, as they both have the same right/obligation, neither of the two parties has to pay a premium to the other when the contract is stipulated, that is to say, *forward contracts are free*[1]. The party who must sell the asset at maturity holds the short position, while the party who must buy the asset is the holder of the long position. Forward contracts are traded OTC (over the counter) and most commonly on commodities or market indexes, such as currency exchange rates, interest rates and volatilities. In the case that the underlying asset is an index, forward contracts are also called **swaps** (e.g., currency swaps, interest rate swaps, volatility swaps, etc.). Let us give two examples.

Example of forward contract on a commodity. Consider a farmer who grows wheat and a miller who needs wheat to produce flour. Clearly, the farmer interest is to sell the wheat for the highest possible price, while the miller interest is to pay the least possible for the wheat. The price of the wheat depends on many economical and non-economical factors (such as whether conditions, which affect the quality and quantity of harvests) and it is therefore quite volatile. As a form to reduce risks, the farmer and the miller stipulate a forward contract on the wheat in the winter (before the plantation, which occurs in the spring) with expiration date in the end of the summer (when harvest occurs) in order to lock the future price of the wheat at a value which is convenient for both of them.

Example of a currency swap. Suppose that a car company in Sweden promises to deliver a stock of 100 cars to another company in the United States in exactly one month. Suppose that the price of each car is fixed in Swedish crowns, say 100.000 crowns. Clearly the American company will benefit by an increase of the exchange rate crown/dollars and will be damaged in the opposite case. To avoid possible high losses, the American company stipulates a forward contract (currency swap) on 100×100.000=ten millions Swedish crowns

[1] In fact, the terminology used for forward contracts is "to enter a forward contract" and not "to buy/sell a forward contract".

expiring in one month and which gives the company the right *and* the obligation to buy ten millions crowns for a price in dollars agreed upon today. The other party of the forward contract could be a company exposed to the opposite risk, i.e., to an increase of the exchange rate crown/dollars.

As it is clear from these examples, one purpose of forward contracts is to share risks. Irrespective of the movement of the underlying asset in the market, its price at time T for the holders of the forward contract will be K. We may define the pay-off for a long position in a forward contract as

$$Y_{\mathrm{long}} = (\Pi(T) - K),$$

while for the holder of the short position the pay-off is

$$Y_{\mathrm{short}} = (K - \Pi(T)).$$

Here $\Pi(T)$ denotes the market price of the asset at time T. Note that one of Y_{long} or Y_{short} is certainly going to be negative (except in the unlike event that $\Pi(T) = K$), which means that one of the two parties of a forward contract is always going to incur in a loss. If the loss is very large, this party could become insolvent, i.e., unable to fulfill the contract, and then both parties will end up losing. As explained below, the risk of insolvency is greatly reduced by trading futures contracts instead of forward contracts.

The delivery price agreed by the two parties in a forward contract may be seen as a pondered estimation of the value that the underlying asset will have at the time T in the future. In this respect, K is also called the **forward price** of the asset. More precisely, the T-forward price $\mathrm{For}(t, T)$ of an asset $\mathcal{U}$ at time $t < T$ is the strike price of a forward contract on $\mathcal{U}$ stipulated at time t and with maturity T, while the current, actual price $\Pi(t)$ of the asset is called the **spot** price[2]. Clearly the forward price is unlikely to be a good estimation for the price of the asset at time T, since the consensus on this value is limited to the participants of the forward contract and different parties may agree to different delivery prices. The delivery price of futures contracts on the asset, which we define below, gives a better and more commonly accepted estimation for the future value of an asset.

When an arbitrage-free model for the value of forward contracts is available, the fair (arbitrage-free) forward price of the asset $\mathcal{U}$ may be defined as the value of K which makes the theoretical arbitrage-free price of the forward equal to zero. Let us see two examples.

Forward price in market models with constant interest rate

Consider a market model which consists of the asset $\mathcal{U}$ and a risk-free asset with constant simply compounded spot rate R. We do not assign a specific dynamics for the price $\Pi(t)$ of the asset $\mathcal{U}$, however we assume that the market is arbitrage-free (e.g., by requiring that the dominance principle holds, see [2]).

[2] Note however that some assets, such as commodities, do not have an official spot price, but only a future price, as defined in Section 4.3.

Theorem 19. *In an arbitrage-free market with constant spot rate R, the forward price of the asset $\mathcal{U}$ is given by*

$$\mathrm{For}(t,T) = \Pi(t)(1+R)^t. \tag{4.11}$$

Proof. The proof is very simple. First we remark that in an arbitrage-free market having a long position on a forward contract with maturity T and delivery price K is equivalent to hold a portfolio which is long 1 share of the call on the asset and short 1 share of the put on the asset, both options with maturity T and strike price K. Indeed, irrespective of the price of the asset at time T, this portfolio entails that we *have* to buy the asset at time T for the price K. Hence if we denote by $F(t), C(t), P(t)$ the arbitrage-free value at time $t < T$ of the forward contract, the call option and the put option respectively, then it must hold

$$C(t) - P(t) = F(t).$$

On the other hand, by the put-call parity[3],

$$C(t) - P(t) = \Pi(t) - \frac{K}{(1+R)^t}.$$

Hence $F(t) = 0$ if and only if $K = \Pi(t)(1+R)^t$, which completes the proof. □

Forward price in market models with stochastic interest rate

When the delivery time is far in the future, it is not realistic to assume that interest rates remain constant. The uncertainty of money market returns affects heavily the value of the asset forward price. Assume for instance that the market model for the underlying asset and for the spot rate is given by the binomial model in Section 4.1. That is to say, we assume that the processes $\{\Pi(t), t = 0, \dots, T\}$ and $\{R(t), t = 0, \dots, T-1\}$ satisfy

$$\Pi(t) = \Pi(0) \exp\left[t\left(\frac{u+d}{2}\right) + \left(\frac{u-d}{2}\right) M_t\right], \quad t = 0, 1, 2, \dots T, \tag{4.12a}$$

$$R(t) = R(0) + \theta(t)\sigma M_t, \quad t = 0, 1, 2, \dots, T-1, \tag{4.12b}$$

where $\{M_t\}_{t=0,\dots,T}$ is a generalized Markov process with transition probabilities given by (4.7), i.e., we assume that the market model is complete.

Theorem 20. *The forward price of the asset in the complete market* (4.12) *is given by*

$$\mathrm{For}(t,T) = \frac{\Pi(t)}{B(t,T)}, \tag{4.13}$$

where $B(t,T)$ is given by (4.9)*, that is to say, $B(t,T)$ is the risk-neutral price at time t of the ZCB expiring at time T and computed with the martingale probability measure.*

[3] Remember that the put-call parity holds in arbitrage-free markets, e.g., as a consequence of the dominance principle, see [2].

Proof. Using the pay-off $Y_{\text{long}} = \Pi(T) - K$ of a long position on the forward contract and Definition 9, the value $F(t)$ of the forward contract at time t is given by

$$F(t) = D(t)^{-1}\widetilde{\mathbb{E}}[D(T)(\Pi(T)-K)|M_0,\dots,M_t] = D(t)^{-1}\widetilde{\mathbb{E}}[D(T)\Pi(T)|M_0,\dots,M_t]-KB(t,T).$$

As $\{\Pi^*(t), t = 0,\dots,T\}$ is a martingale in the risk-neutral probability measure, then

$$\widetilde{\mathbb{E}}[D(T)\Pi(T)|M_0,\dots,M_{T-1}] = D(T-1)\Pi(T-1).$$

Hence

$$\begin{aligned}\widetilde{\mathbb{E}}[D(T)\Pi(T)|M_0,\dots,M_{T-2}] &= \widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[D(T)\Pi(T)|M_0,\dots,M_{T-1}]|M_0,\dots M_{T-2}] \\ &= \widetilde{\mathbb{E}}[D(T-1)\Pi(T-1)|M_0,\dots,M_{T-2}].\end{aligned}$$

Iterating the previous identity we obtain

$$\widetilde{\mathbb{E}}[D(T)\Pi(T)|M_0,\dots,M_t] = \widetilde{\mathbb{E}}[D(t+1)\Pi(t+1)|M_0,\dots,M_t] = D(t)\Pi(t).$$

Hence $F(t) = \Pi(t) - KB(t,T)$ and so $F(t) = 0$ if and only if $K = \mathrm{For}(t,T)$ is given by (4.13). □

Remark 20. The formula (4.13) holds in any complete market model, not only in the example studied in this section. Moreover (4.13) reduces to (4.11) when $R(t)$ is a deterministic constant.

For instance, in the 3-period market model considered in Section 4.1, we have found $B(0,3) = 0.8766$, see (4.10). Hence the 3-forward price at time $t = 0$ of the stock in that market is $\mathrm{For}(0,T) = S(0)/B(0,3) = 10/0.8731 = 11.4077$.

4.3 Futures

Futures contracts are standardized forward contracts, i.e., rather than being OTC, they are negotiated in regularized markets. On of the most interesting aspects of futures contracts is that they make trading on commodities open for anyone. To this regard we remark that commodities, e.g. oil, wheat, gold, etc, are most often sold through long term contracts, such as forward and futures contracts, and therefore they do not usually have an “official spot price”, but only a future delivery price (commodities “spot markets” exist, but their role is marginal for the discussion in this section).

Futures markets are markets in which the object of trading are futures contracts. Unlike forward contracts, all futures contracts in a futures market are subject to the same regulation, and so in particular all contracts on the same asset $\mathcal{U}$ with the same time of maturity T have the same delivery price, which is called T-**future price** of the asset and denoted by $\mathrm{Fut}(t,T)$. Thus $\mathrm{Fut}(t,T)$ is the delivery price in a futures contract on the asset $\mathcal{U}$ with time of maturity T which is stipulated at time $t < T$. Futures markets have been existing for more than 300 years and nowadays the most important ones are the Chicago Mercantile Exchange (CME), the New York Mercantile Exchange (NYMEX), the Chicago Board of Trade (CBOT) and the International Exchange Group (ICE).

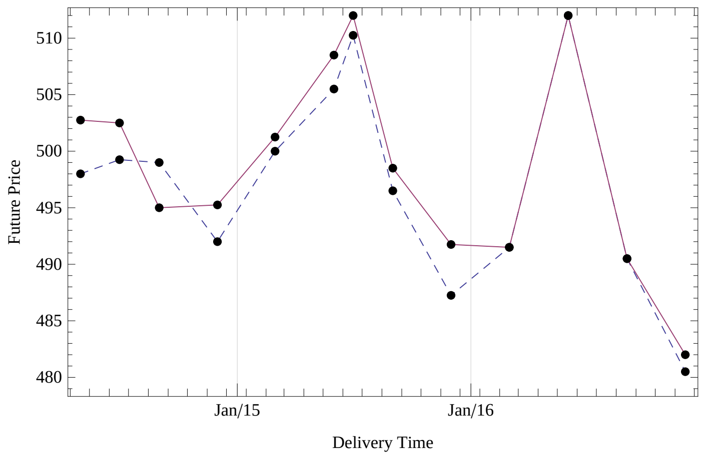


Figure 4.1: Futures price of corn on May 12, 2014 (dashed line) and on May 13, 2014 (continuous line) for different delivery times

In a futures market, anyone (after a proper authorization) can enter a futures contract. More precisely, holding a long position in a futures contract in the futures market consists in the agreement to receive as a cash flow the change in the future price of the underlying asset during the time in which the position is held, while a short position on the same contract receives the opposite cash flow. Notice that the cash flow may be positive or negative. In a long position the cash flow is positive when the future price goes up and it is negative when the future price goes down. Moreover, in order to eliminate the risk of insolvency, the cash flow is distributed in time through the mechanism of the **margin account**. For example, assume that at $t = 0$ we open a long position in a futures contract expiring at time T. At the same time, we need to open a margin account which contains a certain amount of cash (usually, 10 % of the current value of the T-future price for each contract opened). At $t = 1$ day, the amount $\mathrm{Fut}(1, T) - \mathrm{Fut}(0, T)$ will be added to the account, if it positive, or withdrawn, if it is negative. The position can be closed at any time $t < T$ (multiple of days), in which case the total amount of cash flown in the margin account is

$$\begin{gathered}(\mathrm{Fut}(t, T) - \mathrm{Fut}(t-1, T)) + (\mathrm{Fut}(t-1, T) - \mathrm{Fut}(t-2, T)) + \\ \cdots + (\mathrm{Fut}(1, T) - \mathrm{Fut}(0, T)) = (\mathrm{Fut}(t, T) - \mathrm{Fut}(0, T)).\end{gathered}$$

(In fact, if the margin account becomes too low, and the investor does not add new cash to it, the position will be automatically closed by the exchange market).

If a long position is held up to the time of maturity, then the holder of the long position should buy the underlying asset. However futures contracts are often **cash settled** and not **physically settled**, which means that the underlying asset is not delivered and the owner of the Futures contract is paid-off by the cash-flow.

Our next purpose is to introduce the definition of arbitrage-free future price of an asset. Our

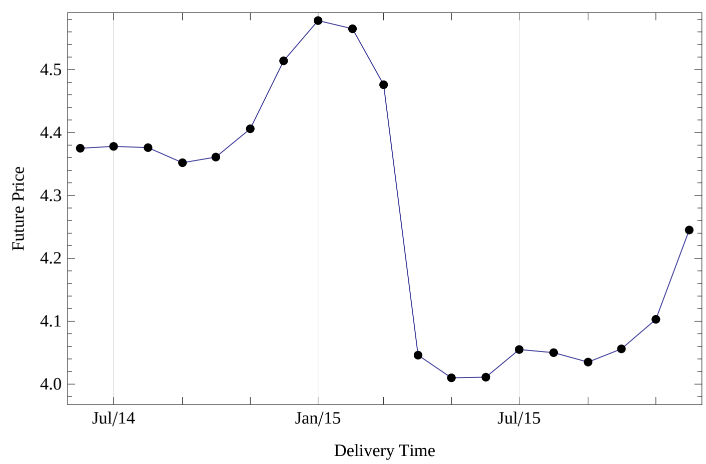


Figure 4.2: Futures price of natural gas on May 13, 2014 for different delivery times

strategy is to show that any reasonable definition should satisfy 3 natural conditions and then show that these conditions define uniquely the future price as a stochastic process.

For simplicity we argue under the assumption that the underlying asset and the money market make up the complete market model studied in Sections 4.1 and 4.2. Hence the price process $\{\Pi(t), t = 0, \ldots, T\}$ of the underlying asset and the interest rate of the money market $\{R(t), t = 0, \ldots, T-1\}$ are given by (4.12), where $\{M_t\}_{t=0,\ldots,T}$ is a generalized Markov process with transition probabilities given by (4.7). As the process $\{M_t\}_{t=0,\ldots,T}$ contains all the information about the state of the market, we are naturally led to impose the following first condition on the future price.

Assumption 1. The future price $\mathrm{Fut}(t, T)$ is measurable with respect to $M_0, \ldots, M_t$.

For the next assumption we need to define the concept of self-financing portfolio process invested in the futures contract and the money market. Consider a portfolio process that, at time $t < T$, consists of $h(t)$ shares of the futures contract expiring at time T and $h_{t+1}(t)$ shares of the bond maturing at time $t + 1$ (which is the position in the money market). We assume that the portfolio process is predictable, i.e., the positions at time t are measurable with respect to $M_0, \ldots, M_t$. As futures contracts have zero value, the value of the portfolio at time t is simply the money market account:

$$V(t) = h_{t+1}(t)B(t, t+1) = \frac{h_{t+1}(t)}{1 + R(t)}.$$

At time $t + 1$ the portfolio generates the cash flow

$$C(t+1) = h_{t+1}(t) + h(t)(\mathrm{Fut}(t+1, T) - \mathrm{Fut}(t, T)) = V(t)(1+R(t)) + h(t)(\mathrm{Fut}(t+1, T) - \mathrm{Fut}(t, T)).$$

In a self-financing portfolio this cash should be immediately re-invested in the money market[4]. Hence $C(t+1) = h_{t+2}(t+1)B(t+1,t+2) = V(t+1)$. It follows that

$$\begin{aligned} h(t)(\mathrm{Fut}(t+1,T) - \mathrm{Fut}(t,T)) &= V(t+1) - (1+R(t))V(t) = V(t+1) - \frac{D(t)}{D(t+1)}V(t) \\ &= D(t+1)^{-1}[D(t+1)V(t+1) - D(t)V(t)]. \end{aligned}$$

Hence

$$h(t)D(t+1)(\mathrm{Fut}(t+1,T) - \mathrm{Fut}(t,T)) = V^*(t+1) - V^*(t). \tag{4.14}$$

Definition 10. *A predictable portfolio process invested in the futures contract and the money market is said to be self-financing if its discounted value satisfies* (4.14)*.*

Now, by the arbitrage-free principle, the market under study should be arbitrage-free. We have seen that this condition can be achieved by imposing that the discounted value of predictable self-financing portfolio processes is a martingale. This holds in particular if

$$\widetilde{\mathbb{E}}[V^*(t+1)|M_0,\dots,M_t] = V^*(t),$$

for all $t = 0,\dots,T-1$. Hence, taking the conditional expectation of both sides of (4.14) with respect to $M_0,\dots,M_t$, we obtain

$$h(t)D(t+1)\widetilde{\mathbb{E}}[\mathrm{Fut}(t+1,T) - \mathrm{Fut}(t,T)|M_0,\dots,M_t] = \widetilde{\mathbb{E}}[V^*(t+1) - V^*(t)|M_0,\dots,M_t] = 0,$$

where we used that $h(t)$ and $D(t+1)$ are measurable with respect to $M_0,\dots,M_t$ and thus can be taken out from the conditional expectation. By Assumption 1 we have

$$\widetilde{\mathbb{E}}[\mathrm{Fut}(t+1,T) - \mathrm{Fut}(t,T)|M_0,\dots,M_t] = \widetilde{\mathbb{E}}[\mathrm{Fut}(t+1,T)|M_0,\dots,M_t] - \mathrm{Fut}(t,T).$$

Hence the requirement that the market is arbitrage-free leads naturally to assume the following

Assumption 2. The future price is a martingale with respect to the process $\{M_t\}_{t=0,\dots,N}$, that is

$$\widetilde{\mathbb{E}}[\mathrm{Fut}(t+1,T)|M_0,\dots,M_t] = \mathrm{Fut}(t,T), \quad t = 0,\dots,T-1.$$

The last natural assumption is that the future price at maturity $t = T$ should coincide with the spot price $\Pi(T)$ of the asset, i.e.,

Assumption 3. $\mathrm{Fut}(T,T) = \Pi(T)$.

Theorem 21. *There is only one stochastic process* $\{\mathrm{Fut}(t,T), t = 0,\dots,T\}$ *that satisfies Assumptions 1-3, namely*

$$\mathrm{Fut}(t,T) = \widetilde{\mathbb{E}}[\Pi(T)|M_0,\dots,M_t]. \tag{4.15}$$

[4]This is the only possibility, as changing position on the futures contract costs nothing.

Proof. The proof that (4.15) satisfies 1-3 follows easily by the properties of the conditional expectation and is left as an exercise. Now, Assumptions 2-3 imply directly that (4.15) holds at time $t = T - 1$, for

$$\mathrm{Fut}(T-1,T) = \widetilde{\mathbb{E}}[\mathrm{Fut}(T,T)|M_0,\dots,M_{T-1}] = \widetilde{\mathbb{E}}[\Pi(\mathrm{T})|M_0,\dots,M_{T-1}],$$

where we used Assumption 2 with $t = T - 1$ in the first equality, and assumption 3 in the second equality. Now the proof of the theorem can be easily completed by induction. $\square$

Exercise 25. *Give all the details missing in the above proof of Theorem 21.*

As an example, consider again the 3-period model in Section 4.1. The 3-future price at time $t = 0$ of the stock in that market is

$$\begin{aligned}
&\mathrm{Fut}(0,3) = \widetilde{\mathbb{E}}[S(3)] = q_1(0)q_2(1)q_3(2)12.3368\\
&+ [q_1(0)q_2(1)(1-q_3(2)) + q_1(0)(1-q_2(1))q_3(0) + (1-q_1(0))q_2(-1)q_3(0)]10.7251\\
&+ [q_1(0)(1-q_2(1))(1-q_3(0)) + (1-q_1(0))q_2(-1)(1-q_3(0))\\
&+ (1-q_1(0))(1-q_2(-1))q_3(-2)]9.3239\\
&+ (1-q_1(0))(1-q_2(-1))(1-q_3(-2))6.1059 = 11.4295
\end{aligned}$$

Recall that the 3-forward price at time $t = 0$ of the same stock is 11.4077, which was computed at the end of Section 4.2. The general relation between forward and future price of an asset is given in the following theorem.

Theorem 22. *Denote* $\widetilde{\mathbb{E}}_t[\cdot] = \widetilde{\mathbb{E}}[\cdot|M_0,\dots,M_t]$*. The* **Forward-Future spread** *of an asset, i.e., the difference between its forward and future price, satisfies*

$$\mathrm{For}(t,T) - \mathrm{Fut}(t,T) = \frac{1}{\widetilde{\mathbb{E}}_t[D(T)]}\{\widetilde{\mathbb{E}}_t[D(T)\Pi(T)] - \widetilde{\mathbb{E}}_t[D(T)]\widetilde{\mathbb{E}}_t[\Pi(T)]\}.$$

Moreover if the interest rate of the money market is deterministic then $\mathrm{Fut}(t,T) = \mathrm{For}(t,T)$*, for all* $t = 0,\dots,T$*.*

Exercise 26. *Prove the theorem. TIP: Use the definition of Forward/Futures price, the bond price formula (4.9) and the fact that the discounted price of the underlying asset is a martingale.*

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