Exam: Finansiell Risk, MVE 220/MSA400
Wednesday, Januari 9, 14.00-18.00
Jour: Edvin Wedin ankn 5325

Allowed material: List of Formulas, Chalmers allowed calculator.

Problems 1-4: Multiple choice, only hand in table with answers
Only one correct answer. Correct answer gives 5 points, no answer ("don't know") gives 0 points and wrong answer gives -1 point (more than one answer automatically gives -1 point).

Uppgift	a	b	c	d	e	f (Don't know)	Points
1	☐	☐	☐	☒	☐	☐	
2	☐	☐	☐	☒	☐	☐	
3	☐	☐	☒	☐	☐	☐	
4	☐	☐	☐	☐	☒	☐	

Problems 5-10: Hand in full solutions

Good Luck!

Q5) a) The probability density function is $f(x) = \frac{d}{dx}F(x) = \frac{1}{\sigma} e^{-x/\sigma}$.
Hence the log likelihood function, $l(\cancel{\alpha})$, is given by

$$l(\cancel{\alpha}) = \sum_{i=1}^{n} \log f(x_i)$$

$$= \sum_{i=1}^{n} \left\{ -\log\sigma - \frac{x_i}{\sigma} \right\}$$

$$= -n\log\sigma - \frac{1}{\sigma}\sum_{i=1}^{n} x_i$$

b) $$\frac{d}{d\sigma} l(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} x_i =: \bar{X}$$

since, e.g., $l(0) = l(\infty) = \infty$ this maximises the likelihood function, and hence $\hat{\sigma} = \bar{X}$ is the maximum likelihood estimator of σ.

c) $\bar{X} = 1.45$ is the maximum likelihood estimator of σ. The median of the distribution is obtained as the solution of the equation

$$0.5 = F(x_{0.5}) = 1 - e^{-\frac{x_{0.5}}{\sigma}}$$

$$\iff$$

$$e^{-\frac{x_{0.5}}{\sigma}} = 0.5$$

$$x_{0.5} = -\sigma \log(0.5)$$

Inserting the estimated value of σ one obtains the maximum likelihood estimator of $x_{0.5}$ as

$$\hat{X}_{0.5} = -1.45 \log(0.5) = 1.01$$

Q6) a) It is easier to work with

$$\bar{F}(x) := 1 - F(x) = \left(1 + \gamma \frac{x}{\sigma}\right)_+^{-1/\gamma}.$$

$$\bar{F}_u(x) = 1 - F_u(x) = P(X - u > x \mid X > u)$$

$$= \frac{P(X > x + u \text{ and } X > u)}{P(X > u)}$$

$$= \frac{\bar{F}(x+u)}{\bar{F}(u)}$$

$$= \frac{\left(1 + \gamma \frac{x+u}{\sigma}\right)^{-1/\gamma}}{\left(1 + \gamma \frac{u}{\sigma}\right)^{-1/\gamma}}$$

$$= \left(1 + \frac{\gamma \frac{x}{\sigma}}{1 + \gamma \frac{u}{\sigma}}\right)^{-1/\gamma}$$

$$= \left(1 + \gamma \frac{x}{\sigma + \gamma u}\right)^{-1/\gamma}.$$

Hence $F_u(x)$ is a GP distribution with scale parameter $\sigma + \gamma u$ and shape parameter γ

b) For $\gamma = 0$, $\bar{F}(x) = e^{-x/\sigma}$ and hence

$$\bar{F}_u(x) = \frac{e^{-(x+u)/\sigma}}{e^{-u/\sigma}} = e^{-x/\sigma},$$

so again F_u is a GP distribution, now with scale parameter σ and shape parameter $\gamma = 0$ (so the exponential distribution is "memoryless": the conditioning doesn't change the distribution)

For $\gamma < 0$, both the computations and the result is the same as in a)

(Q7) a) Let X have the distribution of a hourly loss. Then

$$(*) \quad P(X > 55.4) = P(X > 55.4 \mid X > 30)\,P(X > 30)$$

$$= P(X - 30 > \underbrace{55.4 - 30}_{=25.4} \mid X > 30)\,P(X > 30)$$

Here $P(X > 30)$ may be estimated by the frequency of of exceedances of 30, i.e. by $152/17531 = 0.00867$, and $P(X - 30 > 25.4 \mid X > 30)$ is estimated by $\bar{F}(25.4)$ where $\bar{F}(x)$ is the fitted GP distribution of the excesses. Thus, with ^ denoting estimated values

$$\hat{\bar{F}}(25.4) = \left(1 + \hat{\gamma}\,\frac{25.4}{\hat{\sigma}}\right)^{-1/\hat{\gamma}}$$

$$= \left(1 + 0.1845\,\frac{25.4}{7.440}\right)^{-1/0.1845}$$

$$= 0.07081$$

Hence the probability that an hourly loss exceed 55.4 bp is estimated to be

$$0.07081 \cdot 0.00867 = 0.00061$$

b) The proportion of losses which exceed 55.4 is $11/17531 \approx 0.00063$.

Hence, this estimate is quite similar to the estimate in a). However it is more variable (e.g. the estimate changes to 0.00068 if 11 is changed to 12). Instead the estimate in a) may be wrong (but still less variable) if the GP model does not fit the data.)

c) An asymptotically normally distributed test of the hypotheses that $\gamma = 0$ uses the test quantity

$$\frac{0.1845}{\sqrt{0.01024}} = 1.83$$

since this is larger than 1.64 but smaller than 1.96, a two-sided test rejects at the significance level 0.1 but not at the significance level 0.05.

Task 8: Financial risk, 2019-09-09 (8.1)

Let L_m be the portfolio credit loss in a mixed binomial logit-normal model. We want to compute the probability $P[30 \cdot 10^6 < L_m < 80 \cdot 10^6]$ by using LPA-theory.

Let $L_m = 10^6 \cdot \ell \cdot N_m$

$$a_0 = a \cdot 10^6$$
$$b_0 = b \cdot 10^6$$
$$\ell = 0.6$$

where $a = 30$ and $b = 80$.

So we want to use the LPA-formula to compute $P[a_0 < L_m < b_0]$. But note that

$$P[a_0 < L_m < b_0] = P[a \, 10^6 < 10^6 \cdot \ell \cdot N_m < b \cdot 10^6]$$

$$= P\left[\frac{a}{\ell \cdot m} < \frac{N_m}{m} < \frac{b}{\ell m}\right] \qquad (1)$$

Note that $\frac{N_m}{m}$ is approximatly (8.2)

a continuous random variable when m is large. which in (1) implies that

$$P[a_0 < L_m < b_0] = P\left[\frac{a}{lm} < \frac{N}{m} < \frac{b}{lm}\right]$$

$$\approx P\left[\frac{a}{lm} < \frac{N_m}{m} \leq \frac{b}{lm}\right] =$$

$$= P\left[\frac{N_m}{m} \leq \frac{b}{l \cdot m}\right] - P\left[\frac{N_m}{m} \leq \frac{a}{lm}\right]$$

$\approx F\left(\frac{b}{lm}\right) - F\left(\frac{a}{l \cdot m}\right)$ when m is large and

where $F(x) = P[p(Z) \leq x]$

Hence, we have that when m is large then

$$P[a_0 < L_m < b_0] \approx F\left(\frac{b}{l \cdot m}\right) - F\left(\frac{a}{l \cdot m}\right) \quad (2)$$

where $F(x) = P[p(Z) \leq x]$. In our case $Z \sim N(0,1)$ and $p(x)$ is given by

$$p(x) = \frac{1}{1 + e^{-\mu - \sigma x}} \quad (*)$$

Note that $p(x)$ is strictly increasing (8.3)
since $p'(x) = \frac{d}{dx}\left((1+e^{-\mu-\sigma x})^{-1}\right)$
$= (-1)(-\sigma e^{-\mu-\sigma x})(1+e^{-\mu-\sigma x})^{-2} = \frac{\sigma e^{-\mu-\sigma x}}{(1+e^{-\mu-\sigma x})^2} > 0$
so $p'(x) > 0$ for all $x \in \mathbb{R}$. Hence, $p(x)$ is strictly increasing, and since $p(x)$ is ~~also~~ continuous we know that $p^{-1}(x)$ exsist, is continuous and is well-defined. Hence, we have that

$$F(x) = P[p(Z) \leq x] = P[Z \leq p^{-1}(x)] = N(p^{-1}(x))$$

so $F(x) = N(p^{-1}(x))$ (3)

Next we find an expression for $p^{-1}(x)$. To find $p^{-1}(\cdot)$, we solve for x in the equation $y = p(x) \Leftrightarrow x = p^{-1}(y)$.

So for $y \in (0,1)$ we have that (8.4)

$$y = p(x) \Leftrightarrow y = \frac{1}{1+e^{-\mu-\sigma x}} \Leftrightarrow \frac{1}{y} = 1+e^{-\mu-\sigma x}$$

$$\Leftrightarrow \frac{1-y}{y} = e^{-\mu-\sigma x} \Leftrightarrow -\mu-\sigma x = \ln\left(\frac{1-y}{y}\right)$$

$$\Leftrightarrow x = -\frac{1}{\sigma}\left(\ln\left(\frac{1-y}{y}\right)+\mu\right)$$

Hence, $p^{-1}(y) = -\frac{1}{\sigma}\left(\ln\left(\frac{1-y}{y}\right)+\mu\right)$

or $p^{-1}(x) = \frac{1}{\sigma}\left(\ln\left(\frac{x}{1-x}\right)-\mu\right)$ (4)

Thus, (2) and (3) implies that

$$P[a_0 < L_m < b_0] \approx N\left(p^{-1}\left(\frac{b}{lm}\right)\right) - N\left(p^{-1}\left(\frac{a}{l \cdot m}\right)\right) \quad (5)$$

where $p^{-1}(x)$ is given by (4).

We know the value of μ but not the value of σ. However, we know that

$$VaR_{95\%}(L) = C \quad (6)$$

where $C = C_0 \cdot 10^6$ and $C_0 = 172.3$

and where the VaR-computation (8.5)
in Eq. (6) is done using the LPA-
-approach. So we will use (6) to find
the parameter σ.
To do this we need to derive an
explicit expression for $VaR_\alpha(L)$ in
the mixed binomial logit normal
modell.
By definition we have that

$$VaR_\alpha(L_m) = F^{\leftarrow}_{L_m}(\alpha) \quad (7)$$

where $F_{L_m}(x) = P[L_m \leq x]$ (8)

and $L_m = 10^6 \cdot \ell \cdot N_m$ with $N_m = \sum_{i=1}^{m} X_i$
where $X_i = 1$ if obligor 1 defaults up to time
T and $X_i = 0$ otherwise. Furthermore,
$F^{\leftarrow}_{L_m}(x)$ is the generalized inverse to
the function $F_{L_m}(x)$.

By linearity of VaR we have that (8.6)

$$VaR_\alpha(L_m) = VaR_\alpha(10^6 \cdot l \cdot N_m)$$
$$= 10^6 \cdot VaR_\alpha(l \cdot N_m) \quad (9)$$

So combining (6) and (9) we get

$$10^6 \cdot VaR_\alpha(l \cdot N_m) = C = C_0 \cdot 10^6 \Leftrightarrow$$
$$VaR_\alpha(l \cdot N_m) = C_0 \quad (10)$$

Hence, in view of (9) & (10) we can without loss of generality assume that $L_m = l \cdot N_m$.

From LPA-theory we know that

$$F_{L_m}(x) = P[L_m \leq x] = P\left[\frac{N_m}{m} \leq \frac{x}{l \cdot m}\right] \to F\left(\frac{x}{l \cdot m}\right)$$

when $m \to \infty$ and where $F(x) = P[p(Z) \leq x]$

In the logit-normal case we have

$$F(x) = P[p(Z) \leq x] \overset{(3)}{=} N(p^{-1}(x)) \quad (11)$$

Since

$$F_{L_m}(x) \to F\left(\frac{x}{l \cdot m}\right) \text{ as } m \to \infty \quad (12)$$

this implies that

$$F_{L_m}(x) \approx F\left(\frac{x}{l \cdot m}\right) \text{ for large } m, \quad (13)$$

which in turn implies that

$$F^{\leftarrow}_{L_m}(x) \approx l \cdot m \cdot F^{\leftarrow}(x) \quad (14)$$

for large m. In the logit-normal case $F(x)$ is continuous so $F^{\leftarrow}(x) = F^{-1}(x)$ and (14) can then be rewritten as

$$F^{-1}_{L_m}(x) \approx l \cdot m \, F^{-1}(x)$$

that is

$$VaR_{\alpha}(L) \approx l \cdot m \, F^{-1}(x) \quad (15)$$

when $F(x) = P[p(Z) \leq x]$ is continuous.

To find $F^{-1}(x)$ we solve for x in the equation $y = F(x) \Leftrightarrow x = F^{-1}(y)$

Thus, in the logit-normal case we use (11) to get that

$$y = F(x) \Leftrightarrow y = N(p^{-1}(x)) \Leftrightarrow x = p(N^{-1}(y))$$

Hence, $F^{-1}(x) = p(N^{-1}(x))$ (16)

Thus, (15) and (16) then implies that in the logit-normal case we have (in the LPA-case)

$$VaR_\alpha(L) \approx l \cdot m \cdot p(N^{-1}(x)) \quad (17)$$

So from (*) on p. 8.2 together with (17) we get

$$VaR_\alpha(L) \approx \frac{l \cdot m}{1 + e^{-(\mu + \sigma N^{-1}(\alpha))}} \quad (18)$$

So combining (10) & (18) with $L_m = l \cdot N_m$ we have the equation

$$\frac{l \cdot m}{1 + e^{-(\mu + \sigma N^{-1}(\alpha))}} = C_0$$

8.9

which implies that

$$\frac{l\cdot m}{C_0} = 1 + e^{-(\mu + \sigma N^{-1}(\alpha))} \iff$$

$$\ln\left(\frac{l\cdot m}{C_0} - 1\right) = -\mu - \sigma N^{-1}(\alpha) \iff$$

$$\sigma = -\frac{1}{N^{-1}(\alpha)}\left(\mu + \ln\left(\frac{l\cdot m}{C_0} - 1\right)\right) \quad (19)$$

So with $\mu = -2.5371$, $l = 0.6$, $m = 1000$, $C_0 = 172.3$, $\alpha = 0.95$ we get that

$$\sigma = 0.9897.$$

Given the numerical values of μ and σ we can now compute $P[a_0 < L_m < b_0]$ in Equation (5), that is

$$P[a_0 < L_m < b_0] \approx N\left(p^{-1}\left(\frac{b}{l m}\right)\right) - N\left(p^{-1}\left(\frac{a}{l\cdot m}\right)\right) \quad (20)$$

where $p^{-1}(x)$ is given by (4)

So with $a = 30$, $b = 80$, $l = 0.6$, $m = 1000$, $\sigma = 0.9897$ and $\mu = -2.5371$

(8.10)

We get that

$$p^{-1}\left(\frac{b}{l\cdot m}\right) = p^{-1}\left(\frac{80}{600}\right) = p^{-1}\left(\frac{2}{15}\right) \overset{(4)}{=} 0.6722$$

$$p^{-1}\left(\frac{a}{l\cdot m}\right) = p^{-1}\left(\frac{30}{600}\right) = p^{-1}\left(\frac{1}{20}\right) \overset{(4)}{=} -0.4116$$

Hence

$$N\left(p^{-1}\left(\frac{b}{l\cdot m}\right)\right) = N(0.6722) \approx 0.7493 \quad (21)$$

$$N\left(p^{-1}\left(\frac{a}{l\cdot m}\right)\right) = N(-0.4116) \approx 0.3403 \quad (22)$$

So (21) & (22) in (20) (or Eq (5))
finally gives that

$$P[a_0 < L_m < b_0] \approx 0.4090 = 40.90\%$$

Answer: 40.90%

Task 9: Financial risk, 2019-01-09 (9.1)

We have a credit portfolio in a mixed binomial modell inspired by the Merton framework and we want to compute $VaR_\alpha(L)$ for this portfolio using the LPA-approach. We therefore first need to derive this $VaR_\alpha(L)$-formula and to do this we first need the following general observations.

For $i = 1, 2, \dots, m$ let

$$X_i = \begin{cases} 1 & \text{if obligor } i \text{ defaults within T-years} \\ 0 & \text{otherwise} \end{cases}$$

and let Z be a random variable such that

$$p(Z) = P[X_i = 1 \mid Z]$$

and define $F(x)$ as

$$F(x) = P[p(Z) \leq x] \text{ for } x \in [0,1]. \quad (1)$$

Next, define N_m as $N_m = \sum_{i=1}^{m} X_i$ so that N_m is the number of defaults in the portfolio within T-years (in our case T = 1 year). Furthermore, let $L_m = l \cdot N_m$ be the total credit loss in the portfolio within T years, where l is the individual credit loss which by linearity of VaR w.l.o.g is in percent that is $l \in [0,1]$. Then, the LPA-theory gives that

$$P\left[\frac{N_m}{m} \leq x\right] \to F(x) \quad \text{as } m \to \infty \qquad (2)$$

for $x \in [0,1]$ where $F(x)$ is same as in (1). Hence, we have that

$$F_{L_m}(x) = P[l \cdot N_m \leq x] = P\left[\frac{N_m}{m} \leq \frac{x}{lm}\right] \to F\left(\frac{x}{l \cdot m}\right) \quad (3)$$

as $m \to \infty$. So if m is large, (3) implies

$$F_{L_m}(x) \approx F\left(\frac{x}{l \cdot m}\right) \quad (4)$$

9.3

If $p(Z)$ is a continuous random variable then (4) implies that

$$F_{L_m}^{\leftarrow}(y) \approx l m F^{-1}(y)$$

since $y = F_{L_m}(x) \Leftrightarrow x = F_{L_m}^{\leftarrow}(y)$ (5)

and $y = F\left(\frac{x}{l \cdot m}\right) \Leftrightarrow x = l \cdot m \cdot F^{-1}(y)$ (6)

so (4), (5) & (6) then gives that

$$F_{L_m}^{\leftarrow}(y) \approx l \cdot m \cdot F^{-1}(y) \quad (7)$$

when m is large. By definition we have

$$VaR_\alpha(L_m) = F_{L_m}^{\leftarrow}(\alpha) \quad (8)$$

so (7) in (8) then implies (when m is "large")

$$VaR_\alpha(L_m) \approx l \cdot m \cdot F^{-1}(\alpha) \quad (9)$$

In the mixed binomial model inspired by the Merton framework we have

$$F(x) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\, N^{-1}(x) - N^{-1}(\bar{p})\right)\right) \quad (10)$$

9.4

To find $F^{-1}(\cdot)$ we solve for x in the equation $y = F(x)$ and in the Merton setting (10) then gives

$$y = F(x) \Leftrightarrow y = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,N^{-1}(x) - N^{-1}(\bar{p})\right)\right)$$

$$\Leftrightarrow N^{-1}(y) = \frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,N^{-1}(x) - N^{-1}(\bar{p})\right)$$

$$\Leftrightarrow N^{-1}(x) = \frac{\sqrt{\rho}\,N^{-1}(y) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}$$

$$\Leftrightarrow \quad x = N\left(\frac{\sqrt{\rho}\,N^{-1}(y) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right)$$

Hence, $$F^{-1}(y) = N\left(\frac{\sqrt{\rho}\,N^{-1}(y) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right) \qquad (11)$$

So (9) and (11) implies that

$$\mathrm{VaR}_\alpha(L_m) \approx l \cdot m \cdot N\left(\frac{\sqrt{\rho}\,N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right) \qquad (12)$$

when m is large.

9.5

From Eq (12) we see that we need $\bar{p}$ and ρ to compute $VaR_\alpha(L_m)$ via the LPA-approach. We know ρ (and l and m) but we don't know $\bar{p}$. Recall that $\bar{p} = P[X_i = 1]$ and in the Merton framework we also have that (by definition of a default in the mixed binomial Merton modell)

$$\bar{p} = P[X_i = 1] = P[V_{T,i} < D_i] \quad (13)$$

where for any $t \geq 0$ the asset value $V_{t,i}$ is given by (for obligor i)

$$V_{t,i} = V_{0,i} e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i B_{t,i}} \quad (14)$$

and $B_{t,i} = \sqrt{\rho}\, W_{t,0} + \sqrt{1-\rho}\, W_{t,i}$ (15)

and $W_{t,0}, W_{t,1}, \dots, W_{t,m}$ are independent standard Brownian motions (Wiener process). By independence we conclude that (15) implies that also $B_{t,i}$ is a Brownian motion (Wiener process)

9.6

So for any t it holds that

$$B_{t,i} \sim N(0,t) \qquad (16)$$

i.e $B_{t,i}$ is normally distributed with mean zero and variance t

So (13), (14) and (16) then implies that

$$\bar{p} = P[X_i = 1] = P[V_{T,i} < D_i]$$

$$= P\left[V_{0,i} e^{(\mu_i - \frac{1}{2}\sigma_i^2)T + \sigma_i B_{T,i}} < D_i\right]$$

$$= \left\{ \begin{array}{l} \text{By homogeneity we have that} \\ V_{0,i} = V_0,\ \mu_i = \mu,\ \sigma_i = \sigma,\ D_i = D \end{array} \right\} =$$

$$= P\left[V_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_{T,i}} < D\right] = \left\{ \begin{array}{l} \ln(x) \text{ is} \\ \text{strictly} \\ \text{increasing} \end{array} \right\}$$

$$= P\left[\ln V_0 + (\mu - \tfrac{1}{2}\sigma^2)T + \sigma B_{T,i} < \ln D\right]$$

$$= P\left[\sigma B_{T,i} \leq \ln \frac{D}{V_0} - (\mu - \tfrac{1}{2}\sigma^2)T\right]$$

$$= \left\{ B_{T,i} \overset{d}{=} \sqrt{T} \cdot X \text{ where } X \sim N(0,1) \right\} =$$

9.7

$$= P\left[X \le \frac{\ln \frac{D}{V_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right]$$

$$= N\left(\frac{\ln \frac{D}{V_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

Hence, we have that

$$\bar{p} = N\left(\frac{\ln \frac{D}{V_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \qquad (16)$$

We know that $R = \frac{V_0}{D} = 1.1$ and $\ln \frac{D}{V_0} = -\ln \frac{V_0}{D} = -\ln R = -\ln 1.1$.

We also know that $\mu = 0.05$, $\sigma = 0.2$ and $T = 1$ and all these numbers plugged into (16) gives that

$$\bar{p} = 0.2655 = 26.55\%$$

Hence, with $\bar{p} = 0.2655$ and (9.8)

$\rho = 0.15$, $l = 0.6$ and $m = 1000$

we use Equation (12) to obtain

that $VaR_{95\%}(L)$ via the CPA-approach

gives

$$VaR_{95\%}(L) = 302.7 \text{ monetary units}$$

Answer: $VaR_{95\%}(L) = 302.7$

Task 10: Financial risk, 2019-01-09. 10.1

From the LPA-theory we know that (see also in solutions to task 8-9)

$$\mathrm{VaR}_\alpha(L) \approx l \cdot m \cdot F^{-1}(\alpha) \quad (1)$$

where $F(x) = P[p(Z) \leq x]$ (2)

and $p(Z) = P[X_i = 1 | Z]$

with the same notation as in task 8-9.

Furthermore, $ES_\alpha(L)$ is defined as

$$ES_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(L)\, du \quad (3)$$

and if m is large the LPA-formula in Equation (1) then implies that

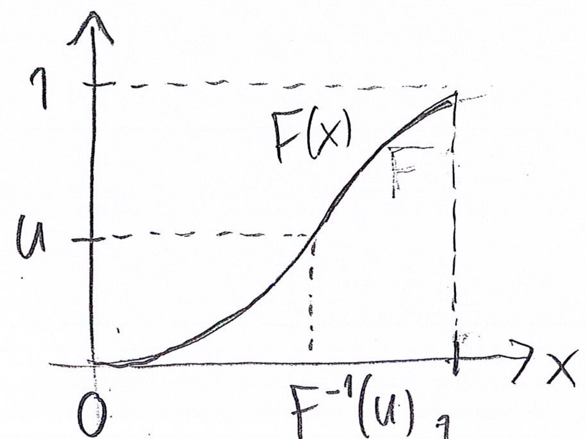
$$ES_\alpha(L) \approx \frac{l \cdot m}{1-\alpha} \int_\alpha^1 F^{-1}(u)\, du \quad (4)$$

where m is "large".

10.2

Since $p(z)$ is a continuous random variable and $p(x) \in [0,1]$ for all x, we have that $F(x)$ is a continuous and strictly increasing function ~~where~~ $F(x) \in [0,1]$. Since $F(x)$ is strictly increasing and continuous, then $F^{-1}(x)$ is well-defined, strictly increasing and continuous. Furthermore, since $F(x) \in [0,1]$ and $p(x) \in [0,1)$ we have that $F^{-1}(x) \in [0,1]$ and is defined for $x \in [0,1]$, see Figure 1.

Figure 1



10.3

Furthermore, we thus have that

$$\lim_{u \uparrow 1} F^{-1}(u) = 1 \quad (5)$$

and that $F^{-1}(u) \leq 1$ for all $u \in [0,1]$

Hence, we have that

$$\int_\alpha^1 \underbrace{F^{-1}(u)}_{\geq F^{-1}(\alpha)} du \geq \int_\alpha^1 F^{-1}(\alpha) du = F^{-1}(\alpha) \int_\alpha^1 du = F^{-1}(\alpha)(1-\alpha)$$

so

$$F^{-1}(\alpha)(1-\alpha) \leq \int_\alpha^1 F^{-1}(u) du \quad (6)$$

for all $\alpha \in [0,1]$.

But since $F^{-1}(u) \leq 1$ for all $u \in [0,1]$ we also have that

$$\int_\alpha^1 \underbrace{F^{-1}(u)}_{\leq 1} du \leq \int_\alpha^1 1 \cdot du = 1-\alpha$$

that is

$$\int_\alpha^1 F^{-1}(u) du \leq 1-\alpha \quad (7)$$

So (6) and (7) implies that

$$(1-\alpha)F^{-1}(\alpha) \leq \int_{\alpha}^{1} F^{-1}(u)\,du \leq 1-\alpha \qquad (8)$$

Note that the inequalities in Equation (8) will be preserved if we multiply all quanties with the positive constant

$$\frac{1}{(1-\alpha)\cdot F^{-1}(\alpha)}$$

that is

$$1 \leq \frac{1}{(1-\alpha)F^{-1}(\alpha)} \int_{\alpha}^{1} F^{-1}(u)\,du \leq \frac{1}{F^{-1}(\alpha)} \qquad (9)$$

But note that

$$\frac{1}{(1-\alpha)F^{-1}(\alpha)} \cdot \int_{\alpha}^{1} F^{-1}(u)\,du = \frac{1m}{1m \cdot F^{-1}(\alpha)} \cdot \frac{1m}{(1-\alpha)} \int_{\alpha}^{1} F^{-1}(u)\,du$$

$$= \{\text{Eq}(1) \,\&\, \text{Eq}(4)\} = \frac{1}{VaR_{\alpha}(L)} \cdot ES_{\alpha}(L) \qquad (10)$$

So (10) in (9) finally renders (10.5) that

$$1 \leq \frac{ES_\alpha(L)}{VaR_\alpha(L)} \leq \frac{1}{F^{-1}(\alpha)} \quad (11)$$

for any $\alpha \in [0,1)$ which is what we wanted to show. $\square$