FTS16-3-1

Financial Time Series

Solution to Exam 3

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Problem 1

$X := (X_t, t \in \mathbb{Z})$ given by

$$X_t = Y_t (Z_t + Z_{t-1}), \quad t \in \mathbb{Z}$$

- $Y := (Y_t, t \in \mathbb{Z})$, $Z := (Z_t, t \in \mathbb{Z})$
- $Y \perp Z$, stationary
- $Z \sim WN(0, \sigma_Z^2)$
- Y has ACVF γ_Y given by $\gamma_Y(h) = 2^{-|h|}$, $h \in \mathbb{Z}$

$\boxed{8\frac{1}{2}}$ a) Let $X = (X_t, t \in \mathbb{Z})$ be a time series with $\mathrm{Var}(X_t) < +\infty$ for all $t \in \mathbb{Z}$. The time series is called (weakly) stationary if $\frac{1}{2}$

$\frac{1}{2}$ (i) $\exists \mu \in \mathbb{R} : \forall t \in \mathbb{Z} : \mathbb{E}(X_t) = \mu$ and

$\frac{1}{2}$ (ii) $\forall r, s, h \in \mathbb{Z} : \gamma_X(r,s) = \gamma_X(r+h, s+h)$,

where $\gamma_X(r,s) := \mathrm{Cov}(X_r, X_s) = \mathbb{E}((X_r - \mathbb{E}(X_r))(X_s - \mathbb{E}(X_s)))$

A stochastic process $X := (X_t, t \in \mathbb{Z})$ is called a white noise with mean μ and variance σ^2 if it is a stationary process with $\frac{1}{2}$

$\frac{1}{2}$ • $\mathbb{E}(X_t) = \mu \quad \forall t \in \mathbb{Z}$

$\frac{1}{2}$ • $\gamma_X(h) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{else} \end{cases}$

To show stationarity, we check the 3 conditions:

(i) $\boxed{1}$

$$\mathbb{E}(X_t) = \mathbb{E}(y_t(z_t + z_{t-1})) \overset{(1/2)\ \perp\!\!\!\perp}{=} \mathbb{E}(y_t)\,\mathbb{E}(z_t + z_{t-1})$$

$$\overset{lin.}{=} \mathbb{E}(y_t)\left(\mathbb{E}(z_t) + \mathbb{E}(z_{t-1})\right)$$

$$\overset{z\sim WN(0,\sigma^2)}{\underset{(1/2)}{=}} \mathbb{E}(y_t)\cdot(0+0)$$

$$= 0$$

$$\Rightarrow \mathbb{E}(X_t) = 0 \quad \forall t \in \mathbb{Z}.$$

(ii) $\boxed{2}$

$$\mathrm{Var}(X_t) \overset{\mathbb{E}(X_t)=0}{=} \mathbb{E}(X_t^2) \overset{(1/2)}{=} \mathbb{E}(y_t^2(z_t+z_{t-1})^2)$$

$$\overset{(1/2)\ \perp\!\!\!\perp,\ lin.}{=} \mathbb{E}(y_t^2)\left(\mathbb{E}(z_t^2) + 2\,\mathbb{E}(z_t z_{t-1}) + \mathbb{E}(z_{t-1}^2)\right)$$

$$\overset{(1/2)\ WN}{=} \left(\mathrm{Var}(y_t) + \mathbb{E}(y_t)^2\right)\left(2\cdot\mathrm{Var}(z_t) + 2\cdot 0\right)$$

$$= \left(\underbrace{\mathrm{Var}(y_t)}_{\substack{<+\infty \\ y\ stat.}} + \underbrace{\mathbb{E}(y_t)^2}_{\substack{<+\infty \\ y\ stat.}}\right) 2\,\underbrace{\sigma^2}_{<\infty} \underset{(1/2)}{<} +\infty$$

(iii) $\boxed{2\tfrac{1}{2}}$

$$\gamma_X(r+h, s+h) \overset{\mathbb{E}(X_t)=0}{=} \mathbb{E}(X_{r+h}X_{s+h})$$

$$\overset{(1/2)}{=} \mathbb{E}\left(y_{r+h}(z_{r+h} + z_{r+h-1})\,y_{s+h}(z_{s+h} + z_{s+h-1})\right)$$

$$\overset{(1/2)\ \perp\!\!\!\perp}{=} \mathbb{E}(y_{r+h}y_{s+h})\,\mathbb{E}\big(z_{r+h}z_{s+h} + z_{r+h}z_{s+h-1} + z_{r+h-1}z_{s+h} + z_{r+h-1}z_{s+h-1}\big)$$

$$\overset{lin.,\ stat\ |\ z\sim WN(0,\sigma^2)}{\underset{(1/2)}{=}} \left(\gamma_y(r+h-s+h) + \mathbb{E}(y_{r+h})\,\mathbb{E}(y_{s+h})\right)$$
$$\Big(\gamma_z(r+h-(s+h)) + \gamma_z(r+h-(s+h-1)) + \gamma_z(r+h-1-(s+h)) + \gamma_z(r+h-1-(s+h-1))\Big)$$

$$\overset{\mathbb{E}(y_t)=\mu}{\underset{(1/2)}{=}} \left(\gamma_y(r-s) + \mathbb{E}(y_r)\,\mathbb{E}(y_s)\right)$$
$$\Big(\gamma_z(r-s) + \gamma_z(r-(s-1)) + \gamma_z(r-1-s) + \gamma_z(r-1-(s-1))\Big)$$

$$\overset{comp.\ back\ as\ before}{\underset{(1/2)}{=}} \mathbb{E}(X_r X_s) = \gamma_X(r,s)$$

b) $\boxed{1}$

$$\gamma_X(h) \overset{\text{for any } t}{=} \mathbb{E}(X_{t+h}X_t)$$

$$\overset{a)(iii)}{\underset{(1/2)}{=}} \left(\gamma_y(-h) + \mu_y^2\right)\left(\gamma_z(h) + \gamma_z(h+1) + \gamma_z(h-1) + \gamma_z(h)\right)$$

$$\overset{(1/2)}{=} \left(2^{-|h|} + \mu_y^2\right)\begin{cases} 2\sigma_z^2 & h=0 \\ \sigma_z^2 & h=\pm 1 \\ 0 & \text{else}\end{cases} \overset{\mu_y=0}{=} \begin{cases} 2\sigma_z^2 & h=0 \\ \frac{1}{2}\sigma_z^2 & h=\pm 1 \\ 0 & \text{else}\end{cases}$$

c) A time series $\tilde{X} = (\tilde{X}_t, t \in \mathbb{Z})$ is called a moving average (½)

[3½] process of order q (MA(q) process) if it is stationary

and if for all $t \in \mathbb{Z}$

$$(½)\quad \tilde{X}_t = V_t + \sum_{j=1}^{q} \theta_j V_{t-j},$$

where $V \sim WN(0, \sigma_V^2)$. (½)

So an MA(1) process has the representation

$$\tilde{X}_t = V_t + \theta V_{t-1}$$

for some $\theta \in \mathbb{R} \setminus \{0\}$.

- $\mathbb{E}(\tilde{X}_t) \overset{\text{lin.}}{=} \mathbb{E}(V_t) + \theta\, \mathbb{E}(V_{t-1})$

(½) $\overset{V \sim WN}{=} 0 + \theta\, 0 = 0$

- $\gamma_{\tilde{X}}(h) \overset{\text{for all } t}{=} \mathbb{E}(X_{t+h} X_t)$

(½) $= \mathbb{E}\big((V_{t+h} + \theta V_{t+h-1})(V_t + \theta V_{t-1})\big)$

$\overset{\text{lin}}{=} \mathbb{E}(V_{t+h} V_t) + \theta\, \mathbb{E}(V_{t+h} V_{t-1}) + \theta\, \mathbb{E}(V_{t+h-1} V_t) + \theta^2 \mathbb{E}(V_{t+h-1} V_{t-1})$

(½) $= (1+\theta^2)\, \gamma_V(h) + \theta\big(\gamma_V(h+1) + \gamma_V(h-1)\big)$

(½) $= \begin{cases} (1+\theta^2)\,\sigma_V^2 & h = 0 \\ \theta\, \sigma_V^2 & h = \pm 1 \\ 0 & \text{else} \end{cases}$

(½)

d) We already know that X is stationary. It remains

[6½] to show a representation $X_t = V_t + \theta V_{t-1}$ for some

$V \sim WN(0, \sigma_V^2)$ and $\theta \in \mathbb{R} \setminus \{0\}$.

Since $\mathbb{E}(X_t) = 0 = $ MA(1) (see c)), we have to match the (½)

ACVF, i.e., (½)

$$\gamma_X(h) = \gamma_{\tilde{X}}(h) \quad \forall h \in \mathbb{Z}.$$

(½) $[\mu_y = 0]$

This leads to

(1) $\begin{cases} 2\, \sigma_Z^2 = (1+\theta^2)\, \sigma_V^2 \\ \frac{1}{2}\, \sigma_Z^2 = \theta\, \sigma_V^2 \end{cases}$

$\theta \neq 0$ $(\tfrac{1}{2})$ $\Rightarrow \quad \frac{1+\theta^2}{\theta} = 4$

$\Leftrightarrow \quad \theta^2 - 4\theta + 1 = 0$

$\Rightarrow \quad \boxed{\theta_{1,2} = 2 \pm \sqrt{4-1} = 2 \pm \sqrt{3}}$ (1)

and

$\boxed{\sigma_V^2 = \frac{1}{2\theta}\sigma_Z^2 = \frac{\sigma_Z^2}{2(2\pm\sqrt{3})}}$ (1)

So choosing $\theta = 2+\sqrt{3}$ or $\theta = 2-\sqrt{3}$ and $V \sim WN(0, \frac{\sigma_Z^2}{2\theta})$, $(\tfrac{1}{2})$
we obtain a representation

$$X_t = V_t + \theta V_{t-1} \quad (\tfrac{1}{2})$$

that has the same mean and ACVF.

(e) An MA(q) process is causal if there exists a real-
[2] valued sequence $(\psi_j, j \in \mathbb{N}_0)$ s.t. $\sum_{j=0}^{\infty} |\psi_j| < +\infty$ $(\tfrac{1}{2})$ and

$$(\tfrac{1}{2}) \quad \tilde{X}_t = \sum_{j=0}^{\infty} \psi_j V_{t-j}, \quad t \in \mathbb{Z},$$

$\leftarrow$ or $\psi_0 = 1$, $\psi_1 = 2 \pm \sqrt{3}$ $(\tfrac{1}{2})$ $\Rightarrow |\psi_0| + |\psi_1| = 3 \pm \sqrt{3} < +\infty$ $(\tfrac{1}{2})$

Lemma (adapted to MA(q) process):

$$\tilde{X} \text{ is causal} \Leftrightarrow \forall z \in \mathbb{C} \text{ s.t. } |z| \leq 1.$$
$$1 - z \cdot 0 \neq 0$$

$\Rightarrow$ An MA(q) and especially our MA(1) process X are causal.

(f) An MA(q) process $\tilde{X}$ is invertible if there exists a real-
[4] valued sequence $(\pi_j, j \in \mathbb{N}_0)$ s.t. $\sum_{j=0}^{\infty} |\pi_j| < +\infty$ $(\tfrac{1}{2})$ and

$$(\tfrac{1}{2}) \quad V_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}, \quad t \in \mathbb{Z}.$$

(1) *Lemma*

$\tilde{X}$ is invertible $\iff$ $\forall z \in \mathbb{C},\ |z| \leq 1$

$$1 + \sum_{j=1}^{q} \theta_j z^j \neq 0$$

Consider $\theta_1 = 2+\sqrt{3}$ and $\theta_2 = 2-\sqrt{3}$ separately.

- $1+\theta_1 z = 1+(2+\sqrt{3})z = 0$

  $\iff \quad z = -\frac{1}{2+\sqrt{3}}$

  and $|z| = \frac{1}{2+\sqrt{3}} < 1$ (1/2)

  $\Rightarrow$ X is not invertible (1/2)

- $1+\theta_2 z = 1+(2-\sqrt{3})z = 0$

  $\iff \quad z = -\frac{1}{2-\sqrt{3}}$

  and $|z| = \frac{1}{2-\sqrt{3}} > 1$ (1/2)

  $\Rightarrow$ X is invertible (1/2)

$\boxed{4}$ g) Let

$\hat{X}_n = b_n^{\ell}(X^{n-1})$ be the best linear one-step predictor of X_n,

then

(1) $$\hat{X}_{n+1} = \begin{cases} 0 & \text{for } n=0 \\ \sum_{j=1}^{n} \theta_{nj}(X_{n+1-j} - \hat{X}_{n+1-j}) & \text{for } n \geq 1 \end{cases}$$

Innovations algorithm

(1/2) Compute the coefficients $\theta_{n1}, \dots, \theta_{nn}$ recursively from the equations (1/2) (1/2)

$$\nu_0 := \gamma_X(0) = (1+\theta^2)\sigma_V^2$$

(1) $$\theta_{n(n-k)} := \frac{1}{\nu_k}\left(\gamma_X(n-k) - \sum_{j=0}^{k-1} \theta_{k(k-j)}\theta_{n(n-j)}\nu_j\right), \quad 0 \leq k < n$$

and (1/2) $\nu_n := \gamma_X(0) - \sum_{j=0}^{n-1} \theta_{n(n-j)}^2 \nu_j$

h) (1/2) $\overline{X}_2 = \frac{x_1+x_2}{2}$ sample mean

[3 1/2]

$$\hat\gamma_X(0) = \frac{1}{2}\sum_{t=1}^{2}(x_t - \overline{X}_2)^2 \overset{(1/2)}{=} \frac{1}{2}\left(\left(x_1 - \frac{x_1+x_2}{2}\right)^2 + \left(x_2 - \frac{x_1+x_2}{2}\right)^2\right)$$
$$= \frac{1}{2}\left(\frac{(x_1-x_2)^2}{4} + \frac{(x_2-x_1)^2}{4}\right)$$
$$= \frac{1}{8}\, 2\, (x_1-x_2)^2$$
$$\overset{(1/2)}{=} \frac{(x_1-x_2)^2}{4}$$

$$\hat\gamma_X(1) = \frac{1}{2}\sum_{t=1}^{1}(x_{t+1} - \overline{X}_2)(x_t - \overline{X}_2) \overset{(1/2)}{=} \frac{1}{2}\left(x_2 - \frac{x_1+x_2}{2}\right)\left(x_1 - \frac{x_1+x_2}{2}\right)$$
$$= \frac{1}{2}\cdot\frac{(x_2-x_1)(x_1-x_2)}{4} \overset{(1/2)}{=} -\frac{(x_1-x_2)^2}{8}$$

$$\hat\gamma_X(-1) = \gamma_X(|-1|) \overset{(1/2)}{=} -\frac{(x_1-x_2)^2}{8}$$

$\hat\gamma_X(h) = 0$ for $|h| > 1$ (1/2)

i) Plug in $\hat\gamma_X$ instead of γ_X and compute (1/2)

[4]

$$\hat\theta = \hat\theta_{11} \quad (1/2)$$

and $\hat\sigma_V^2 = \hat\nu_1$ (1/2)

The innovations algorithm sets

$$\hat\nu_0 := \hat\gamma_X(0) = \frac{(x_1-x_2)^2}{4} \quad (1/2)$$

$$\hat\theta_{11} \overset{(1/2)}{:=} \frac{1}{\hat\nu_0}(\hat\gamma_X(1) - 0) = \frac{4}{(x_1-x_2)^2}\cdot\left(-\frac{(x_1-x_2)^2}{8}\right) \overset{(1/2)}{=} -\frac{1}{2}$$

$$\hat\nu_1 \overset{(1/2)}{:=} \hat\gamma_X(0) - \hat\theta_{11}^2\hat\nu_0 = \left(1-\frac{1}{4}\right)\frac{(x_1-x_2)^2}{4} \overset{(1/2)}{=} \frac{3}{16}(x_1-x_2)^2$$

$$\Rightarrow \quad \hat\theta = -\frac{1}{2}$$
$$\hat\sigma_V^2 = \frac{3}{16}(x_1-x_2)^2$$

j) We are now interested in $\hat X_3$, then

[1 1/2]

$$\hat X_3 \overset{(1/2)}{=} \sum_{j=1}^{2}\theta_{2j}(X_{2+1-j} - \hat X_{2+1-j}) \overset{(1/2)}{=} \theta_{21}(X_2 - \overbrace{\hat X_2}^{\theta_{11}(X_1-\hat X_1)}) + \theta_{22}(X_1 - \overbrace{\hat X_1}^{0})$$
$$\overset{(1/2)}{=} \theta_{21}(X_2 - \theta_{11}X_1) + \theta_{22}X_1$$
$$\overset{(1/2)}{=} \theta_{21}X_2 + (\theta_{22} - \theta_{21}\theta_{11})X_1$$

We have to compute θ_{21}, θ_{22}, and θ_{11}

Our fitted model from i) is

$$X_t = V_t - \frac{1}{2} V_{t-1} \qquad \frac{1}{2}$$

$$\text{with } V \sim WN\left(0, \frac{3}{16}(x_1 - x_2)^2\right) \qquad \frac{1}{2}$$

$$\overset{c)}{\Longrightarrow} \quad \gamma_X(h) = \begin{cases} \frac{5}{4} \cdot \frac{3}{16}(x_1 - x_2)^2 & h = 0 \\ -\frac{1}{2}\frac{3}{16}(x_1 - x_2)^2 & h = \pm 1 \\ 0 & \text{else} \end{cases} \qquad 1$$

is the ACVF of the fitted model.

The innovations algorithm computes

$$\nu_0 := \gamma_X(0) \qquad \frac{1}{2}$$

$$\theta_{22} := \frac{1}{\gamma_X(0)}\left(\overset{0}{\overset{\parallel}{\gamma_X(2)}} - 0\right) = 0 \qquad \frac{1}{2}$$

$$\nu_2 := \gamma_X(0) - \sum_{j=0}^{1} \theta^2_{2(2-j)} \nu_j$$

$$\overset{\frac{1}{2}}{=} \gamma_X(0) - 0 - \theta_{21}^2 \nu_1 \overset{\frac{1}{2}}{=} \gamma_X(0) - \frac{\gamma_X(0)^{\cancel{2}}\gamma_X(1)^2}{(\gamma_X(0)^2 - \gamma_X(1)^2)^{\cancel{2}}} \cdot \frac{\cancel{\gamma_X(0)^2 - \gamma_X(1)^2}}{\cancel{\gamma_X(0)}}$$

$$\overset{\frac{1}{2}}{=} \gamma_X(0) - \frac{\gamma_X(0)\gamma_X(1)^2}{\gamma_X(0)^2 - \gamma_X(1)^2}$$

$$\theta_{21} \overset{\frac{1}{2}}{:=} \frac{1}{\nu_1}\left(\gamma_X(1) - \theta_{11} \cdot \theta_{22}\nu_0\right) \overset{\frac{1}{2}}{=} \frac{\gamma_X(1)}{\nu_1} \overset{\frac{1}{2}}{=} \frac{\gamma_X(0)\gamma_X(1)}{\gamma_X(0)^2 - \gamma_X(1)^2}$$

$$\theta_{11} \overset{\frac{1}{2}}{:=} \frac{1}{\gamma_X(0)}\left(\gamma_X(1) - 0\right) = \frac{\gamma_X(1)}{\gamma_X(0)}$$

$$\nu_1 \overset{\frac{1}{2}}{:=} \gamma_X(0) - \theta_{11}^2 \cdot \nu_0 \overset{\frac{1}{2}}{=} \gamma_X(0) - \frac{\gamma_X(1)^2}{\gamma_X(0)^{\cancel{2}}}\cancel{\gamma_X(0)}$$

$$= \gamma_X(0) - \frac{\gamma_X(1)^2}{\gamma_X(0)}$$

$$\overset{\frac{1}{2}}{=} \frac{\gamma_X(0)^2 - \gamma_X(1)^2}{\gamma_X(0)}$$

$$\Rightarrow \hat{X}_3 = \theta_{21} X_2 - \theta_{21}\theta_{11} X_1$$

$$\overset{\frac{1}{2}}{=} \frac{\gamma_X(0)\gamma_X(1)}{\gamma_X(0)^2 - \gamma_X(1)^2}\left(X_2 - \frac{\gamma_X(1)}{\gamma_X(0)} X_1\right)$$

$$\overset{\frac{1}{2}}{=} \frac{\frac{15}{64} \cdot \left(-\frac{3}{32}\right)}{\frac{15^2}{64^2} - \frac{9}{32^2}} \cdot \left(X_2 + \frac{\frac{3}{32}}{\frac{15}{64}} X_1\right) = -\frac{10}{21} X_2 - \frac{10}{21} \cdot \frac{2}{5} X_1 = \boxed{-\frac{10}{21} X_2 - \frac{4}{21} X_1} \qquad \frac{1}{2}$$

$$-\frac{3^2 \cdot 5 \cdot 2^{12}}{2^{11}(225 - 36)}$$

$$= -\frac{10}{21}$$

k) The mean squared error is

$\boxed{2\tfrac{1}{2}}$

$$MSE(\hat{X}_3, X_3) \overset{\frac{1}{2}}{=} v_2$$

$$\overset{\frac{1}{2}}{=} \frac{15}{64}(x_1-x_2)^2 - \left(-\frac{\cancelto{5}{10}}{\cancelto{7}{21}}\right)\cdot\left(-\frac{\cancelto{1}{3}}{\cancelto{16}{32}}\right)(x_1-x_2)^2$$

$$= \left(\frac{15}{64} - \frac{5}{7\cdot 16}\right)(x_1-x_2)^2$$

$$\overset{\frac{1}{2}}{=} \frac{105-20}{64\cdot 7}(x_1-x_2)^2$$

$$= \frac{85}{448}(x_1-x_2)^2$$

This error does not include the model fitting error which is in the parameters $\hat{\theta}$ and $\hat{\sigma}_V^2$. 1

l) We have

$\boxed{9}$

$$b_3^{\ell}(X^2) = a_1 X_2 + a_2 X_1 \quad 1$$

and

$$\mathbb{E}\left(\left(X_3 - b_3^{\ell}(X^2)\right)X_2\right) \overset{\frac{1}{2}}{=} \mathbb{E}\left((X_3 - a_1X_2 - a_2X_1)X_2\right)$$

$$\overset{\frac{1}{2}}{=} \gamma_X(1) - a_1\gamma_X(0) - a_2\gamma_X(1) = 0$$

$$\mathbb{E}\left(\left(X_3 - b_3^{\ell}(X^2)\right)X_1\right) \overset{\frac{1}{2}}{=} \mathbb{E}\left((X_3 - a_1X_2 - a_2X_1)X_1\right)$$

$$\overset{\frac{1}{2}}{=} \underbrace{\gamma_X(2)}_{=0} - a_1\gamma_X(1) - a_2\gamma_X(0) = 0$$

$$\Leftrightarrow \begin{cases} a_2 = \dfrac{\gamma_X(1) - a_1\gamma_X(0)}{\gamma_X(1)} \\ a_2 = -\dfrac{a_1\gamma_X(1)}{\gamma_X(0)} \end{cases}$$

$$\Rightarrow 1 - \frac{\gamma_X(0)}{\gamma_X(1)}a_1 = -\frac{\gamma_X(1)}{\gamma_X(0)}a_1$$

$$\Leftrightarrow a_1 = \frac{1}{\frac{\gamma_X(0)}{\gamma_X(1)} - \frac{\gamma_X(1)}{\gamma_X(0)}} = \frac{\gamma_X(0)\gamma_X(1)}{\gamma_X(0)^2 - \gamma_X(1)^2}$$

$$a_2 = -a_1\frac{\gamma_X(1)}{\gamma_X(0)} = -\frac{\gamma_X(1)^2}{\gamma_X(0)^2 - \gamma_X(1)^2} \quad 2$$

$$\overset{j)\ 2}{\Longrightarrow} \left|\hat{X}_3 = -\frac{10}{21}x_2 - \frac{4}{21}x_1\right| \quad 1$$

$\Rightarrow$ Same result as j. 1 Clear by uniqueness.