Problem 1

Let a time series model be given by

$$X_t - \frac{1}{2}X_{t-1} = Z_t$$

for $t \in \mathbb{N}$, where $(Z_t, t \in \mathbb{N})$ is a sequence of independent, standard normally distributed random variables, independent of X_0 and X_0 satisfies that $\mathbb{E}(X_0) = 0$ and $\mathbb{E}(X_0^2) = 4/3$.

(a) Give the definition of a time series and a time series model.

(b) Define what it means that a time series is weakly stationary.

(c) Define the autocovariance function of a stationary time series.

(d) Show that the given time series is weakly stationary by doing the following:

(i) Compute $\mathbb{E}(X_t)$.

(ii) Compute $\mathbb{E}(X_t^2)$. (*Hint: Start with* $\mathbb{E}(X_1^2)$ *and use this result to compute* $\mathbb{E}(X_t^2)$ *OR compute* $\mathbb{E}(X_t^2)$ *directly by using the properties of the geometric series.*)

(iii) Compute $\mathbb{E}(X_{t-j}X_t)$. (*Hint: Start with* $j = 1$ *and continue to compute the other values recursively.*)

(iv) Conclude that you have shown stationarity and write down the autocovariance function.

(e) Define the best linear predictor of a stationary time series.

(f) Assume that you have observed X_1 and X_2. Compute the best linear predictor P_2X_3 of X_3.

(g) Define the mean squared error of a best linear predictor and show that it is 1 for P_2X_3.

(h) Give the algorithm that predicts X_3 for observed X_1 and X_2 using parametric bootstrap.

(29 points)

Problem 2

(a) Give the definition of an ARMA(p, q) process.

(b) Define when an ARMA(p, q) is causal. Give an equivalent characterization.

(c) Define when an ARMA(p, q) is invertible. Give an equivalent characterization.

(d) Determine which of the following ARMA processes are causal and which of them are invertible, where $(Z_t, t \in \mathbb{Z})$ is a sequence of independent, standard normally distributed random variables:

(i) $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$,

(ii) $X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1}$,

(iii) $X_t + 1.6X_{t-1} = Z_t - 0.4Z_{t-1} + 0.04Z_{t-2}$.

(15 points)

Problem 3

Let $X = (X_t, t \in \mathbb{N})$ be a time series.

(a) Define the sample mean $\bar{X}_n$ of X for some finite $n \in \mathbb{N}$.

(b) What does the sample mean estimate if X is a sequence of independent, identically distributed random variables? Show that it is an unbiased estimator of that quantity.

(c) Write down the classical decomposition model and define all components.

(d) Explain one algorithm to eliminate seasonal components from data that is assumed to follow the classical decomposition model.

(e) Define when X is called a GARCH(p, q) process and explain how it is connected to an ARMA$(\max\{p, q\}, q)$ process.

(16 points)