Stochastic differential equations are frequently used in finance to model prices. Assume that you are given data and you were told that the data can be modeled by the stochastic differential equation

$$X_t = X_{t_0} + \int_{t_0}^{t} \mu X_s \,\mathrm{d}s + \int_{t_0}^{t} \sigma \,\mathrm{d}B_s,$$

where $t > t_0$ with $t_0 \in \mathbb{Z}$, μ and σ are real numbers, $(B_t, t \in \mathbb{R})$ is a Brownian motion, and X_{t_0} is the solution at time t_0. Furthermore, you are told that a good approximation of the solution can be found with the *Euler Maruyama scheme* given in recursive form by

$$X_t = (1+\mu)X_{t-1} + \sigma Z_t, \tag{1}$$

where the stochastic process $Z := (Z_t, t \in \mathbb{Z})$ is IID$(0,1)$ noise such that $Z_t \sim \mathcal{N}(0,1)$ for all t and it models the increments of the Brownian motion. We know further that X_{t_0} at time t_0 satisfies $\mathbb{E}(X_{t_0}) = 0$ and $\mathbb{E}(X_{t_0}^2) = -\sigma^2/(2\mu + \mu^2)$, e.g., because we observed it. We assume that $-2 < \mu < 0$ and that $\mu \neq -1$.
The goal of this exam is to fit the parameters μ and σ using the data and an appropriate time series model from the lecture to derive predictors of the future development of the price modeled by the stochastic differential equation. This task is split into the following problems to be solved by you.

Problem 1
Assume that $\mathsf{Cov}(X_t, Z_{t+h}) = 0$ for all $t \in \mathbb{Z}$ and all $h > 0$. Show that $X = (X_t, t \in \mathbb{Z})$ is weakly stationary by doing the following:

(a) Compute $\mathbb{E}(X_t)$ for all $t \in \mathbb{Z}$.

(b) Compute $\mathbb{E}(X_t^2)$ for all $t \in \mathbb{Z}$.

(c) Compute $\mathbb{E}(X_t X_{t-j})$ for all $t, j \in \mathbb{Z}$.

(d) Put the previous results together to conclude that X is weakly stationary. Derive the mean and the autocovariance function.

(15 points)

Problem 2

(a) Show that X is an AR(1) process.

(b) Is X a causal process? Do not forget to refer to all definitions and results that you use in your derivation.

(c) Is X an invertible process? Do not forget to refer to all definitions and results that you use in your derivation.

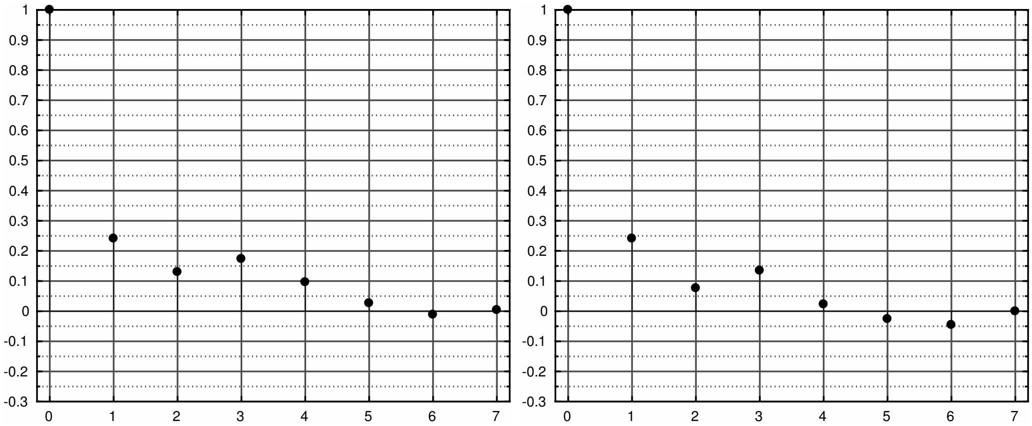
(10 points)

Problem 3

Suppose you are given two data sets and that you are asked to judge if the given model (1) is appropriate for either of the data sets. For the first data set you compute a sample mean $\bar{x} = -0.0817$ from $n = 380$ samples, and for the second data set you get the sample mean $\bar{x} = 0.0377$ from $n = 100$ samples. You also compute sample ACFs and PACFs for the data sets, which you draw in Figures 1 and 2.

(a) Compute confidence intervals for the data and draw them in Figures 1 and 2. Remember to hand in this part of the exam paper along with your handwritten solutions.

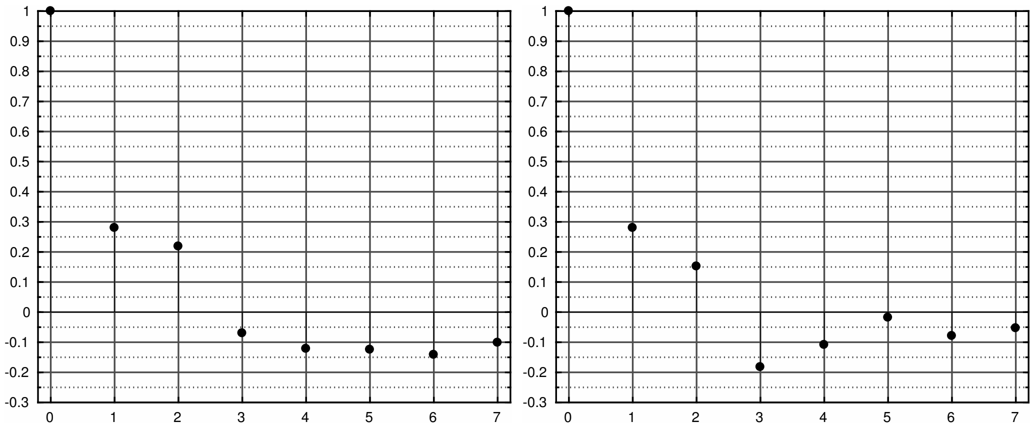
(b) Conclude if any and if so which of the data sets should be modeled with the suggested model (1).



(a) Sample ACF (b) Sample PACF

Figure 1: The sample ACFs and PACFs of data set 1.

(9 points)



(a) Sample ACF (b) Sample PACF

Figure 2: The sample ACFs and PACFs of data set 2.

Problem 4

Assume that you are given a data sample of observations of size $n = 1000$. It is used to compute estimators $\hat{\mu}$ and $\hat{\sigma}$ of the parameters in the AR(1) model (1). You know that the sample $x^n := (x_1, \dots, x_n)$ has sample mean $\bar{x} = -0.03$ and the sample autocovariance function up to lag 5 is

$\hat{\gamma}(0)$	$\hat{\gamma}(1)$	$\hat{\gamma}(2)$	$\hat{\gamma}(3)$	$\hat{\gamma}(4)$	$\hat{\gamma}(5)$
2.11	0.55	0.20	-0.08	-0.02	-0.00

Furthermore, you are given the information that $x_n = 0.35$ and $x_{n-1} = 1.36$. Show that Yule Walker estimation leads to $\hat{\mu} = -0.74$ and $\hat{\sigma}^2 = 1.97$ (rounded to two digits).

(5 points)

Problem 5

Having fitted parameters to the model problem (1) in Problem 4, the goal of this problem is to compute predictors.

(a) Use the sample autocovariance function from Problem 4 but not the fitted model (1) to compute the best linear predictors for X_{n+1} and X_{n+2} given the observations of (X_{n-1}, X_n).

(b) Use $\hat{\mu}$ and $\hat{\sigma}$ and the model (1) instead to compute best linear predictors for X_{n+1} and X_{n+2} given the observations of $(X_1, \dots, X_n)$. What do you observe if you compare your result with the one from (a)?

(21 points)