MSA 410 / TMS 087

Financial Time Series

Exam 1, June 2, 2017
Läsår 2016/17, lp 4
Examiner Annika Lang

Given the TS

$$X_t = (1+\mu) X_{t-1} + \sigma z_t,$$

where

- $z = (z_t, t \in \mathbb{Z})$ IID(0,1) and $z_t \sim N(0,1)$ $\forall t \in \mathbb{Z}$
- $t_0 \in \mathbb{Z}$ fixed with
  - * $\mathbb{E}(X_{t_0}) = 0$
  - * $\mathbb{E}(X_{t_0}^2) = -\frac{\sigma^2}{2\mu+\mu^2}$
- $-2 < \mu < 0$, $\mu \neq -1$

[15] Problem 1

Assume that $\mathrm{Cov}(X_t, z_{t+h}) = 0$ $\forall t \in \mathbb{Z}$, $h > 0$

[3] (a) Let $t > t_0$, then

$$\begin{aligned}
\mathbb{E}(X_t) &= \mathbb{E}((1+\mu)X_{t-1} + \sigma z_t) \\
&\overset{\text{lin. } \mathbb{E}}{=} (1+\mu)\,\mathbb{E}(X_{t-1}) + \sigma\,\mathbb{E}(z_t) \qquad \tfrac{1}{2} \\
&\overset{z_t \sim N(0,1)}{=} (1+\mu)\,\mathbb{E}(X_{t-1}) + 0 \\
&= \dots \quad \text{[recursively]} \\
&= (1+\mu)^{t-t_0}\,\mathbb{E}(X_{t_0}) \qquad \tfrac{1}{2} \\
&\overset{\mathbb{E}(X_{t_0})=0}{=} 0
\end{aligned}$$

Let next $t < t_0$. Observe that

$$X_t = (1+\mu) X_{t-1} + \sigma z_t$$

is equivalent to

$$(1+\mu) X_{t-1} = X_t - \sigma z_t \qquad (1)$$

and since $\mu \neq -1$ equivalent to

$$X_{t-1} = (1+\mu)^{-1} X_t - \frac{\sigma}{1+\mu} z_t$$

or with $t \to t+1$

$$X_t = (1+\mu)^{-1} X_{t+1} - \frac{\sigma}{1+\mu} z_{t+1}$$

Therefore we obtain

$$\begin{aligned}
\mathbb{E}(X_t) &= \mathbb{E}\left((1+\mu)^{-1} X_{t+1} - \frac{\sigma}{1+\mu} z_{t+1} \right) \\
&\overset{\text{lin } \mathbb{E}}{=} (1+\mu)^{-1} \mathbb{E}(X_{t+1}) - \frac{\sigma}{1+\mu} \mathbb{E}(z_{t+1}) \\
&\overset{z_{t+1} \sim N(0,1)}{=} (1+\mu)^{-1} \mathbb{E}(X_{t+1}) - 0 \\
&= \dots \quad [\text{recursively}] \\
&= (1+\mu)^{-(t_0-t)} \mathbb{E}(X_{t_0}) \qquad (1) \\
&\overset{\mathbb{E}(X_{t_0})=0}{=} 0
\end{aligned}$$

$\Rightarrow$ We have shown

$$\mathbb{E}(X_t) = 0 \qquad \forall t \in \mathbb{Z}$$

$\boxed{5\tfrac{1}{2}}$

(b) Observe first the general recursion

$$\begin{aligned}
\mathbb{E}(X_t^2) &= \mathbb{E}\left(((1+\mu) X_{t-1} + \sigma z_t)^2 \right) \\
&= \mathbb{E}\left((1+\mu)^2 X_{t-1}^2 + (1+\mu)\sigma X_{t-1} z_t + \sigma^2 z_t^2 \right) \qquad (\tfrac{1}{2}) \\
&\overset{\text{lin } \mathbb{E}}{=} (1+\mu)^2 \mathbb{E}(X_{t-1}^2) + (1+\mu)\sigma \mathbb{E}(X_{t-1} z_t) + \sigma^2 \mathbb{E}(z_t^2) \qquad (\tfrac{1}{2}) \\
&\overset{z_t \sim N(0,1)}{=} (1+\mu)^2 \mathbb{E}(X_{t-1}^2) + (1+\mu)\sigma \operatorname{Cov}(X_{t-1}, z_t) + \sigma^2 \qquad (\tfrac{1}{2}) \\
&\overset{\substack{\text{Assume} \\ \operatorname{Cov}(X_{t-1}, z_t)=0}}{=} (1+\mu)^2 \mathbb{E}(X_{t-1}^2) + \sigma^2 \qquad (\tfrac{1}{2})
\end{aligned}$$

Set $t = t_0 + 1$, then

$$\mathbb{E}(X_{t_0+1}^2) = (1+\mu)^2 \mathbb{E}(X_{t_0}^2) + \sigma^2$$

$$= (1+\mu)^2 \left(-\frac{\sigma^2}{2\mu+\mu^2}\right) + \sigma^2$$
$$= -(1+2\mu+\mu^2)\frac{\sigma^2}{2\mu+\mu^2} + \sigma^2$$
$$= -\frac{\sigma^2}{2\mu+\mu^2} - \frac{(2\mu+\mu^2)\sigma^2}{2\mu+\mu^2} + \sigma^2$$
$$= -\frac{\sigma^2}{2\mu+\mu^2} \qquad (1)$$

and therefore $\mathbb{E}(X_{t_0+1}^2) = \mathbb{E}(X_{t_0}^2)$ and recursively for all $t \geq t_0$. $(\frac{1}{2})$

To get results for $t < t_0$, observe first that

$$-\frac{\sigma^2}{2\mu+\mu^2} = \mathbb{E}(X_{t_0}^2) \overset{\text{t=to in the prev. comp.}}{=} (1+\mu)^2\, \mathbb{E}(X_{t_0-1}^2) + \sigma^2 \qquad (\tfrac{1}{2})$$

which is equivalent (since $\mu \neq -1$) to

$$\mathbb{E}(X_{t_0-1}^2) = \left(-\frac{\sigma^2}{2\mu+\mu^2} - \sigma^2\right)\frac{1}{(1+\mu)^2}$$
$$= -\frac{\sigma^2(\cancel{1+2\mu+\mu^2})}{2\mu+\mu^2}\cdot\frac{1}{\cancel{1+2\mu+\mu^2}}$$
$$= -\frac{\sigma^2}{2\mu+\mu^2} = \mathbb{E}(X_{t_0}^2) \qquad (1)$$

Recursively we therefore obtain that
$$\mathbb{E}(X_t^2) = \mathbb{E}(X_{t_0}^2) \quad \forall\, t \leq t_0 \qquad (\tfrac{1}{2})$$
and all together

$$\mathbb{E}(X_t^2) = -\frac{\sigma^2}{2\mu+\mu^2} \qquad \forall\, t \in \mathbb{Z}$$

$\boxed{3}$ (c) Let $j > 0$, then

$$\mathbb{E}(X_t X_{t+j}) = \mathbb{E}\left(\left((1+\mu)X_{t-1} + \sigma Z_t\right)X_{t+j}\right)$$
$$\overset{\text{lin}}{=} (1+\mu)\,\mathbb{E}(X_{t-1}X_{t+j}) + \sigma\,\mathbb{E}(Z_t X_{t+j})$$
$$\overset{\mathrm{Cov}(X_{t+j},Z_t)=0}{=} (1+\mu)\,\mathbb{E}(X_{t-1}X_{t+j}) + \sigma\cdot 0 \qquad (1)$$

$$= (1+\mu)\, \mathbb{E}(X_{t-1} X_{t-j})$$

$$= \cdots \qquad \text{[recursively]}$$

$$= (1+\mu)^j \; \mathbb{E}(X_{t-j}^2)$$

$$\overset{(b)}{=} -(1+\mu)^j \frac{\sigma^2}{2\mu+\mu^2} \qquad 1$$

For $j < 0$, set

$$t' = t-j > t'+j = t'-(-j) = t$$

and plug in to obtain

$$\mathbb{E}(X_t X_{t-j}) = \mathbb{E}(X_{t'-(-j)} X_{t'})$$

$$= -(1+\mu)^{-j} \frac{\sigma^2}{2\mu+\mu^2} \qquad \tfrac{1}{2}$$

So in conclusion

$$\mathbb{E}(X_t X_{t-j}) = -(1-\mu)^{|j|} \frac{\sigma^2}{2\mu+\mu^2} \qquad \tfrac{1}{2}$$

$\boxed{3\tfrac{1}{2}}$

(d) • We have seen in (a) that the mean is constant for all t

$$\mathbb{E}(X_t) = 0 \qquad \tfrac{1}{2}$$

• We have seen in (b) that

$$Var(X_t) = \mathbb{E}(X_t^2) = -\frac{\sigma^2}{2\mu+\mu^2} < +\infty \qquad \tfrac{1}{2}$$

since $\mu < 0$. $\tfrac{1}{2}$

• We have seen in (c) that

$$Cov(X_t, X_{t-j}) = \mathbb{E}(X_t X_{t-j})$$

$$= -(1+\mu)^{|j|} \frac{\sigma^2}{2\mu+\mu^2}$$

just depends on the distance $|t-t-j| = |j|$ and not on t. 1

$$\gamma_X(h) = -(1+\mu)^{|h|} \frac{\sigma^2}{2\mu+\mu^2} \qquad 1$$

is the ACVF for all $h \in \mathbb{Z}$.

10

Problem 2

3 (a) We have shown in Problem 1 that X is stationary. (1/2)

It remains to show that X has a representation

$$X_t - \phi_1 X_{t-1} = \tilde{Z}_t, \quad (1/2)$$

where $\tilde{Z}_t \sim WN(0, \tilde{\sigma}^2)$ for some $\tilde{\sigma}^2 > 0$.

Set

$$\phi_1 := 1+\mu \quad (1/2)$$

$$\tilde{Z}_t := \sigma Z_t \quad (1/2)$$

Since $Z_t \sim \mathcal{N}(0,1)$ and $Z \sim \mathrm{IID}(0,1)$,

$$\tilde{Z}_t \sim \mathcal{N}(0,\sigma^2) \quad \text{and} \quad \tilde{Z} \sim \mathrm{IID}(0,\sigma^2) \quad (1/2)$$

and since

$$\mathrm{IID}(0,\sigma^2) \Rightarrow WN(0,\sigma^2)$$

$$\left.\begin{array}{l} \tilde{Z} \sim WN(0,\sigma^2) \\ \text{Therefore we are done setting} \\ \tilde{\sigma}^2 := \sigma^2. \end{array}\right\} \quad (1/2)$$

4 (b) We know that X is causal

$$\overset{\text{def.}}{\Longleftrightarrow} \exists (\psi_j, j \in \mathbb{N}_0) \text{ s.t. } \sum_{j=0}^{\infty} |\psi_j| < +\infty \text{ and}$$

$$(1) \quad X_t = \sum_{j=0}^{\infty} \psi_j \tilde{Z}_{t-j} \quad \forall t \in \mathbb{Z}$$

$$(1) \quad \overset{\text{Lemma}}{\Longleftrightarrow} 1 - \phi_1 z \neq 0 \quad \forall z \in \mathbb{C}, \ |z| \leq 1$$

Using (a), we obtain

$$1 - (1+\mu) z = 0$$

$$\Longleftrightarrow (1+\mu) z = 1$$

$$\overset{\mu \neq -1}{\Longleftrightarrow} z = \frac{1}{1+\mu} \quad (1/2)$$

Since

$$-2 < \mu < 0$$

$$\iff \quad -1 < 1+\mu < 1 \qquad \frac{1}{2}$$

$$\iff \quad \frac{1}{1+\mu} > 1 \quad \text{or} \quad \frac{1}{1+\mu} < -1$$

$$\iff \quad \left| \frac{1}{1+\mu} \right| > 1 \qquad \frac{1}{2}$$

Therefore

$$1 + (1+\mu) z = 0 \quad \overset{\mu \neq -1}{\iff} \quad z = \frac{1}{1+\mu}$$

with

$$|z| > 1,$$

i.e., X is causal $\forall\ \mu \in (-2, 0) \setminus \{-1\}$ $\frac{1}{2}$

$\boxed{3}$

(c) X is invertible

$$\overset{\text{def}}{\iff} \quad \exists\ (\pi_j,\ j \in \mathbb{N}_0) \text{ s.t. } \sum_{j=0}^{\infty} |\pi_j| < +\infty \text{ and} \qquad 1$$

$$\tilde{z}_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \qquad \forall t \in \mathbb{Z}$$

Set

$$\left.\begin{aligned} \pi_0 &:= 1, \\ \pi_1 &:= -(1+\mu), \\ \pi_j &:= 0 \qquad \forall j > 1, \end{aligned}\right\} \quad 1$$

then

$$\tilde{z}_t = X_t - (1+\mu) X_{t-1} = \pi_0 X_t + \pi_1 X_{t-1}$$

and

$$|\pi_0| + |\pi_1| = 1 + |-(1+\mu)|$$

$$< 2 < +\infty, \qquad \frac{1}{2}$$

i.e., X is invertible by definition. $\frac{1}{2}$

9 Problem 3

5 (a) For an AR(1) process, the PACF for all lags $|h|>1$ is zero. (1)

By the CLT, all lags $|h|>1$ are approx. indep. normally distributed and 95% of the values should fall within the bounds $\pm \frac{1.96}{\sqrt{n}}$ of the sample PACF (1)

data set 1: $\pm \frac{1.96}{\sqrt{380}} \approx \pm 0.1005$ (1)

data set 2: $\pm \frac{1.96}{\sqrt{100}} \approx \pm 0.1960$ (1)

The same bounds hold for the sample ACF, but it is just known to be decreasing (and known from Problem 1).

lines: put in in sample PACF. (1)

4 (b) For data set 1, 1 out of 6 for lags >1 of the sample PACF is outside the confidence interval, i.e., 16.67% which is $>>$ 5% (for 95% confidence). (1)

$\Rightarrow$ data set 1 is not good to be modeled as AR(1). (1)

For data set 2, all lags >1 of the sample PACF fall within the 95% confidence interval (and the sample ACF is decreasing outside the confidence intervals). (1)

$\Rightarrow$ data set 2 can be modeled as AR(1). (1)

$\boxed{5}$

Problem 4

Yule - Walker estimation sets

$$\hat{\phi}_1 := \hat{R}_1^{-1}\,\hat{\rho}(1), \qquad 1$$

$$\hat{\sigma}^2 := \hat{\gamma}(0)\left(1 - \hat{\rho}(1)\,\hat{R}_1^{-1}\,\hat{\rho}(1)\right), \qquad 1$$

where

$\left.\begin{array}{ll} \hat{\rho} : & \text{sample ACF} \\ \hat{\gamma} : & \text{sample ACVF} \\ \hat{R}_1 = (\hat{\rho}(0)) = 1 : & \text{sample autocorrelation matrix} \end{array}\right\} 1$

$$\Rightarrow \quad \hat{\phi}_1 = 1 \cdot \hat{\rho}(1) = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} \overset{!}{=} 1 + \hat{\mu} \qquad \tfrac{1}{2}$$

$$\Leftrightarrow \quad \hat{\mu} = 1 - \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} = -0{,}74 \qquad \tfrac{1}{2}$$

and

$$\hat{\sigma}^2 = \hat{\gamma}(0)\left(1 - \hat{\rho}(1)^2\right) = \hat{\gamma}(0)\left(1 - \frac{\hat{\gamma}(1)^2}{\hat{\gamma}(0)^2}\right) \qquad \tfrac{1}{2}$$

$$= \frac{\hat{\gamma}(0)^2 - \hat{\gamma}(1)^2}{\hat{\gamma}(0)} = \hat{\gamma}(0) - \frac{\hat{\gamma}(1)^2}{\hat{\gamma}(0)}$$

$$= 1{,}97 \qquad \tfrac{1}{2}$$

21

Problem 5

9½ (a) We are looking for the best representations

$$X_{n+1} = a_0 + a_1 X_n + a_2 X_{n-1} \quad (1) \quad ½$$

and

$$X_{n+2} = a_0' + a_1' X_n + a_2' X_{n-1} \quad (2) \quad ½$$

These can be found with Proposition 2.3.5

$$a_0 := \overline{X_n}\,(1 - a_1 - a_2) \quad ½$$

and

$$\begin{pmatrix} \hat\gamma(0) & \hat\gamma(1) \\ \hat\gamma(1) & \hat\gamma(0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \hat\gamma(1) \\ \hat\gamma(2) \end{pmatrix} \quad 1$$

$$\Leftrightarrow \begin{cases} \hat\gamma(0)\, a_1 + \hat\gamma(1)\, a_2 = \hat\gamma(1) \\ \hat\gamma(1)\, a_1 + \hat\gamma(0)\, a_2 = \hat\gamma(2) \end{cases}$$

$\hat\gamma(1) \neq 0$
$\hat\gamma(0) \neq 0$

$$\Longleftrightarrow \begin{cases} a_1 + \frac{\hat\gamma(1)}{\hat\gamma(0)} a_2 = \frac{\hat\gamma(1)}{\hat\gamma(0)} \\ a_1 + \frac{\hat\gamma(0)}{\hat\gamma(1)} a_2 = \frac{\hat\gamma(2)}{\hat\gamma(1)} \end{cases}$$

$$\overset{(1)-(2)}{\Longrightarrow} \underbrace{\left(\frac{\hat\gamma(1)}{\hat\gamma(0)} - \frac{\hat\gamma(0)}{\hat\gamma(1)}\right)}_{\neq 0} a_2 = \frac{\hat\gamma(1)}{\hat\gamma(0)} - \frac{\hat\gamma(2)}{\hat\gamma(1)} \quad 1$$

$$\Rightarrow \quad a_2 = \frac{\frac{\hat\gamma(1)}{\hat\gamma(0)} - \frac{\hat\gamma(2)}{\hat\gamma(1)}}{\frac{\hat\gamma(1)}{\hat\gamma(0)} - \frac{\hat\gamma(0)}{\hat\gamma(1)}} = 0.03 \quad ½$$

and

$$a_1 = \frac{\hat\gamma(1)}{\hat\gamma(0)} - \frac{\hat\gamma(1)}{\hat\gamma(0)} a_2 = 0.25 \quad ½$$

and

$$a_0 = \overline{X_n}\,(1 - a_1 - a_2) = -0.02 \quad ½$$

For X_{n+2} the linear system reads

$$\begin{pmatrix}\hat\gamma(0) & \hat\gamma(1)\\ \hat\gamma(1) & \hat\gamma(0)\end{pmatrix}\begin{pmatrix}a'_1\\ a'_2\end{pmatrix} = \begin{pmatrix}\hat\gamma(2)\\ \hat\gamma(3)\end{pmatrix} \qquad (1)$$

which leads with the same computations to

$$a'_2 \overset{(1/2)}{=} \frac{\frac{\hat\gamma(2)}{\hat\gamma(0)} - \frac{\hat\gamma(3)}{\hat\gamma(1)}}{\frac{\hat\gamma(1)}{\hat\gamma(0)} - \frac{\hat\gamma(0)}{\hat\gamma(1)}} = -0.07 \qquad (1/2)$$

$$a'_1 \overset{(1/2)}{=} \frac{\hat\gamma(2)}{\hat\gamma(0)} - \frac{\hat\gamma(1)}{\hat\gamma(0)}\, a'_2 = 0.11 \qquad (1/2)$$

and

$$a'_0 = x_n\,(1 - a'_1 - a'_2) = -0.03 \qquad (1/2)$$

$$\Rightarrow \quad b^e_{n+1}(X_{n-1}, X_n) = -0.02 + 0.25\,X_n + 0.03\,X_{n-1} = 0.11 \qquad (1/2)$$

$$b^e_{n+2}(X_{n-1}, X_n) = -0.03 + 0.11\,X_n - 0.07\,X_{n-1} = -0.08 \qquad (1/2)$$

$\boxed{11\tfrac{1}{2}}$

(b) We know from Proposition 2.3.3 that the best predictor given $X^n = (X_1, \dots, X_n)$ is the conditional expectation, i.e.,

$$\begin{aligned}
b_{n+1}(X^n) &= \mathbb{E}(X_{n+1} \mid (X_1,\dots,X_n)) \qquad (1)\\
&= \mathbb{E}((1+\hat\mu)X_n + z_{n+1} \mid (X_1,\dots,X_n))\\
&\overset{\text{lin. cond. exp.}}{=} (1+\hat\mu)\,\mathbb{E}(X_n \mid (X_1,\dots,X_n)) + \mathbb{E}(z_{n+1} \mid (X_1,\dots,X_n))\\
&\overset{\substack{(1)\ X_n \text{ is } X_n\text{-meas.}\\ (1)\ z_{n+1} \perp\!\!\!\perp (X_1,\dots,X_n)}}{=} (1+\hat\mu)\,X_n + \mathbb{E}(z_{n+1})\\
&= (1+\hat\mu)\,X_n = 0.26\,X_n \qquad (1/2)\\
&\overset{\substack{\text{since lin.}\\ \text{fction of } X^n\\ (1)}}{=} b^e_{n+1}((X_1,\dots,X_n)) = 0.09
\end{aligned}$$

Similarly

$$\begin{aligned}
b_{n+2}((X_1,\dots,X_n)) &= \mathbb{E}(X_{n+2} \mid (X_1,\dots,X_n))\\
&= \mathbb{E}((1+\hat\mu)X_{n+1} + z_{n+2} \mid (X_1,\dots,X_n))\\
&= \mathbb{E}((1+\hat\mu)((1+\hat\mu)X_n + z_{n+1}) + z_{n+2} \mid (X_1,\dots,X_n)) \qquad (1)
\end{aligned}$$

$$\overset{\text{lin. cond. exp.}}{=} (1+\hat{\mu})^2\, \mathbb{E}(X_n \mid (X_1,\dots,X_n)) + (1+\hat{\mu})\,\mathbb{E}(Z_{n+1} \mid (X_1,\dots,X_n)) + \mathbb{E}(Z_{n+2} \mid (X_1,\dots,X_n))$$

① $Z_{n+2} \perp\!\!\!\perp (X_1,\dots,X_n)$, Z_{n+1}
① X_n is X_n-meas.

$$= (1+\hat{\mu})^2 X_n + (1+\hat{\mu})\,\mathbb{E}(Z_{n+1}) + \mathbb{E}(Z_{n+2})$$

$$= (1+\hat{\mu})^2 X_n = 0.07\, X_n \quad \tfrac{1}{2}$$

since linear fction ① $= b^e_{n+2}((X_1,\dots,X_n)) = 0.02$

We observe that

- the best linear predictor using the model just depends on X_n while
- the best linear predictor using the sample ACVF depends on X_n and X_{n-1} ①

Reason: sample ACVF is just an approximation that does not precisely "recover" the model ½

	b^e_{n+1}	b^e_{n+2}
(a)	0.11	−0.08
(b)	0.09	0.02

① Overall are the computed numbers are very similar, i.e., $b^e_{n+1} \approx 0.1$ and $b^e_{n+2} \approx 0$.

alternative solutions

- Use the innovations algorithm as presented in the lecture notes "Forecasting of ARMA processes"
- Use Proposition 2.3.5 with an abstract $n\times n$ matrix and observe that there is only one solution to this system.

All methods lead due to the uniqueness of $b^e_{n+i}((X_1,\dots,X_n))$, $i=1,2$, to the same result.