Exercises for ARCH and GARCH models

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We recall the formulas for a finite and infinite geometric sum. Let $r \neq 1$ and $n \in \mathbb{N}$. Then

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$$

hence, if $|r| < 1$,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

These formulas are useful in some of the following exercises on ARCH and GARCH models. Note that all (G)ARCH processes below are implicitly assumed to be stationary.

Ex. 1 — Let the stochastic process $X = (X_t, t \in \mathbb{Z})$ be an ARCH(1) *process* with

$$X_t = \sigma_t Z_t,$$

where $Z \sim \mathrm{IID}\,\mathcal{N}(0,1)$,

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2,$$

with $\alpha_0 > 0$, $0 < \alpha_1 < 1$ and Z_t and $(X_{t-j}, j \in \mathbb{N})$ independent for all $t \in \mathbb{Z}$.

a) Show that

$$X_t^2 = \alpha_0 \sum_{j=0}^{n} \alpha_1^j Z_t^2 Z_{t-1}^2 \cdots Z_{t-j}^2 + \alpha_1^{n+1} X_{t-n-1}^2 Z_t^2 Z_{t-1}^2 \cdots Z_{t-n}^2$$

for all $n \in \mathbb{N}$.

b) One can show that the previous task implies that

$$X_t^2 = \lim_{n\to\infty} \alpha_0 \sum_{j=0}^{n} \alpha_1^j Z_t^2 Z_{t-1}^2 \cdots Z_{t-j}^2$$

almost surely. The so called *monotone convergence theorem* says that if $(Y_n, n \in \mathbb{N})$ are non-negative increasing random variables such that $\lim_{n\to\infty} Y_n = Y$ almost surely then $\mathbb{E}[Y] = \mathbb{E}[\lim_{n\to\infty} Y_n] = \lim_{n\to\infty} \mathbb{E}[Y_n]$. Use this to find $\mathbb{E}[X_t^2]$ in another way than in the lecture notes.

c) (Harder exercise) Use the fact that the fourth moment of the white noise $E[Z_t^4] = 3$ to evaluate $\mathbb{E}[X_t^4]$ using the monotone convergence theorem of the previous exercise. Deduce that $\mathbb{E}[X_t^4] < \infty \iff 3\alpha_1^2 < 1$.

d) Find the so called *conditional variance* of the ARCH(1)-model $\mathsf{Var}(X_t|X_{t-1}) = \mathbb{E}[(X_t - \mathbb{E}[X_t])^2|X_{t-1}]$.

Ex. 2 — Let the stochastic process $X = (X_t, t \in \mathbb{Z})$ be an ARCH(1) *process* given by

$$X_t = \sigma_t Z_t$$

and

$$\sigma_t^2 = \frac{1}{2} + \frac{1}{4} X_{t-1}^2$$

for $t \in \mathbb{Z}$, where $Z \sim \mathrm{IID}\,\mathcal{N}(0,1)$ (with 4th moment equal to 3), Z_t is independent of $(X_{t-j}, j \in \mathbb{N})$ for all $t \in \mathbb{Z}$.

a) Show that $X^2 := (X_t^2, t \in \mathbb{Z})$ is weakly stationary by doing the following:
   (i) Show that $\mathbb{E}(X_t^2) = \mathbb{E}(\sigma_t^2)$.
   (ii) Use the previous result and the fact that $\mathbb{E}(X_t^2)$ does not depend on t to compute $\mathbb{E}(X_t^2)$ explicitly.
   (iii) Assume that $\mathbb{E}(X_t^4)$ is constant for all $t \in \mathbb{Z}$. Compute $\mathbb{E}(X_t^4)$.
   (iv) Use the previous result to show that $\mathsf{Cov}(X_t^2, X_{t+h}^2)$ does not depend on t for $h > 0$.
   (v) Conclude that you have shown stationarity and write down the autocovariance function.

b) Show that $\tilde{Z} := (\tilde{Z}_t, t \in \mathbb{N})$ with $\tilde{Z}_t := X_t^2 - \sigma_t^2$ is white noise with mean zero and variance $40/39$.

c) Show that $(X_t^2, t \in \mathbb{Z})$ is a causal AR(1) process with mean $2/3$. (*Hint:* Use the result of the previous task.)

Ex. 3 — Let the stochastic process $X = (X_t, t \in \mathbb{Z})$ be a causal ARCH(p) *process* with $\mathbb{E}[X_t^4] < \infty$.

a) Show that $Y_t = X_t^2/\alpha_0$ satisifies the equations

$$Y_t = Z_t^2\big(1 + \sum_{i=1}^{p} \alpha_i Y_{t-i}\big).$$

b) Show that Y_t satisifies the AR(p) equations

$$\phi(B) Y_t = 1 + \tilde{Z}_t$$

for some white noise $\tilde{Z} := (\tilde{Z}_t, t \in \mathbb{N})$, where $\phi_i = \alpha_i$ for $i = 1, \ldots, p$. *Hint:* Consider the solution to Exercise 2. Can you do something similar here?

Ex. 4 — Let the stochastic process $X = (X_t, t \in \mathbb{Z})$ be a causal GARCH(p, q) *process* with $\mathbb{E}[X_t^4] < \infty$.

a) Let $\tilde{Z}_t = X_t^2 - \sigma_t^2$. Assuming that $\mathsf{Var}(\tilde{Z}_t) < \infty$ (this follows from $\mathbb{E}[X_t^4] < \infty$), show that $(\tilde{Z}_t, t \in \mathbb{Z})$ is mean zero white noise.

b) Show that $(X_t^2, t \in \mathbb{Z})$ satisfies the ARMA(m, q)-equation

$$\phi(B) X_t^2 = \alpha_0 + \theta(B) \tilde{Z}_t$$

where $m = \max(p, q)$ and the ARMA coefficients satisfies $\phi_i = \alpha_i + \beta_i$ for $i = 1, \ldots, m$ and $\theta_i = -\beta_i$ for $i = 1 \ldots q$ where $\beta_i = 0$ for $i > q$ and $\alpha_i = 0$ for $i > p$.