

EXAMINATION: Tentamensskrivning i Matematisk Statistik (TMS061)
(TMS060)

Time: Thursday 30 August 2007

Jour: Viktor Olsson, 0730-888826

Aid: You are allowed to use a scientific calculator and a half page (both sides) of hand written notes

Lab: Depending on the performance in the lab 0-5 points will be added to your test score to provide with your final score.

Grade: You need points for 29 for 5, 22 points for 4 and 15 points for 3.

Motivate all your answers. Good Luck!

- 1) a) $P(B|A) = P(B|A') = 0.2$. Compute $P(B)$. (A' = complement to A).
b) $\text{Var}(X) = 1.5$. Compute $\text{Var}(2 - X)$.
c) X is $Po(\lambda)$. Explain why $2X$ cannot be $Po(2\lambda)$. (6p)
- 2) Let $X_1, X_2, \dots, X_n$ be a random sample.
 - a) What conditions do $X_1, X_2, \dots, X_n$ have to satisfy?
 - b) If $X_1, X_2, \dots, X_n$ are additionally normally distributed $N(\mu, \sigma^2)$ what is the distribution of $\bar{X}$?
 - c) Find $E(\bar{X})$, $\text{Var}(\bar{X})$?
 - d) State the Central Limit Theorem.
(5p)
- 3) X is a geometric random variable with parameter p if $P(X = x) = p(1-p)^{x-1}$, $x = 1, 2, 3, \dots$. For which p is $P(X \leq 2) > 0.75$? (4p).

- 4) Consider the following frequency table:

Values	0	1	2	3	4	5
Observed frequency	75	140	108	66	9	2

Based on 400 observations is a binomial distribution $B(5, 0.3)$ an appropriate model? Perform a χ^2 test with $\alpha = 0.05$. (5p)

- 5) Let X be a random variable with probability density function

$$P(X = x) = (x-1)p^2(1-p)^{x-2}, x = 2, 3, \dots$$

Find the maximum likelihood estimator of p . (5p)

- 6) The probability density function for the time X it takes to assist a customer is $f(x) = \frac{10-x}{50}, 0 < x < 10$.

a) Compute $E(X)$ and $Var(X)$.

b) For another customer Y the time is $8 + X$. Compute $E(Y)$ and $Var(Y)$.

(4p)

- 7) Two different materials can be used for a mechanical construction. To decide which material to use five samples from each material have been tested. The results follow:

Material 1: 5.60, 9.92, 6.03, 5.53, 7.30

Material 2: 4.26, 4.47, 6.79, 6.20, 6.61

Provide with a confidence interval for the difference between the two materials at a confidence level $\alpha = 0.05$. State clearly all the necessary assumptions.

(4p)

8) Six tests for one type of material gave (in kp/mm^2)

6.7, 7.5, 7.2, 7.3, 6.9, 7.6

Suppose that the sample is from a normal population with mean value μ . Test $H_0 : \mu = 7$ against $H_1 : \mu > 7$ at significant level $\alpha = 0.05$.
(3p)

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1a) $P(B) = P(B|A) \cdot P(A) + P(B|A') \cdot P(A') = 0.0730$

$$= P(B|A) \cdot P(A) + P(B|A') \cdot P(A') =$$

$$= P(B|A) \cdot (P(A) + P(A')) =$$

$$= P(B|A) \cdot 1 = \underline{\underline{0.2}}$$

1p

1b) $\text{Var}(2-X) = \text{Var} X = 1.5$

1p

1c) If $X \sim \text{Po}(\lambda) \Rightarrow \text{Var} X = \lambda$

But $\text{Var}(2X) = 4 \cdot \text{Var} X = 4\lambda \neq \text{Var of a } \text{Po}(2\lambda)$

2p

OR $E(2X) = 2\lambda \neq 4\lambda = \text{Var}(2X)$

2 a) independent and identically distributed (1)

b) normal

1

c) $E\bar{X} = E\left(\frac{X_1 + \dots + X_n}{n}\right) = \frac{n \cdot \mu}{n} = \mu$

$$\text{Var} \bar{X} = \text{Var}\left(\frac{X_1 + \dots + X_n}{n}\right) = \frac{n \sigma^2}{n^2} = \frac{\sigma^2}{n}$$

1

~~B) $P(X \geq 2) = 1 - P(X < 2) = 1 - P(X=1) - P(X=0)$~~
 ~~$= 1 - p(1-p) = 1 - p = 0.75$~~
 ~~$p = 0.25$~~

$$3a) P(X \leq 2) = P(X=1) + P(X=2) =$$

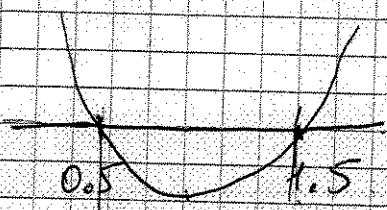
$$p + p(1-p) = p \cdot (1 + 1 - p) = p(2-p) = 2p - p^2 \geq 0.7$$

$$\Rightarrow p^2 - 2p + 0.75 \leq 0$$

$$\Delta = 4 - 4 \cdot 0.75 = 4 - 3.00 = 1 > 0$$

$$p_{1,2} = \frac{2 \pm \sqrt{\Delta}}{2a} = \frac{2 \pm 1}{2} = \begin{matrix} 1.5 \\ 0.5 \end{matrix}$$

3p



so $p \in [0.5, 1]$

$$4) X \sim B(n, p) \Rightarrow P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$P(X=0) = 0.1681$$

$$E_1 = 67.228$$

$$P(X=1) = 0.3601$$

$$E_2 = 144.06$$

$$P(X=2) = 0.3087$$

$$E_3 = 123.48$$

$$P(X=3) = 0.1323$$

$$E_4 = 52.92$$

$$P(X=4) = 0.0283$$

$$E_5 = 11.34$$

$$P(X=5) = 0.0024$$

$$E_6 = 0.9790 < 3$$

$$\tilde{p}_5 = 0.0283 + 0.0024$$

$$\tilde{E}_5 = 12.28$$

5p

$$\chi_0^2 = \sum_{i=1}^5 \frac{(O_i - E_i)^2}{E_i} = 63.194, \quad n-p-1 = 5-0-1 = 4 \text{ df}$$

$$\chi_{0.05, 4}^2 = 9.49$$

Since $\chi_0^2 = 63.194 < 9.49$ Fail to reject H_0

$$5) L(p) = \prod_{i=1}^n (x_i - 1) \cdot p^2 \cdot (1-p)^{x_i - 2} =$$

$$= ~~p^{2n}~~ p^{2n} \prod_{i=1}^n (x_i - 1) (1-p)^{x_i - 2}$$

5

$$\log L(p) = 2n \cdot \log p + \sum_{i=1}^n \log(x_i - 1) + \sum_{i=1}^n (x_i - 2) \log(1-p)$$

$$\frac{\partial \log L(p)}{\partial p} = \frac{2n}{p} + \sum_{i=1}^n \frac{-(x_i - 2)}{1-p} = \frac{2n}{p} - \sum_{i=1}^n \frac{(x_i - 2)}{1-p} =$$

$$= 0 \Rightarrow \frac{2n}{p} = \sum_{i=1}^n \frac{x_i - 2}{1-p} \Rightarrow$$

$$\frac{1-p}{p} = \sum_{i=1}^n \frac{x_i - 2}{2n} \Rightarrow \frac{1}{p} - 1 = ~~\frac{1}{2n}~~ \frac{1}{2} \bar{x} - \frac{2n}{2n} \Rightarrow$$

$$\frac{1}{p} - 1 = \frac{\bar{x}}{2} - 1 \Rightarrow \hat{p} = \frac{2}{\bar{x}}$$

$$6) f(x) = \frac{10-x}{50}, x \in (0, 10).$$

$$EX = \int_0^{10} x f(x) dx = \int_0^{10} x \cdot \frac{10-x}{50} dx =$$

$$= \frac{1}{50} \int_0^{10} 10x - x^2 dx = \frac{1}{50} \cdot \left(\frac{10x^2}{2} - \frac{x^3}{3} \right) \Big|_0^{10} =$$

$$= \frac{1}{50} \left(5 \cdot (100 - 0) - \frac{1000}{3} - 0 \right) = \frac{1}{50} \left(500 - \frac{1000}{3} \right) = \frac{10}{3}$$

$$EX^2 = \int_0^{10} x^2 \cdot \frac{10-x}{50} dx = \frac{1}{50} \int_0^{10} (10x^2 - x^3) dx =$$

$$= \frac{1}{50} \left(\frac{10x^3}{3} - \frac{x^4}{4} \right) \Big|_0^{10} = \frac{1}{50} \left(\frac{10 \cdot 1000}{3} - \frac{10000}{4} \right) =$$

$$= \frac{50}{3}$$

$$\text{Var } X = EX^2 - (EX)^2 = \frac{50}{3} - \frac{100}{9} = \frac{50}{9}$$

$$b) EY = E(8+X) = 8 + EX = 8 + \frac{10}{3} = \frac{34}{3}$$

$$\text{Var } Y = \text{Var}(8+X) = \text{Var } X = \frac{50}{9}$$

7)

Assuming the two samples are independent from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$.

$$\mu_1 - \mu_2 = \bar{X}_1 - \bar{X}_2 \pm t_{0.025}^{(8)} \cdot s_p \sqrt{\frac{1}{5} + \frac{1}{5}} =$$

$$= 6.876 - 5.666 \pm 2.306 \cdot \sqrt{2.431} \cdot \sqrt{\frac{1}{5} + \frac{1}{5}} =$$

$$= \underline{\underline{1.21 \pm 2.27}}$$

$$8) \bar{X} = 7.2$$

$$S = 0.3464$$

$$t_{obs} = \frac{7.2 - 7}{\frac{0.3464}{\sqrt{6}}} = 1.41$$

$$t_{5,0.05} = 2.015 > t_{obs} \quad \text{Cannot reject } H_0 \text{ at } \alpha = 0.05.$$

3d) If $X_1, \dots, X_n$ is a random sample of size n taken from a pop with mean μ and ^{finite} variance σ^2 , and if $\bar{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}, \text{ as } n \rightarrow \infty \text{ is the}$$

standard normal.

(2 p)