EXAMINATION: Tentamensskrivning i Matematisk Statistik (TMS061)

Time: Wednesday 16 January 2008

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Aid: You are allowed to use a scientific calculator and a half page (both sides) of hand written notes

Lab: Depending on the performance in the lab 0-5 points will be added to your test score to provide with your final score.

Grade: You need 21 points for 5, 16 points for 4 and 11 points for 3.

Motivate all your answers. Good Luck!

1) Suppose the random variable X has a normal distribution with mean 3 and variance 9. Let $Y = \frac{1}{3}X - 1$.
a) What are the mean and variance of Y? (2p)

b) What is the probability that Y is at least 1? (1p)

2) a) Suppose that A and B are two events such that: $P(A) = 0.6$ and $P(B) = 0.8$. Are A and B disjoint? Explain. (1.5p)

b) True or false: If A and B are events, then: $P(A \cup B) \geq P(A) + P(B)$. Justify your answer. (1.5p)

3) State in your own words the Central Limit Theorem. (2p)

4) a) Someone is recording the number of clients that arrive at a shop between 3 and 4 every Saturday afternoon for three months. Which distribution best describes the recordings? (1p)

b) What are the expected value and the variance of a Poisson random variable X for which $P(X = 2) = P(X = 3)$? (1p)

5) 500 observations from a random variable X have given 35 zeros, 140 ones, 158 twos, 121 threes and 46 fours. Test using a χ^2 test the hypothesis that the random variable X is binomial with $n = 4$ and $p = 1/2$. (3p)

6) a) The random variable Z is Poisson with mean value 2.4. Compute the probability $P(Z > 2)$.(1p)

b) The random variable Y is normally distributed with mean value $\mu = 3$ and standard deviation $\sigma = 0.8$. Compute $P(Y > 2)$.(1p)

7) For the random variable X with probability density function

$$f(x) = \frac{\lambda^3 x^2}{2} e^{-\lambda x}, \quad x > 0,$$

find the maximum likelihood estimator of λ. (3p)

8) Let X be the random variable that measures the content of a bottle of a specific perfume (in ml). A sample of size 16 has been taken from the this perfume and gave $\bar{x} = 476.4$ and $s = 0.7$ ml. Assume that X is normally distributed and
a) Compute $P(X \leq 475)$. (1p)

b) Construct a confidence interval for the true mean μ for $\alpha = 0.95$. (1p).

9) Let the random variable X have the probability density function $P(X = x) = 0.1 + 0.05x, \quad x = 0, 1, 2, 3, 4$.
a) Compute $E(X)$ and $Var(X)$. (1p)

b) What is the probability $P(X_1 + X_2 > 5)$ if X_1 and X_2 are independent random variables distributed like X? (2p)

10) For a engineering study we have recorded the time it takes two different machines A and B to warm up (in min.). The results are:

$$A : 6.7 \quad 7.2 \quad 5.9 \quad 6.9 \quad 7.0 \quad 6.7 \quad 5.9$$

$$B : 5.4 \quad 5.8 \quad 6.3 \quad 6.2 \quad 5.6 \quad 5.5$$

Assume that the above observations are independent samples from a normal distribution with the same variance. Test the hypothesis that the means of the two distributions are also the same with alternative hypothesis that are different. $\alpha = 0.01$ (3p)

Solutions for TMS061. 16-1-2008.

1) $X \sim N(3, 9)$ and $Y = \frac{1}{3}X - 1$.

a) $E(Y) = E\left(\frac{1}{3}X - 1\right) = \frac{1}{3}E(X) - 1 = \frac{1}{3} \cdot 3 - 1 = 1 - 1 = 0$

$Var(Y) = Var\left(\frac{1}{3}X - 1\right) = \frac{1}{9} \cdot Var X = \frac{1}{9} \cdot 9 = 1$.

b) $P(Y \geq 1) = 1 - P(Y < 1) = 1 - 0.841345 = \underline{\underline{0.158655}}$

2) a) $P(A) = 0.6$, $P(B) = 0.8$. Are A, B disjoint?

For A, B to be disjoint we need $P(A \cap B) = 0$ since $A \cap B = \emptyset$.

This is equivalent to $P(A \cup B) = P(A) + P(B) =$

$= 0.6 + 0.8 = 1.4 > 1$ Which is impossible so $\underline{\underline{NO}}$

b) In general $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

so $P(A \cup B) \leq P(A) + P(B)$ so $\underline{\underline{NO}}$

3). Let $X_1, X_2, \ldots, X_n$ be a <u>random</u> sample of size n from a population <u>with mean μ and variance σ^2</u> and let $\bar{X}$ denote the sample mean. Then the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

for sample size $n \to \infty$ is the standard normal whatever the distribution of the original population is.

4) a) Poisson distr.

b) $P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$, $x = 0, 1, 2, \ldots$

Then $P(X=2) = P(X=3) \Rightarrow$

$$\frac{e^{-\lambda} \cdot \lambda^2}{2!} = \frac{e^{-\lambda} \cdot \lambda^3}{3!} \Rightarrow$$

$$\frac{\lambda^2}{2} = \frac{\lambda^3}{6} \Rightarrow 6\lambda^2 = 2\lambda^3 \Rightarrow \boxed{3 = \lambda}$$

5) $H_0: X \sim B(4, 1/2)$

$H_1: X$ is NOT $B(4, 1/2)$

1

X:	0	1	2	3	4
Frequency:	35	140	158	121	46
$E[\text{Freq} \mid H_0]$	31.25	125	187.5	125	31.25

1

$$T_{obs} = \frac{(35-31.25)^2}{31.25} + \frac{(140-125)^2}{125} + \dots + \frac{(46-31.25)^2}{31.25} = \underline{\underline{13.98}}$$

Under the assumption H_0 is true, $T \sim \chi^2_4$

For $\alpha = 0.05$ $\quad T_{obs} > 9.49$

$\alpha = 0.01$ $\quad T_{obs} > 13.28$.

So H_0 is rejected at both levels. 1

6) a) $P(Z>2) = 1 - P(Z \leq 2) = 1 - e^{-2.4}\left(1 + 2.4 + \frac{2.4^2}{2}\right) \approx$

0.4303

b) $P(Y>2) = 1 - \Phi\left(\frac{2-3}{0.8}\right) = 1 - \Phi(-1.25) = \Phi(1.25) = 0.8944$

7) $\cancel{L(\lambda, x) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{\lambda^3 x_i^2}{2} e^{-\lambda x_i}}$

$$\log L(\lambda; x) = \log \prod_{i=1}^{n} f(x_i) = \log \prod_{i=1}^{n} \frac{\lambda^3 x_i^2}{2} e^{-\lambda x_i} =$$

$$= \log \frac{\lambda^{3n}}{2^n} \prod_{i=1}^{n} x_i^2 e^{-\lambda x_i} = \log \frac{\lambda^{3n}}{2^n} + \log \prod x_i^2 +$$

$$+ \log \prod_{i=1}^{n} e^{-\lambda x_i} =$$

$$= 3n \log \lambda - n \log 2 + \sum_{i=1}^{n} \log x_i^2 + \log e^{-\lambda \sum x_i} =$$

$$\frac{\partial \log L(\lambda;x)}{\partial \lambda} = \frac{\partial}{\partial \lambda}\Big(3n\log\lambda - n\log 2 + \sum_{i=1}^{n}\log x_i^2 -$$

$$-\lambda\sum_{i=1}^{n} x_i\Big) =$$

$$= \frac{3n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \frac{3n}{\lambda} = \sum_{i=1}^{n} x_i \Rightarrow \lambda = \frac{3n}{\sum x_i} \Rightarrow$$

$$\boxed{\hat{\lambda} = \frac{3}{\bar{X}}}$$

8) a) $P(X \leq 475) = P\left(\frac{X-\mu}{\sigma} \leq \frac{475-\mu}{\sigma}\right) =$

$= \Phi\left(\frac{475-\mu}{\sigma}\right)$ which we approximate by

$\Phi\left(\frac{475-\bar{x}}{s}\right) = \cancel{\Phi(}\ 0.0228.$

b) $P\left(-b \leq \frac{\bar{X}-\mu}{s/\sqrt{n}} \leq b\right) = 0.95$ for

$b = 2.1313$ from a t_{n-1} tabel

$(476.02 \leq \mu \leq 476.78)$ a 95% C.I

9) a) $E\,X = \sum_{k=0}^{4} k \cdot P(X=k) = 2.5$ 0.5

$Var(X) = \sum_{k=0}^{4} (k-2.5)^2 \cdot P(X=k) = 1.75$ 0.5

b). $P(X_1 + X_2 > 5) = P(X_1=4, X_2=4) + 2P(X_1=4, X_2=3) +$

$+ 2P(X_1=4, X_2=2) = P(X_1=3, X_2=3) \overset{\text{independence}}{=}$

$= P(X_1=4)^2 + 2P(X_1=4) \cdot P(X_2=3) + 2 \cdot P(X_1=4) \cdot P(X_2=2) +$

$+ P(X_1=3)^2 = 0.4225$

2

10). $H_0: \mu_A = \mu_B$

$H_1: \mu_A \neq \mu_B$

$\bar{X}_A = 6.61$ $\bar{X}_B = 5.80$ $S_p^2 = \frac{6 \cdot S_A^2 + 5 \cdot S_B^2}{11} = 0.210$

$S_A^2 = 0.268$ $S_B^2 = 0.140$

$t = \frac{\bar{X}_A - \bar{X}_B}{S_p \sqrt{1/7 + 1/6}} = 3.19$

For $\alpha = 0.01$ $t_{0.005} = 3.106$ so H_0 can be rejected at $\alpha = 0.01$