Lösningsförslag Tenta 2011-10-21. (TMS136).

1) a) Slumpförsök = Ett försök som kan utfalla på olika sätt när det utförs på samma sätt flera gånger.

b) Utfallsrum = S = { Möjliga utfall från ett slumpför

c) $X : S \to \mathbb{R}$.

d) X, Y är oberoende om $f(x,y) = f(x)f(y)$.

2) $P(\text{kåk}) = \dfrac{\overbrace{\binom{13}{2}}^{\text{2 valörer}}\overbrace{\binom{2}{1}}^{\text{vilken skall vara par/triss}}\underbrace{\binom{4}{2}}_{\text{par}}\underbrace{\binom{4}{3}}_{\text{triss}}}{\binom{52}{5}} = 0.144\ \%$

3) b) $P(X=0 \mid Y=0) = \dfrac{P(X=0, Y=0)}{P(Y=0)} = \dfrac{P(Y=0 \mid X=0)P(X=0)}{P(Y=0\mid X=0)P(X=0) + P(Y=0\mid X=1)P(X=1)}$

$= \dfrac{(1-p)\frac{1}{4}}{(1-p)\frac{1}{4} + p\frac{3}{4}} = \boxed{\dfrac{1-p}{1+2p}}$

a) $P(Y=0) = P(Y=0 \mid X=0)P(X=0) + P(Y=0 \mid X=1)P(X=1) =$
$= (1-p)\frac{1}{4} + p\frac{3}{4} = \dfrac{1+2p}{4}$

$P(Y=1) = 1 - \dfrac{1+2p}{4} = \dfrac{3-2p}{4}$

$\Rightarrow Y \sim \text{Ber}\left(\boxed{\dfrac{3-2p}{4}}\right).$

c) $\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = E[XY] - \frac{3}{4}\cdot\left(\frac{3-2p}{4}\right)$

$E[XY] = \sum_x \sum_y xy f(x,y) = 1\cdot 1\cdot f(1,1) = P(X=1, Y=1) =$
$= P(Y=1 \mid X=1)P(X=1) = (1-p)\frac{3}{4} = \underbrace{\dfrac{3(1-p)}{4}}_{E(XY)}$

$\therefore \text{Cov}(X,Y) = \dfrac{3(1-p)}{4} - \dfrac{3}{4}\left(\dfrac{3-2p}{4}\right) = \underbrace{\dfrac{3(1-2p)}{16}}_{\text{Cov}(X,Y)}$

Kommunikation är möjlig.

$\text{Cov}(X,Y) = 0$ omm $p = \frac{1}{2}$. Om alla bits som skickas förvrängs med s.h. $\frac{1}{2}$ så kan vi inte veta vad som skickats.

d) $\hat{p} = \bar{X} = \frac{1}{n}\sum X_i$, där $X_i = \begin{cases} 1 & \text{om biten förvanskade} \\ 0 & \text{annars.} \end{cases}$

$\hat{p} \sim N(p, p(1-p))$

Så $\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$. Och eftersom n är så stort så är även

$$\frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \sim N(0,1).$$

(Sätt $\alpha = 0.05$.) Dvs. 95% CI

$$0{,}95 = P(A \le p \le B) = P\Big(\underbrace{\frac{\hat{p}-B}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}}_{=-1{,}96} \le Z \le \underbrace{\frac{\hat{p}-A}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}}_{=1{,}96}\Big) \Rightarrow$$

$\Rightarrow$ $A = \hat{p} - 1{,}96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$B = \hat{p} + 1{,}96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$\hat{p} = \frac{341}{10.000} = 0{,}0341.$

$\left.\begin{aligned} a &= 0{,}0305 \\ b &= 0{,}0377. \end{aligned}\right\}$ $\boxed{0{,}0305 \le p \le 0{,}0377.}$

Om du valde 99% CI så blir det

$\boxed{0{,}0298 \le p \le 0{,}0392}$

4.) $T \sim Exp(\lambda) \iff N =$ antal samtal/timma $\sim Poi(\lambda)$.

$P(N \le 55) = P\left(\frac{N-60}{\sqrt{60}} \le \frac{55-60}{\sqrt{60}}\right) \approx \{$Centrala gränsv. satsen$\} = P(Z \le -0{,}65)$

$= 0{,}26.$

5) $X \sim Poi(\lambda)$, $X_1, \dots, X_n$ stickprov.

a) $\mathbb{E}(X) = \lambda \Rightarrow MOM(\lambda) = \overline{X}$.

b) $f(x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$$L(\lambda) = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{x_1!\,x_2!\cdots x_n!}$$

$$\ln L(\lambda) = -n\lambda + \left(\sum x_i\right)(\ln\lambda) - \ln(x_1!\cdots x_n!)$$

$$\frac{d}{d\lambda}\ln L(\lambda) = -n + \frac{\sum x_i}{\lambda} = 0 \Rightarrow \lambda = \frac{\sum x_i}{n}$$

$\therefore MLE(\lambda) = \overline{X}$.

c) $$\mathbb{E}(\hat{\lambda}) = \mathbb{E}\left[k\sum_1^{n-1}(X_{i+1}-X_i)^2\right] = k\sum_1^{n-1}\mathbb{E}\left[(X_{i+1}-X_i)^2\right] =$$

$$= k\sum_1^{n-1}\mathbb{E}\left[X_{i+1}^2 - 2X_{i+1}X_i + X_i^2\right] =$$

$$= k\sum_1^{n-1}\left(\mathbb{E}[X_{i+1}^2] - 2\underbrace{\mathbb{E}(X_{i+1}X_i)}_{=\mathbb{E}(X_{i+1})\mathbb{E}(X_i)\ (\text{ober.!})} + \mathbb{E}[X_i^2]\right) =$$

$$= k\sum_1^{n-1}\left(\underbrace{Var(X_{i+1})}_{\lambda} + \underbrace{\mathbb{E}(X_{i+1})^2}_{\lambda^2} - 2\lambda^2 + \underbrace{Var(X_i)}_{\lambda} + \underbrace{\mathbb{E}(X_i)^2}_{\lambda^2}\right)$$

$$= k\sum_1^{n-1}2\lambda = k(n-1)2\lambda = \lambda \quad \text{om} \quad \boxed{k = \frac{1}{2(n-1)}}.$$

$\hat{\mu} = 1194.5 = \bar{x}$

$\hat{\sigma}^2 = 9{,}2649 = s^2$

$s = 3{,}0438.$

6)

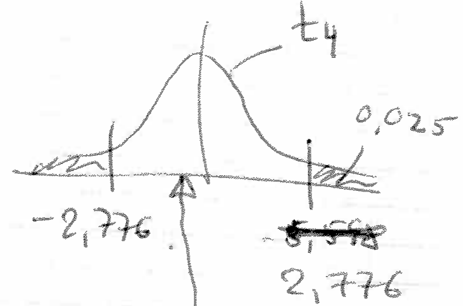
$H_0: \mu = 1195$

$H_1: \mu \neq 1195.$

$\alpha = 0.05.$

Statistika:

$$T_0 = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$$



$$t_0 = \frac{1194.5 - 1195}{3{,}0438/\sqrt{5}} = -0{,}367$$

∴ Vi kan inte förkasta att $\mu = 1195$.

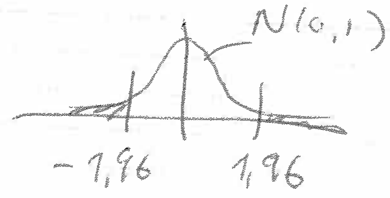
7)

$n = 20, \ \sigma^2 = 2, \ \alpha = 0{,}05.$

$H_0: \mu = 10$

$H_1: \mu \neq 10.$

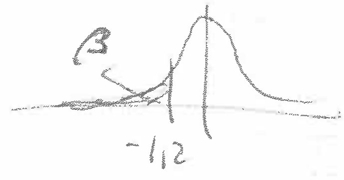
$$Z_0 = \frac{\bar{X} - \mu_0}{\sqrt{2/20}} = \frac{\bar{X} - 10}{\sqrt{2/20}}$$



$$\beta = P(\text{inte förk. } H_0 \mid \mu = 9) = P(-1{,}96 \leq Z_0 \leq 1{,}96 \mid \mu = 9) =$$

$$P\left(-1{,}96 + \frac{1}{\sqrt{2/20}} \leq \underbrace{\frac{\bar{x} - 9}{\sqrt{2/20}}}_{N(0,1)} \leq 1{,}96 + \underbrace{\frac{1}{\sqrt{2/20}}}_{= 3.16}\right) = P(-5{,}12 \leq Z \leq -1{,}2)$$

$$\approx P(Z \leq -1{,}2) = 0{,}115.$$



8) a) $P(\text{svarar Ja!}) =$

$= P(\text{Svarar Ja} \mid \text{Krona})\, P(\text{Krona}) + P(\text{Svarar ja} \mid \text{Klave})\, P(\text{Klave})$

$\cancel{P(}\quad 1 \cdot \frac{1}{2} + p\,\frac{1}{2} = \frac{1}{2}(p+1)$

b) $X_i = \begin{cases} 1, & \text{om svarar Ja!} \quad \frac{1}{2}(p+1) \\ 0 & \text{annars} \quad 1-\frac{1}{2}(p+1). \end{cases}$

$E(X_i) = \frac{1}{2}(p+1).$

Så $p = 2E(X_i) - 1 \Rightarrow$ Momentskattaren:

$\hat{p} = 2\bar{X} - 1.$

$E[\hat{p}] = E[2\bar{X} - 1] = 2E[\bar{X}] - 1 =$

$= 2\,\frac{1}{2}(p+1) - 1 = p.$ ∴ Väntevärdesriktig!

$Var(\hat{p}) = Var(2\bar{X} - 1) = 4\,Var(\bar{X}) = 4\,\frac{\sigma^2}{n} =$

<u>OBS</u>: $\sigma^2 = Var(X_i) = \frac{1}{2}(p+1)\cdot\left(1 - \frac{1}{2}(p+1)\right) =$

$= \frac{1}{2}\left[p+1 - \frac{1}{2}(p+1)^2\right] = \frac{1}{4}\left[1-p^2\right]$

Alltså: $Var(\hat{p}) = \frac{1-p^2}{n}.$

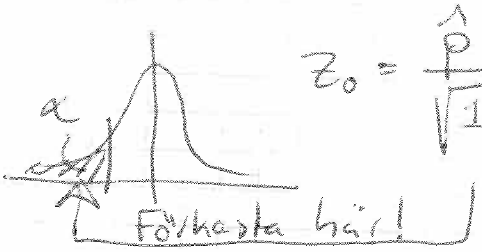
Så om n stort:

$\hat{p} \sim N\left(p, \frac{1-p^2}{n}\right).$

d) $H_0: p = p_0$

$H_1: p < p_0$

α = nånting.



$Z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{1-p_0^2}{n}}} \sim N(0,1).$