Lösningar TMS136 2010-10-22

(1) a) $X: S \to \mathbb{R}$ b) $f(x,y) = f(x)f(y)$ c) $X_1 \dots X_n$ oberoende & likafördelade med X.

~~b) Täthet~~ ~~Massfunktion~~

~~$f(x) \geq 0$~~ ~~$f(x) \geq 0$~~

~~$\int_x f(x)\,dx = 1$~~ ~~$\sum_x f(x) = 1$~~

~~$F(x) = \int_{-\infty}^{x} f(x)\,dx = P(X \leq x)$~~ ~~$f(x) = P(X = x)$~~

~~d) För en täthet är $f(x) \neq P(X=x) = 0$.~~

(2) (x,y) likformig på $\Rightarrow f(x,y) = \frac{1}{9}$

a) Beroende, ty tex. $Y = 1 \Rightarrow X = 0$ medan
$Y = 0 \Rightarrow X = -1; -0{,}5; 0; 0{,}5; 1$.

b) $$\mathrm{Cov}(X,Y) = E(XY) - \underbrace{E(X)}_{\substack{=0 \\ \text{p.g.a symmetri}}} E(Y) = E(XY)$$

$$E(XY) = \sum_y \sum_x xy\, f(x,y) = \frac{1}{9} \sum_y y \Big(\underbrace{\sum x}_{=0} \Big) = 0$$

$\boxed{\therefore \rho = 0}$

c) Att $\rho = 0$ beror på att X och Y inte är <u>linjärt</u> beroende.

(3) X_i = Bok nr. i:s tjocklek. $\mu = E(X_i) = 1{,}8$
$\sigma^2 = Var(X_i) = 0{,}7$

n = antalet böcker

$\sum_1^n X_i$ = antal hyllcentimeter.

Observera att $\sum_1^n X_i \sim N(\cdot,\cdot)$ enligt centrala gränsvärdessatsen.

Och $E(\sum X_i) = n\mu$ och $Var(\sum X_i) = n\sigma^2$.

P(P.Englund har fler än 2500 böcker) =

$$= P\left(\sum_1^{2500} X_i < 4600\right) = P\left(\frac{\sum_1^{2500} X_i - 2500\cdot 1{,}8}{\sqrt{2500\cdot 0{,}7}} < \frac{4600 - 2500\cdot 1{,}8}{\sqrt{2500\cdot 0{,}7}}\right) =$$

$$= P(Z < 2{,}39) \approx 0{,}99.$$

(4)

$H_0: \mu = 0$

$H_1: \mu > 0$

$\alpha = 0{,}05.$

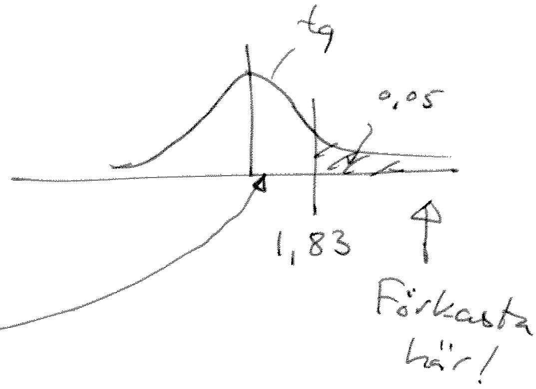
$D_i = X_i - Y_i$

$\bar{D} = \cancel{\bar{X}_1 - \bar{Y}_2} = \frac{1}{n}\sum (X_i - Y_i) =$

$= \frac{1}{n}\sum X_i - \frac{1}{n}\sum Y_i.$

$n = 10.$

$$T_0 = \frac{\bar{D} - 0}{318/\sqrt{n}} \sim t_{n-1} = t_9$$



$$t_0 = \frac{3}{318/\sqrt{10}} = 0{,}03$$

∴ Vi kan inte förkasta H_0.
Och P-värdet är jättestort, dvs data talar alls inte emot H_0.

(5) $N(t)$ = antal kvalitetsprogr. på TV3 på tiden t (timmar)

T = tiden mellan bra program.

$N(t) \sim Poi(0{,}01 \cdot t)$

$T \sim Exp(0{,}01)$.

a) $P(N(6)=0) = \dfrac{e^{-0{,}01\cdot 6}(0{,}01\cdot 6)^0}{0!} = e^{-0{,}06} = 0{,}94.$

b) Pga minneslöshetsegenskapen $= 0{,}06$.

c) $0{,}9 = P(T<t) = \int_0^t \lambda e^{-\lambda s}\,ds = -e^{-\lambda s}\Big|_0^t = 1-e^{-\lambda t}$

$\Rightarrow \quad e^{-\lambda t} = 0{,}1$ (logaritmera)

$\ln e^{-\lambda t} = \ln 0{,}1$

$-\lambda t = \ln 0{,}1$

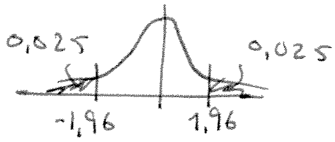
$t = -\dfrac{\ln 0{,}1}{\lambda} = 230\ h \approx 10$ dygn.

$\boxed{6}$ a) $H_0: \mu = 1000$

$H_1: \mu \neq 1000$

$\boxed{\alpha = 0{,}05}$ $\sigma = 0{,}68$, $n = 5$.

$$Z_0 = \frac{\bar{X} - 1000}{\sigma/\sqrt{n}} \sim N(0,1)$$



$\bar{X} = 1000{,}28$

$$Z_0 = \frac{1000{,}28 - 1000}{0{,}68/\sqrt{5}} = 0{,}92$$

$\therefore$ Vi kan ej dra slutsatsen att vågen har ett mätfel.

b) $\beta = P(\text{inte förk. } H_0 \mid H_0 \text{ falsk}) =$

$$= P\left(-1{,}96 \leq \frac{\bar{X} - 1000}{\sigma/\sqrt{n}} \leq 1{,}96 \;\middle|\; \mu = 1000{,}63\right) =$$

$$= P\Bigg(\underbrace{-1{,}96 - \frac{0{,}63}{\sigma/\sqrt{n}}}_{-4{,}03} \leq \underbrace{\frac{\bar{X} - 1000{,}63}{\sigma/\sqrt{n}}}_{\sim N(0,1)} \leq \underbrace{1{,}96 - \frac{0{,}63}{\sigma/\sqrt{n}}}_{=-0{,}11}\Bigg) \approx$$

$$\approx P(Z \leq -0{,}11) = 0{,}46$$

$\therefore$ Styrkan $= 1 - \beta = 0{,}54$.

$\boxed{7}$ $X \sim f(x) = e^{-(x-\theta)}$, $x \geq \theta$, $X_1, \ldots, X_n$ stickprov.

a) Observera att:

$$E(X) = \int_\theta^\infty x f(x)\,dx = \int_\theta^\infty x e^{-(x-\theta)}\,dx = \left\{\begin{matrix}\text{Part.}\\ \text{int}\end{matrix}\right. = \underbrace{-xe^{-(x-\theta)}\Big|_\theta^\infty}_{\theta} + \underbrace{\int_\theta^\infty f(x)\,dx}_{=1.}$$

$$= \theta + 1.$$

d.v.s. $\theta = E(X) - 1$

och alltså: $\hat{\theta}_m = \bar{x} - 1$.

b) $E(\hat{\theta}_m) = E(\bar{X} - 1) = E(\bar{X}) - 1 = E(X) - 1 = \theta$.

c) $Var(\hat{\theta}_m) = Var(\bar{X} - 1) = Var(\bar{X}) = \dfrac{\sigma^2}{n}$

där $\sigma^2 = Var(X) = \cancel{\int_\theta^\infty x^2 f(x)\,dx - \theta^2} = E(X^2) - E(X)^2$

$$= E(X^2) - (\theta + 1)^2$$

och $E(X^2) = \int_\theta^\infty x^2 e^{-(x-\theta)}\,dx = \left\{\begin{matrix}\text{Part}\\ \text{int}\end{matrix}\right. = -x^2 e^{-(x-\theta)}\Big|_\theta^\infty + 2\underbrace{\int_\theta^\infty x f(x)\,dx}_{=E(X)}$

$$= \theta^2 + 2(\theta + 1).$$

Alltså:

$$\sigma^2 = \theta^2 + 2(\theta+1) - (\theta+1)^2 = 1.$$

och

$$Var(\hat{\theta}_m) = \frac{1}{n}.$$

d) $L(\theta) = f(x_1)\dots f(x_n) = e^{-(x_1-\theta)} \cdots e^{-(x_n-\theta)} = e^{-\sum x_i + n\theta}$.

$\ln L(\theta) = -\sum x_i + n\theta$

Observera att $\ln L(\theta)$ maximeras av det största tillåtna θ, som är det minsta av $x_1, \dots, x_n$.

Alltså: $\hat{\theta} = \min\limits_{1 \le i \le n}(X_i)$.

e) $P(\hat{\theta} \le x) = P(\min(X_i) \le x) = 1 - P(\min(X_i) \geq x)$

$P(\min(X_i) \geq x) = P(X_1 \geq x, X_2 \geq x, \dots, X_n \geq x) = \{ X_i$ oberoende!

$= P(X_1 \geq x)^n = \left(\int_x^\infty f(x)\,dx \right)^n = \left(\int_x^\infty e^{-(x-\theta)}\,dx \right)^n = \left(-e^{-(x-\theta)} \Big|_x^\infty \right)^n$

$= \left(e^{-(x-\theta)}\right)^n = e^{-n(x-\theta)}$

$\therefore \quad F_{\hat{\theta}}(x) = P(\hat{\theta} \le x) = 1 - e^{-n(x-\theta)}$

och alltså ~~att~~ har vi:

$f_{\hat{\theta}}(x) = n e^{-n(x-\theta)}$.

f) $E(\hat{\theta}) = \int_\theta^\infty x\, f_{\hat{\theta}}(x)\,dx = \int_\theta^\infty x n e^{-n(x-\theta)}\,dx = \{$ Part. int $=$

$= -x e^{-n(x-\theta)} \Big|_\theta^\infty + \underbrace{\int_\theta^\infty e^{-n(x-\theta)}\,dx}_{= \frac{1}{n}\int n e^{-n(x-\theta)}\,dx = \frac{1}{n}\int f_{\hat{\theta}}(x) = \frac{1}{n}} =$

$= \theta + \frac{1}{n}$. Alltså ej v.v.r.

g) $\text{Var}(\hat{\theta}) = \mathbb{E}(\hat{\theta}^2) - \mathbb{E}(\hat{\theta})^2 = \mathbb{E}(\hat{\theta}^2) - \left(\theta + \frac{1}{n}\right)^2$

$$\mathbb{E}(\hat{\theta}^2) = \int_\theta^\infty x^2 n e^{-n(x-\theta)} dx = \{ PI \} =$$

$$= -x^2 e^{-n(x-\theta)} \Big|_\theta^\infty + \frac{2}{n} \underbrace{\int_\theta^\infty x n e^{-n(x-\theta)} dx}_{\mathbb{E}(\hat{\theta}) = \theta + \frac{1}{n}} =$$

$$= \theta^2 + \frac{2}{n}\left(\theta + \frac{1}{n}\right).$$

$$\text{Var}(\hat{\theta}) = \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2} - \left(\theta + \frac{1}{n}\right)^2 =$$

$$= \cancel{\theta^2} + \cancel{\frac{2\theta}{n}} + \frac{2}{n^2} - \cancel{\theta^2} - \cancel{\frac{2\theta}{n}} - \frac{1}{n^2} = \frac{1}{n^2}.$$

h) Momentskattaren $\hat{\theta}_m$ är väntevärdesriktig medans $\hat{\theta}$ har ett bias på $\frac{1}{n}$.

Men för hyfsat stora n har $\hat{\theta}$ mycket mindre varians.

i) Observera att för n stort (≥ 30) är båda skattarna approximativt normalfördelade.

Dvs. $\hat{\theta}_m \sim N(\theta, \frac{1}{n})$

$\hat{\theta} \sim N(\theta + \frac{1}{n}, \frac{1}{n^2})$

$$P(\theta - 0{,}1 \leq \hat{\theta}_m \leq \theta + 0{,}1) = P\Big(-0{,}1 \cdot \sqrt{n} \leq \underbrace{\frac{\hat{\theta}_m - \theta}{\sqrt{1/n}}}_{\sim N(0,1)} \leq 0{,}1 \cdot \sqrt{n}\Big) = P(-0{,}58 \leq Z \leq 0{,}58) =$$

$$= \Phi(0{,}58) - \Phi(-0{,}58) = \boxed{0{,}41.}$$

$$P(\theta - 0{,}1 \leq \hat{\theta} \leq \theta + 0{,}1) = P\Big(n(-0{,}1 - \frac{1}{n}) \leq \underbrace{\frac{\hat{\theta} - (\theta + \frac{1}{n})}{\sqrt{1/n^2}}}_{\sim N(0,1)} \leq n(0{,}1 - \frac{1}{n})\Big) =$$

$$= P(-4 \leq Z \leq 2) = \boxed{0{,}98}$$