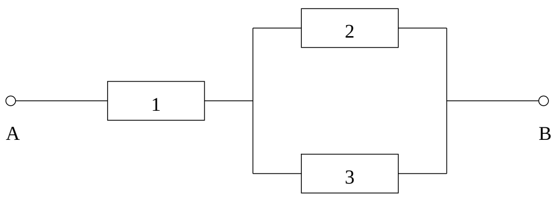
Chapter 5

Reliability and Survival

5.1 Systems of components

We will study *systems* that in a simple case might look like in Figure 5.1 below:

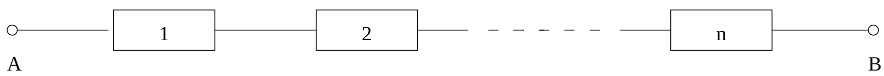
Figure 5.1. A simple system.



In the system in Figure 5.1, the *component* with component number i may be *healthy*, for $i = 1, 2, \ldots$, otherwise it is *unhealthy* or *dead*[6]. The system is healthy if there is a path from the point A to the point B which only passes healthy components. Otherwise the system is unhealthy.

Example 5.1. The *series coupling* in Figure 5.2 below is healthy if and only if all its components are healthy.

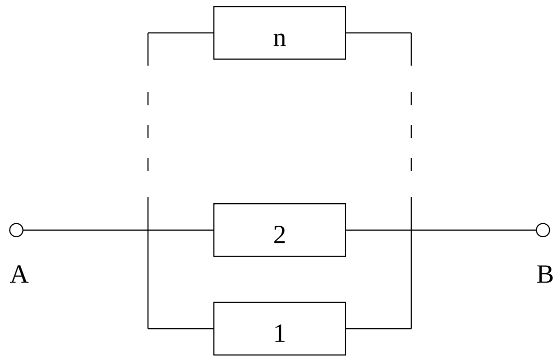
Figure 5.2. Series coupling.



[6] Rest in peace.

Example 5.2. The *parallell coupling* in Figure 5.3 below is healthy if and only if at least one of its components is healthy.

Figure 5.3. parallell coupling.



A system can be built by means of a finite number of series couplings and parallell couplings. See Figures 5.4 and 5.5 below for an example of how this works in a practical application.

We will study systems the components with component number $i = 1, 2, \ldots$ of which are healthy with certain *health probabilities* $p_1, p_2, \ldots$. Unless otherwise is stated, the components of a system are assumed to be independent of each other.

Example 5.3. The system in the Figure 5.4 below is a series coupling of two systems with health probabilities p_1 and $1 - (1 - p_2)(1 - p_3)$, respectively. The health probability for the whole system is $p_1[1 - (1-p_2)(1-p_3)]$.

Figure 5.4. A simple system with health probabilities.

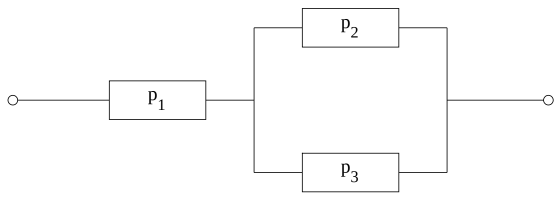
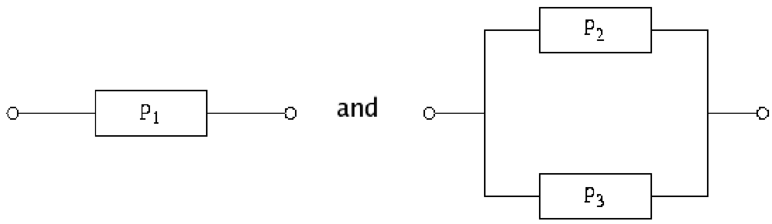


Figure 5.5. Composition of a simple system as a series coupling of a single component with a parallell coupling.



The components with component numbers $i = 1, 2, \ldots$ of a system have certain *life lengths* $T_1, T_2, \ldots$. The life lengths are modeled as *random variables* that are mutually independent, unless otherwise is stated.

The relation between the health probability and the life length T_i of the component with component number i is as follows:

$$\begin{aligned} p_i &= p_i(t) \\ &= \mathbf{P}\{\text{component with component number } i \text{ is healthy at time } t\} \\ &= \mathbf{P}\{T_i > t\} \\ &= 1 - F_{T_i}(t) = p. \end{aligned}$$

Here $F_{T_i}(t) = \mathbf{P}\{T_i \le t\}$, $i = 1, \ldots, n$, are the *distribution functions* of the life lengths $T_1, T_2, \ldots$. These distribution functions in turn are assumed to be continuous, unless otherwise is stated.

Definition 5.1. *The* survival function R_T *of a system with life length* T *is given by*

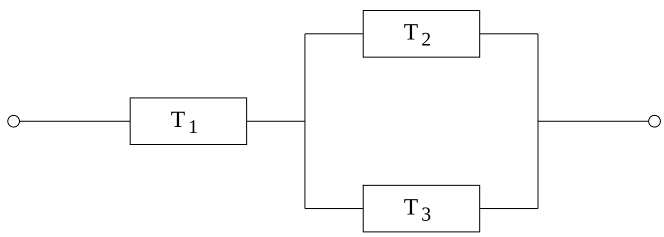
$$R_T(t) = \mathbf{P}\{\textit{the system is healthy at time } t\} = \mathbf{P}\{T > t\} = 1 - F_T(t)$$

for $t > 0$*, where* $F_T(t)$ *is the distribution function of the life length.*

Example 5.4. The life length of the system in the Figure 5.6 below is given by $T = \min\{T_1, \max[T_2, T_3]\}$, see also Example 5.3. Hence the survival function for the system is given by

$$\begin{aligned} R_T(t) &= \mathbf{P}\{T > t\} \\ &= \mathbf{P}\{T_1 > t\}\,[1 - (1 - \mathbf{P}\{T_2 > t\})\,(1 - \mathbf{P}\{T_3 > t\})] \\ &= R_{T_1}(t)\,[1 - (1 - R_{T_2}(t))(1 - R_{T_3}(t))]\,. \end{aligned}$$

Figure 5.6. A simple system with life lengths.



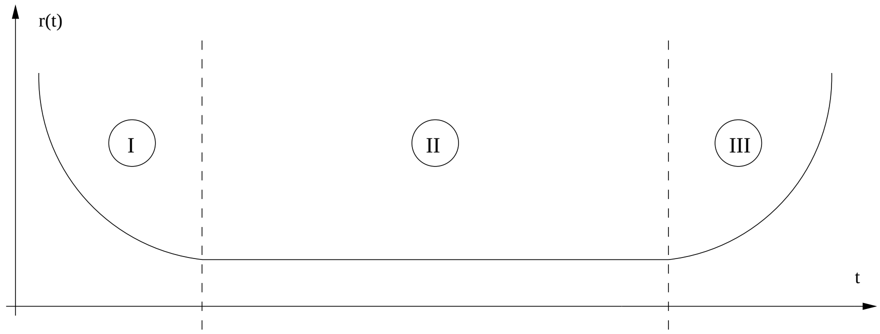
Definition 5.2. *The* death intensity r_T *of a system with life length* T *is given by*

$$r_T(t) = -\frac{d}{dt} \ln\left(R_T(t)\right) \quad \textit{for } t > 0.$$

Definition 5.3. *A system with life length* T *has* increasing failure rate*, IFR, if the death intensity is increasing* $r'(t) \ge 0$ *for* $t > 0$*. A system with life length* T *has* decreasing failure rate*, DFR, if the death intensity is decreasing* $r'(t) \le 0$ *for* $t > 0$*.*

Example 5.5. It is quite common that life lengths have failure rates that are neither IFR or DFR, but instead follow a so called *bath tub curve*, BTC, the principal appearance of which is displayed in Figure 5.7 below. In that figure the region I corresponds to an early phase with a comperatively high probability of unhealth (as for example, small children). The region II corresponds to a component that has survived these early hazards and has settled at a lesser death intensity (as for example, grown up people). Finally, the region III corresponds to a aged component, where the death intensity increases with accumlated age (as for example, aged people).

Figure 5.7. Example of a life length with a death intensity that displays a BTC shape.



5.2 More on systems

The following theorem explains that the death intensity really is the (infinitesimal) intensity at which unhealth occurs:

Theorem 5.1. *For a life length T with death intensity r_T, we have*

$$\mathbf{P}\{T \le t+h | T > t\} = r_T(t)h + o(h) \quad \textit{as} \;\; h \downarrow 0.$$

Proof. Writing F_T and f_T for the distribution function and probability density function of the life length T, respectively, we have

$$\begin{aligned}\mathbf{P}\{T \le t+h | T > t\} &= \frac{F_T(t+h) - F_T(t)}{1 - F_T(t)} = \frac{f_T(t)h + o(h)}{R_T(t)} \\ &= -\frac{d}{dt} \ln(R_T(t))h + o(h) = r_T(t)h + o(h). \quad \square\end{aligned}$$

Theorem 5.2. *A function $r : (0, \infty) \to [0, \infty)$ is a death intensity if and only if*

$$\int_0^\infty r(t)dt = \infty.$$

In that case the corresponding survival function is given by

$$R(t) = \exp\left\{-\int_0^t r(s)ds\right\}.$$

Proof. If r is a death intensity of a system with survival function R_T, then a differentiation of the function

$$R(t) = \exp\left\{-\int_0^t r(s)ds\right\}$$

gives

$$-\frac{d}{dt}\ln(R(t)) = -\frac{-r(t)\exp\left\{-\int_0^t r(s)ds\right\}}{\exp\left\{-\int_0^t r(s)ds\right\}} = r(t).$$

As the function $-\ln(R(t))$ has the same derivative as the function $-\ln(R_T(t))$, namely the death intensity $r(t)$, it follows that the functions $-\ln(R(t))$ and $-\ln(R_T(t))$ can differ only by a additive constant, so that the functions $R(t)$ and $R_T(t)$ differ only by a multiplicative constant. Since $R(0) = 1 = \mathbf{P}\{T > 0\} = R_T(0)$, we conclude that the functions $R(t)$ and $R_T(t)$ are equal. Finally, as

$$\lim_{t\to\infty} \exp\left\{-\int_0^t r(s)ds\right\} = \lim_{t\to\infty} R_T(t) = \lim_{t\to\infty} \mathbf{P}\{T > t\} = 0,$$

we must have $\int_0^\infty r(s)ds = \infty$.

Conversely, if $\int_0^\infty r(s)ds = \infty$ and we define the function

$$R(t) = \exp\left\{-\int_0^t r(s)\,ds\right\},$$

then $R(t)$ in decreasing with $R(0) = 1$ and $R(\infty) = 0$, so that $F(t) = 1 - R(t)$ is increasing with $F(0) = 0$ and $F(\infty) = 1$. This makes F a probability distribution function, so that R is a survival function. □

Theorem 5.3. *For a life length T we have the following formula for expectations*

$$\mathbf{E}\{T^n\} = \int_0^\infty R_T(t^{1/n})\,dt \quad \textit{for } n > 0.$$

Proof. By integration by parts and a change of variable in the integral, we get

$$\mathbf{E}\{T^n\} = \int_0^\infty t\left(-\frac{d}{dt}R_{T^n}(t)\right)dt = [t\,R_{T^n}(t)]_0^\infty + \int_0^\infty R_{T^n}(t)dt = \int_0^\infty R_T(t^{1/n})dt. \quad \square$$

A life length T with a constant death intensity $r_T(t) = \lambda$ has the *lack of memory* property (cf. Theorem 5.1). By Theorem 5.2, a life length T lacks memory if and only if T is exponentially $\exp(\lambda)$ distributed with parameter λ, i.e., $R_T(t) = e^{-\lambda t}$.

Example 5.6. The second simplest form of death intensity, after a constant one, is a polynomial death intensity $r_T(t) = ba^b t^{b-1}$. In this case, Theorem 5.2 gives that the survival function is $R_T(t) = \exp\{-(a\,t)^b\}$, that is, a Weibull distribution with parameters a and b, Weibull(a,b)[7]. And so we have the expectation

$$\mathbf{E}\{T^n\} = \int_0^\infty R_T(t^{1/n})dt = \int_0^\infty \exp\left\{-a^b t^{b/n}\right\} dt = \frac{\Gamma(n/b+1)}{a^n}$$

by Theorem 5.3 (where Γ denotes the gamma function), because

```
In[1]:= Integrate[Exp[-a^b*t^(b/n)], {t,0,Infinity},
        Assumptions -> a>0 && b>0 && n>=1]
Out[1]= a^(-n) Gamma[(b+n)/b]
```

If the life lengths $T_1, \ldots, T_n$ are exponentially $\exp(\lambda)$ distributed, then their sum $T \equiv T_1 + \ldots + T_n$ is gamma(n,λ) distributed with parameters n and λ. Hence the corresponding probability density function is given by

$$f_T(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}\, e^{-\lambda t} \quad \text{for } t > 0,$$

giving the survival function

$$R_T(t) = \sum_{k=0}^{n-1} \frac{\lambda^k t^k}{k!}\, e^{-\lambda t} \quad \text{for } t > 0.$$

To achieve a high health probability of a system, the system may be equipped with more components than are actually needed for its health, if all the components are healthy. In other words, the system is not a pure series coupling, but a series coupling of subsystems, some of which are parallell couplings, to achieve higher health probability.

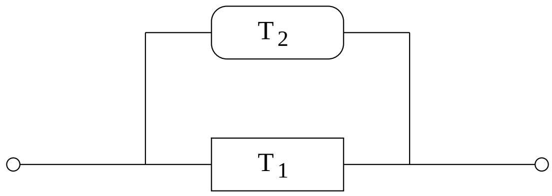
A component that is not required for the health of a system, when all other components of the system are healthy, is called a *redundant component*, RC.

A *warm redundant components*, WRC, is incorporated with the system already from the start of the system, while a *cold redundant component*, CRC, is incorporated with the system first at the time at which it is required for the health of the system.

[7]It should be noted that there is a lot of variation in the parametrization of the Weibull distribution, so that, e.g., what is denoted Weibull(a,b) by us could be denoted Weibull$(b,1/a)$ in a software package.

Example 5.7. Figure 5.8 below depicts a system where a first component with life length T_1 is supported by a second redundant component with life length T_2.

Figure 5.8. A system where a first component is supported by a second redundant component.



For the life length T of the system we have $T = \max\{T_1, T_2\}$ when the redundant component is warm, so that

$$R_T(t) = 1 - (1 - R_{T_1}(t))(1 - R_{T_2}(t)).$$

If the redundant component is cold, we get $T = T_1 + T_2$ instead, so that,

$$R_T(t) = 1 - \int_0^t (1 - R_{T_1}(t - x))R_{T_2}(x)r_{T_2}(x)\,dx.$$

A quantity of great interest for a system, is the probability that component with component number $i = 1, 2, \ldots$ causes the death (unhealth) of the system. That probability, in turn, coincides with the probability that the life length of component with component number $i = 1, 2, \ldots$ is equal to the life length of the whole system!

Primarily, component that have high probabilities to cause the death (unhealth) of the system, are those who should be supported by (warm or cold) redundant components.

Example 5.8. For the system in Figure 5.6, we have

$$\begin{aligned}
&\mathbf{P}\{\text{component with number 1 causes death}\}\\
&\quad = \mathbf{P}\left\{T_1 \le \max[T_2, T_3]\right\}\\
&\quad = \int_0^\infty \mathbf{P}\{\max(T_2, T_3) \ge t\} f_{T_1}(t)dt\\
&\quad = \int_0^\infty (1 - F_{T_2}(t)F_{T_3}(t)) f_{T_1}(t)dt
\end{aligned}$$

$$= \int_0^\infty (1 - (1 - R_{T_2}(t))(1 - R_{T_3}(t)))\, R_{T_1}(t)r_{T_1}(t)dt.$$

(This probability must be 2/3 when the life lengths T_1, T_2 and T_3 are identically distributed.)

5.3 Laboration

5.3.1 Software

The laboration is to be done in Mathematica. Useful functions:

Integrate, Derivative, D, FindRoot, FindMaximum, ListPlot

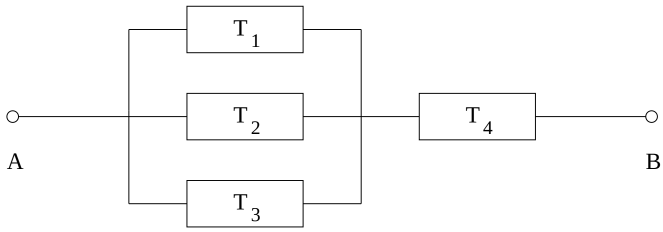
You will not be able to get far with Mathematica without functions. A function is defined by **funcname[par_] = expression**. For example, you can define a function "f" through **f[x_, y_]:= x+y**.

When using maximization/minimization procedures, make sure to apply suitable constraints.

5.3.2 Tasks

1. In the system in Figure 5.9 below, the first three components have life lengths T_1, T_2, T_3 that are Weibull$(1, \frac{1}{2})$ distributed, while the fourth component have a life length T_4 that is $\exp(\frac{1}{2})$ distributed.

   Figure 5.9. A system with four components.

   - Find the expected lifelength $\mathbf{E}\{T\}$ of the system.
   - Plot the death rate $r_T(t)$, $t \in (0, 10)$, for the system: Is the system IFR, DFR or neither?
   - Find the probability that it is component with component number 4 that causes the death of the system.

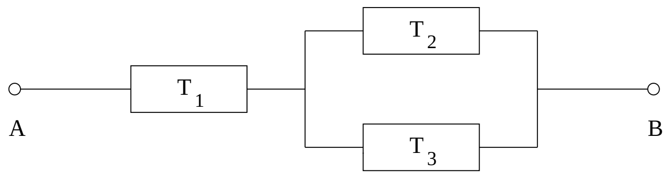
2. (2p)

   - Redo the first problem in previous task, first with the component with component number 4 doubled with a warm redundant $\exp(\frac{1}{2})$ distributed component, and then with the component with component number 4 doubled with a cold redundant $\exp(\frac{1}{2})$ distributed component. Plot the death rates $r_T(t)$ (including the one from task 1).
   - For which values of the parameter $\rho < \frac{1}{2}$ does a change of the component with component number 4 to a $\exp(\rho)$ distributed component, have the same effect on the expected life length $\mathbf{E}\{T\}$ of the system, as have the incorporation of a warm and cold redundant $\exp(\frac{1}{2})$ distributed component?

3. (2p)

In the system in Figure 5.10 below, the first component has a life length T_1 that is Weibull($\mu, \frac{1}{3}$) distributed, while the second and third components have life length T_2 and T_3 that are Weibull($\lambda, \frac{1}{3}$) distributed.

Figure 5.10. Yet another system with three components!



The cost of a Weibull($\gamma, \frac{1}{3}$) distributed component is $1/5 + 1/\gamma$ monetary units. Display graphically the values of the parameters λ and μ, that maximizes the expected life length $\mathbf{E}\{T\}$ of the system, at the total costs $1, 2, \ldots, 10$ monetary units, respectively, of the system. Also plot the expected life length $\mathbf{E}\{T\}$ as a function of the costs of $1, 2, \ldots, 10$ monetary units, for the optimal values of the parameters λ and μ.

To get a pass task 1 has to be completed.