

## TMS 165/MSA350 Stochastic Calculus

### Solved Exercises for Chapters 1-2 in Klebaner's book

**Exercise 1.** Explain why the definition of variation of a function according to Equation 1.7 in Klebaner's book is more generally valid than that according to Equation 1.9.

**Solution.** Equation 1.7 defines the variation as the supremum of a set of real numbers [namely the set of all values the sum in (1.7) can take for different partitions], and a supremum of a set of real numbers is always a well-defined quantity. Now we may consider the function  $g : [0, 1] \rightarrow \mathbb{R}$  given by  $g(\frac{1}{2}) = 1$  and  $g(x) = 0$  for  $x \in [0, 1] \setminus \{\frac{1}{2}\}$ , which has a well-defined variation  $V_g([0, 1]) = 2$  according to (1.7), but which does not have a well-defined variation  $V_g([0, 1])$  according to (1.9).

**Exercise 2.** Prove Equations 1.16 and 1.17 in Klebaner's book for quadratic variation.

**Solution.** Clearly, (1.17) follows readily from using the definition (1.15) of quadratic variation together with a few simple algebraic manipulations. Further, we may derive (1.16) from (1.17) together with symmetry as

$$\begin{aligned} \frac{1}{2}(\lfloor f+g, f+g \rfloor - \lfloor f, f \rfloor - \lfloor g, g \rfloor) &= \frac{1}{2}(\lfloor f, f+g \rfloor + \lfloor g, f+g \rfloor - \lfloor f, f \rfloor - \lfloor g, g \rfloor) \\ &= \frac{1}{2}(\lfloor f+g, f \rfloor + \lfloor f+g, g \rfloor - \lfloor f, f \rfloor - \lfloor g, g \rfloor) \\ &= \frac{1}{2}(\lfloor f, f \rfloor + \lfloor g, f \rfloor + \lfloor f, g \rfloor + \lfloor g, g \rfloor - \lfloor f, f \rfloor - \lfloor g, g \rfloor) \\ &= \frac{1}{2}(\lfloor g, f \rfloor + \lfloor f, g \rfloor) \\ &= \lfloor f, g \rfloor. \end{aligned}$$

**Exercise 3.** Calculate the Stieltjes integral of a continuous function  $f : [0, \infty) \rightarrow \mathbb{R}$  with respect to the function  $[0, \infty) \ni x \rightsquigarrow g(x) = \lfloor x \rfloor \in [0, \infty)$  (i.e., the integer part of  $x$ ) over an interval  $(a, b] \subseteq [0, \infty)$ .

**Solution.** Selecting integers  $k, \ell \in \mathbb{N}$  such that  $\lfloor a \rfloor = k$  and  $\lfloor b \rfloor = \ell$ , we either have  $k = \ell$ , in which case  $k \leq a < b < k+1$  and  $\int_{(a,b]} f dg = 0$ , or else  $k < \ell$ , in which case  $k \leq a < k+1 \leq \ell \leq b < \ell+1$  and  $\int_{(a,b]} f dg = \sum_{n=\lfloor a \rfloor+1}^{\lfloor b \rfloor} f(n)$ .

**Exercise 4.** Construct a standard normal distributed random variable defined on a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ .

**Solution.** With  $\Omega = \mathbb{R}$ ,  $\mathcal{F}$  the Borel  $\sigma$ -field generated by the intervals in  $\mathbb{R}$  and  $\mathbf{P}\{A\} = \int_A \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$  for  $A \in \mathcal{F}$ , the random variable  $X(\omega) = \omega$  for  $\omega \in \Omega$  is standard normal distributed.

**Exercise 5.** Calculate the expectation  $\mathbf{E}\{X^+\}$  using the definition of the Lebesgue integral for the random variable in Exercise 4.

**Solution.** Using Example 2.9 together with page 33 in Klebaner's book we see that

$$\begin{aligned}
 \mathbf{E}\{X^+\} &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbf{P}\left\{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right\} \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \int_{k/2^n}^{(k+1)/2^n} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \frac{1}{\sqrt{2\pi}} e^{-(k/2^n)^2/2} \frac{1}{2^n} \\
 &= \int_0^\infty x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
 &= 1/\sqrt{2\pi}.
 \end{aligned}$$