TMS165/MSA350 Stochastic Calculus Part I

Written Exam Tuesday 18 October 2011 8.30 am–12.30 am

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Aids: None.

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Througout this exam $B = \{B(t)\}_{t\geq 0}$ is a Brownian motion. And Good Luck to you all!

Task 1. Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ [where Ω is the sample space of all possible outcomes of a random experiment, $\mathcal{F}$ is the σ-field of those subsets of Ω that are events and $\mathbf{P}$ is a probability measure defined on $(\Omega, \mathcal{F})$]. Prove that $\mathbf{E}\{\mathbf{E}\{X\,|\,\mathcal{G}\}\} = \mathbf{E}\{X\}$ for any σ-field $\mathcal{G}$ that is contained in $\mathcal{F}$ and that $\mathbf{E}\{X\,|\,\mathcal{G}\} = \mathbf{E}\{X\}$ in the particular case when $\mathcal{G} = \{\emptyset, \Omega\}$. **(5 points)**

Task 2. Let $B_1 = \{B_1(t)\}_{t\geq 0}$ and $B_2 = \{B_2(t)\}_{t\geq 0}$ be independent Brownian motions. Show that $[B_1, B_2](t) = 0$ for $t \geq 0$. [Hint: Use the polarization identity together with the easily verified fact that $(B_1+B_2)/\sqrt{2}$ is also a Brownian motion.] **(5 points)**

Task 3. Let $g : \mathbb{R} \to \mathbb{R}$ be a two times continuously differentiable function. Show that $[g(B), B](t) = \int_0^t g'(B(s))\,ds$ for $t \geq 0$. **(5 points)**

Task 4. Find a diffusion type SDE that has solution $X(t) = B(t)^3$. **(5 points)**

Task 5. Suppose that we have observed the solution $\{X(t)\}_{t\in[0,10]}$ to the SDE

$$dX(t) = \alpha\,dt + \sigma\,X(t)\,dB(t) \quad \text{for } t \in [0, 10], \quad X(0) = 1,$$

where $\alpha, \sigma > 0$ are unknown constants. How can α be estimated? **(5 points)**

Task 6. Let $\{X(s)\}_{s\in[t,T]}$ solve the SDE

$$dX(s) = \mu(X(s), s)\,ds + \sigma(X(s), s)\,dB(s) \quad \text{for } s \in [t, T], \quad X(t) = x.$$

Show that under suitable technical constraints on the coefficients $\mu(x, t)$ and $\sigma(x, t)$ of the SDE, a solution $f(x, t)$ to the PDE

$$\mu(x,t)\,\frac{\partial f(x,t)}{\partial x} + \frac{\sigma(x,t)^2}{2}\,\frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial f(x,t)}{\partial t} = k(x,t) \quad \text{for } t \in [0, T], \quad f(x, T) = g(x),$$

must take the form $f(x,t) = \mathbf{E}\big\{g(X(T)) - \int_t^T k(X(s), s)\,ds \,\big|\, X(t) = x\big\}$. **(5 points)**

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Solutions to Written Exam Tuesday 18 October 2011

Task 1. By definition we have $\mathbf{E}\{I_A\,\mathbf{E}\{X\,|\,\mathcal{G}\}\} = \mathbf{E}\{I_A\,X\}$ for any event $A\in\mathcal{G}$. Taking $A=\Omega$, so that $I_A(\omega)=1$ for all $\omega\in\Omega$, we get $\mathbf{E}\{\mathbf{E}\{X\,|\,\mathcal{G}\}\} = \mathbf{E}\{X\}$.

By definition $\mathbf{E}\{X\,|\,\{\emptyset,\Omega\}\}$ is the unique $\{\emptyset,\Omega\}$-measurable random variable that satisfies $\mathbf{E}\{I_A\,\mathbf{E}\{X\,|\,\{\emptyset,\Omega\}\}\} = \mathbf{E}\{I_A\,X\}$ for $A\in\{\emptyset,\Omega\}$. It follows that $\mathbf{E}\{X\,|\,\{\emptyset,\Omega\}\} = \mathbf{E}\{X\}$ as $\mathbf{E}\{X\}$ is $\{\emptyset,\Omega\}$-measurable (as is any non-random constant) and $\mathbf{E}\{I_A\,\mathbf{E}\{X\}\} = \mathbf{E}\{I_A\,X\}$ holds trivially for $A=\emptyset$ and $A=\Omega$ with values 0 and $\mathbf{E}\{X\}$, respectively.

Task 2. By polarization together with the fact that BM has quadratic variation t we have $[B_1,B_2](t) = \frac{1}{2}\big([B_1+B_2,B_1+B_2](t) - [B_1,B_1](t) - [B_2,B_2](t)\big) = \frac{1}{2}\big(2\,[(B_1+B_2)/\sqrt{2},(B_1+B_2)/\sqrt{2}](t) - [B_1,B_1](t) - [B_2,B_2](t)\big) = \frac{1}{2}\,(2t-t-t) = 0$.

Task 3. By the definition of quadratic covariation together with Itô's formula we have $d[g(B)(t),B(t)] = dg(B(t))\,dB(t) = \big(g'(B(t))\,dB(t) + \frac{1}{2}\,g''(B(t))\,dt\big)\,dB(t) = g'(B(t)) \times (dB(t))^2 + \frac{1}{2}\,g''(B(t))\,dtdB(t) = g'(B(t))\,dt + 0$, so that $[g(B),B](t) = [g(B)(t),B(t)] = [g(B),B](0) + \int_0^t g'(B(s))\,ds = \int_0^t g'(B(s))\,ds$.

Task 4. By Itô's formula $X(t) = B(t)^3$ satisfies $dX(t) = d(B(t)^3) = 3\,B(t)^2\,dB(t) + 3\,B(t)\,dt = 3\,X(t)^{2/3}\,dB(t) + 3\,X(t)^{1/3}\,dt$. Hence the diffusion type SDE

$$dX(t) = \mu(X(t),t)\,dt + \sigma(X(t),t)\,dB(t) \quad \text{for } t>0, \quad X(0)=x_0,$$

has solution $X(t) = B(t)^3$ for $\mu(x,t) = 3\,x^{1/3}$, $\sigma(x,t) = 3\,x^{2/3}$ and $x_0 = 0$.

Task 5. In the language of Section 10.6 in Klebaner's book, with $dX(t) = \alpha\,dt + \sigma\,X(t)\,dB(t)$ for a $\mathbf{P}$-BM B and $dX(t) = \sigma\,X(t)\,dW(t)$ for a $\mathbf{Q}$-BM W, the likelihood $d\mathbf{P}/d\mathbf{Q} = \exp\big\{\int_0^{10}\big(\alpha/(\sigma^2\,X(t)^2)\big)\,dX(t) - \frac{1}{2}\int_0^{10}\big(\alpha^2/(\sigma^2\,X(t)^2)\big)\,dt\big\}$ has derivative with respect to α given by $\sigma^{-2}\big(\int_0^{10} X(t)^{-2}\,dX(t) - \alpha\int_0^{10} X(t)^{-2}\,dt\big)\,(d\mathbf{P}/d\mathbf{Q})$, which is zero for the maximum likelihood α-estimator $\alpha = \big(\int_0^{10} X(t)^{-2}\,dX(t)\big)/\big(\int_0^{10} X(t)^{-2}\,dt\big)$.

Task 6. If f solves the PDE, then Itô's formula shows that $\mathbf{E}\big\{g(X(T))\,\big|\,X(t)=x\big\} = \mathbf{E}\big\{f(X(T),T)\,\big|\,X(t)=x\big\} = \mathbf{E}\big\{f(X(t),t) + \int_t^T\big(\frac{\partial f}{\partial x}\,(\mu\,ds + \sigma\,dB(s)) + \frac{1}{2}\,\frac{\partial^2 f}{\partial x^2}\,\sigma^2\,ds + \frac{\partial f}{\partial t}\,ds\,\big|\,X(t)=x\big\} = f(x,t) + \mathbf{E}\big\{\int_t^T k\,ds\,\big|\,X(t)=x\big\} + \mathbf{E}\big\{\int_t^T \frac{\partial f}{\partial x}\,\sigma\,dB\,\big|\,X(t)=x\big\} = f(x,t) + \mathbf{E}\big\{\int_t^T k(X(s),s)\,ds\,\big|\,X(t)=x\big\}$ as $\{\int_t^s \frac{\partial f}{\partial x}\,\sigma\,dB\}_{s\in[t,T]}$ is a martingale.