TMS165/MSA350 Stochastic Calculus

Written Exam Monday 4 January 2016 2–6 pm

TEACHER AND JOUR: Patrik Albin, telephone 070 6945709.

AIDS: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Given constants $c, T > 0$, show that $-B(t)$, $B(t+T) - B(T)$, $c\,B(t/c^2)$ and $t\,B(1/t)$ are also Brownian motions. **(5 points)**

Task 2. Let $X(t)$ satisfy the SDE

$$dX(t) = X(t)^2\,dt + X(t)\,dB(t), \quad X(0) = 1.$$

Show that $X(t)$ also satisfies the equation

$$X(t) = \exp\left\{\int_0^t (X(s) - 1/2)\,ds + B(t)\right\}.$$

(5 points)

Task 3. As you know, an Ornstein-Uhlenbeck process is the solution $X(t)$ to the SDE

$$dX(t) = -\mu\,X(t)\,dt + \sigma\,dB(t) \quad \text{for } t > 0,$$

where $\mu, \sigma > 0$ are parameters. Derive an expression for the transition density function $p(t,x,y) = \frac{d}{dy}\mathbf{P}\{X(t+s) \leq y\,|\,X(s) = x\}$ for $0 < s < t$, for this process, for example, making use of the fact that $X(t+s) = \mathrm{e}^{-\mu t}X(s) + \sigma\int_s^{t+s}\mathrm{e}^{\mu(r-t-s)}\,dB(r)$. **(5 points)**

Task 4. Let a time homogeneous diffusion process $X(t)$ be started according to its stationary probability density function π (assuming that it exists), so that $f_{X(0)}(x) = \pi(x)$. Explain why $X(t)$ is a stationary process, that is, why $(X(t_1+h), \ldots, X(t_n+h))$ has the same probability density function as $(X(t_1), \ldots, X(t_n))$ for all choices of $0 < t_1 < \ldots < t_n$, $h \geq 0$ and $n \in \mathbb{N}$. **(5 points)**

Task 5. Let X be a unit mean exponentially distributed random variable under the probability measure $\mathbf{P}$. Which other continuous probability density functions can X be given by means of a change of probability measure? **(5 points)**

Task 6. Describe how Itô-Taylor expansion works for a time homogeneous SDE.

(5 points)

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Solutions to Written Exam Monday 4 January

Task 1. As the four processes listed are all zero-mean Gaussian it is enough to check that they all have the same covariance function $\min\{s,t\}$ as has Brownian motion. That this is so in turn follows from elementary calculations.

Task 2. This is Exercise 5.6 in Klebaner's book: See his solution.

Task 3. Make use of the simple fact that $(X(t+s)\,|\,X(s)=x)$ is $\mathrm{N}\big(\mathrm{e}^{-\mu t}x, \sigma^2(1-\mathrm{e}^{-2\mu t})/(2\mu)\big)$-distributed to conclude that

$$p(t,x,y) = \frac{\sqrt{\mu}}{\sqrt{\pi}\,\sigma\sqrt{1-\mathrm{e}^{-2\mu t}}}\exp\left\{-\frac{(y-\mathrm{e}^{-\mu t}x)^2}{\sigma^2\,(1-\mathrm{e}^{-2\mu t})/\mu}\right\}.$$

Task 4. According to the lecture on applications the joint probability density function of $(X(t_0), X(t_1), \ldots, X(t_n))$ for $0=t_0<t_1<\ldots<t_n$ is given by

$$f_{X(t_0),X(t_1),\ldots,X(t_n)}(x_0,x_1,\ldots,x_n) = \pi(x_0)\prod_{i=1}^{n} p(t_i-t_{i-1},x_{i-1},x_i)$$

when the process is started according to π. By integrating with respect to x_0 and making use of the defining fact that $\int p(t,x_0,x_1)\,\pi(x_0)\,dx_0 = \pi(x_1)$ it follows that the joint probability density function of $(X(t_1),\ldots,X(t_n))$ for $0<t_1<\ldots<t_n$ is given by

$$f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n) = \pi(x_1)\prod_{i=2}^{n} p(t_i-t_{i-1},x_{i-1},x_i).$$

But this expression remains unaltered if $t_1,\ldots,t_n$ is replaced with $t_1+h,\ldots,t_n+h$. Or, in fact, it is enough to note that $f_{X(t_0+h),X(t_1+h),\ldots,X(t_n+h)}(x_0,x_1,\ldots,x_n)$ also do not depend on $h\geq 0$.

Task 5. For any probability density function $f(x)$, $x\geq 0$, on the positive real axis we can introduce a probability measure $\mathbf{Q}(A) = \int_A \Lambda(X)\,d\mathbf{P}$ where $\Lambda(X) = f(X)\,\mathrm{e}^X$, because $\Lambda(X)$ is non-negative with $\mathbf{Q}(\Omega) = \int_\Omega \Lambda(X)\,d\mathbf{P} = \mathbf{E_P}\{\Lambda(X)\} = \int_0^\infty f(x)\,\mathrm{e}^x\,\mathrm{e}^{-x}\,dx = \int_0^\infty f(x)\,dx = 1$. And then X will have density f under $\mathbf{Q}$ as $\mathbf{E_Q}\{\mathrm{e}^{i\omega X}\} = \mathbf{E_P}\{\Lambda(X)\,\mathrm{e}^{i\omega X}\} = \int_0^\infty f(x)\,\mathrm{e}^x\,\mathrm{e}^{i\omega x}\,\mathrm{e}^{-x}\,dx = \int_0^\infty \mathrm{e}^{i\omega x}\,f(x)\,dx$.

Task 6. See page 12 in Stig's lecture notes.