Stochastic Calculus Part II Fall 2008

Hand In 2

1. A stochastic process $\{X(t)\}_{t\geq 0}$ is called *self-similar* with index $\kappa > 0$ (*Hurst parameter*) if:
For $n \in \mathbb{N}$, $\lambda > 0$, and $t_1, \dots, t_n \geq 0$ it holds that
$\big(X(\lambda t_1), \dots, X(\lambda t_n)\big) \stackrel{d}{=} \big(\lambda^\kappa X(t_1), \dots, \lambda^\kappa X(t_n)\big)$

It is an easy exercise to show that Brownian motion $\{B(t)\}_{t\geq 0}$ is a self similar process.
i) What is the Hurst parameter in the case of Brownian motion?
ii) Let $\tau_x = \inf\{t \geq 0 : B(t) = x\}$. Show that $\mathbf{P}(\tau_x > 1) = \mathbf{P}(|x|\sqrt{\tau_1} > 1)$.
(Note that x may be negative).

2. Let $\{X_t\}$ be a martingale and $g(x)$ a convex function, where $\mathbf{E}[|g(X_t)|] < \infty$. Show that $\{g(X_t)\}$ is a submartingale.

3. Let $X_t = e^{-\theta^2 t/2}\cosh(\theta B_t)$, where $\theta \in \mathbb{R}$.
i) Show that $\{X_t\}_{t\geq 0}$ is a martingale.
(Hint: what can you say about $-B_t$?)
ii) Find $E[X_t]$ and $Var(X_t)$.
iii) Is $\{X_t\}_{t\geq 0}$ uniformly integrable? Is it square integrable?