17.7 Reliability

Coherent systems

System functions and reliability

System function $\Phi(\mathbf{x}) = \begin{cases} 1 \text{ if system is functioning for vector } \mathbf{x} \\ 0 \text{ if system is failed for vector } \mathbf{x} \end{cases}$

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where

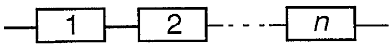
$$x_i = \begin{cases} 1 \text{ if component no } i \text{ is functioning} \\ 0 \text{ if system component no } i \text{ is failed} \end{cases}$$

For *coherent systems* $\mathbf{x} < \mathbf{y} \Rightarrow \Phi(\mathbf{x}) \leq \Phi(\mathbf{y})$.

$p_i = P(x_i = 1) = E[x_i] =$ reliability of component no i

$E[\Phi(\mathbf{x})] = h(p_1, p_2, \ldots, p_n) =$ reliability of system

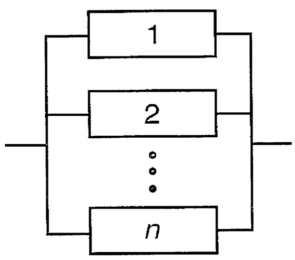
Series system



$$\Phi(\mathbf{x}) = \prod_{i=1}^{n} x_i = \min(x_1, x_2, \ldots, x_n)$$

Reliability (independent components) $= \prod_{i=1}^{n} p_i$

Parallel system



$$\Phi(\mathbf{x}) = 1 - \prod_{i=1}^{n} (1 - x_i) = \coprod_{i=1}^{n} x_i = \max(x_1, x_2, \ldots, x_n)$$

Reliability (independent components) $= \coprod_{i=1}^{n} p_i = 1 - \prod_{i=1}^{n} (1 - p_i)$

k-out-of-n system

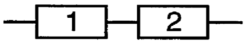
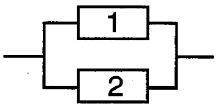
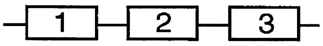
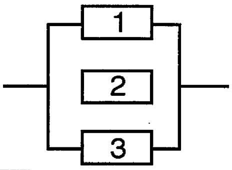
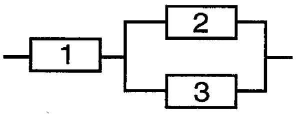
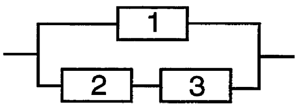
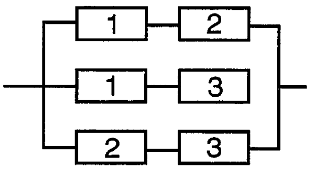
The system is functioning if at least k components are functioning.

$$\Phi(\mathbf{x}) = \begin{cases} 1 \text{ if } \sum_{i=1}^{n} x_i \geq k \\ 0 \text{ if } \sum_{i=1}^{n} x_i < k \end{cases}$$

Reliability (independent components with the same reliability p) =

$$= \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

Specific coherent systems

System	System function	Reliability (independent components)
	x_1x_2	p_1p_2
	$x_1 \amalg x_2=$ $=1-(1-x_1)(1-x_2)$	$p_1 \amalg p_2=$ $=1-(1-p_1)(1-p_2)$
	$x_1x_2x_3$	$p_1p_2p_3$
	$x_1 \amalg x_2 \amalg x_3=$ $=1-(1-x_1)(1-x_2)(1-x_3)$	$p_1 \amalg p_2 \amalg p_3=$ $=1-(1-p_1)(1-p_2)(1-p_3)$
	$x_1(x_2 \amalg x_3)=$ $=x_1(x_2+x_3-x_2x_3)$	$p_1(p_2+p_3-p_2p_3)$
	$x_1 \amalg (x_2x_3)=$ $=x_1+x_2x_3-x_1x_2x_3$	$p_1+p_2p_3-p_1p_2p_3$
	$(x_1x_2) \amalg (x_1x_3) \amalg (x_2x_3)=$ $=1-(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)=$ $=x_1x_2+x_1x_3+x_2x_3-2x_1x_2x_3$	$p_1p_2+p_1p_3+p_2p_3-2p_1p_2p_3$

Life distributions

Basic definitions

X = length of life or time to failure for component or system of components

Distribution function	$F(x)=P(X\le x)$
Survival probability (reliability)	$G(x)=P(X>x)=1-F(x)$
Probability density	$f(x)=F'(x)=-G'(x)$
Failure (hazard) rate	$r(x)=f(x)/G(x)=F'(x)/(1-F(x))$
TTT-transform (*TTT* = Total Time on Test)	$F_{TTT}(x)=\int_0^{H(x)} G(t)dt \Big/ \int_0^{H(1)} G(t)dt,\ H=F^{-1}$

Expected lifetime $\mu = \int_0^\infty x f(x)dx = \int_0^\infty G(x)dx$

Variance $\sigma^2 = \int_0^\infty (x-\mu)^2 f(x)dx$

$$P(X \leq t+h \mid X > t) = r(t)h + o(h)$$

$$G(x) = 1 - F(x) = e^{-\int_0^x r(t)dt}$$

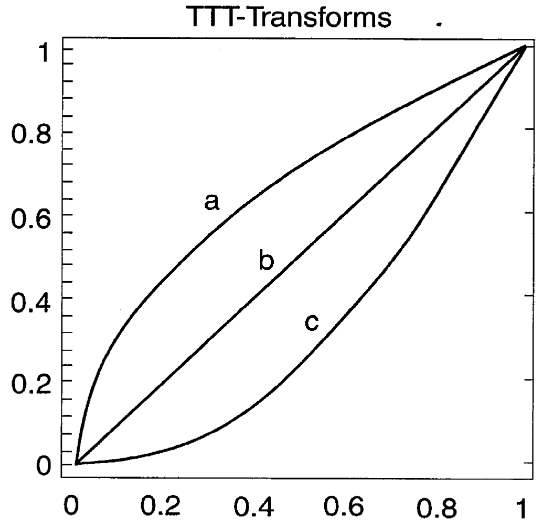
Properties of life distributions

Property	Notation	Definition
Increasing failure rate	IFR	$r(t)$ is increasing in t
Decreasing failure rate	DFR	$r(t)$ is decreasing in t
Increasing failure rate average	IFRA	$\frac{1}{t}\int_0^t r(x)dx$ is increasing in t
Decreasing failure rate average	DFRA	$\frac{1}{t}\int_0^t r(x)dx$ is decreasing in t
New better than used	NBU	$G(x+y) \leq G(x)G(y)$
New worse than used	NWU	$G(x+y) \geq G(x)G(y)$
New better than used in expectation	NBUE	$\int_t^\infty G(x)dx \leq \mu G(t)$
New worse than used in expectation	NWUE	$\int_t^\infty G(x)dx \geq \mu G(t)$
Harmonic new better than used in expectation	HNBUE	$\int_t^\infty G(x)dx \leq \mu e^{-t/\mu}$
Harmonic new worse than used in expectation	HNWUE	$\int_t^\infty G(x)dx \geq \mu e^{-t/\mu}$

IFR $\Rightarrow$ IFRA $\Rightarrow$ NBU $\Rightarrow$ NBUE $\Rightarrow$ HNBUE

DFR $\Rightarrow$ DFRA $\Rightarrow$ NWU $\Rightarrow$ NWUE $\Rightarrow$ HNWUE

TTT-transform and IFR (DFR)



a) TTT-transform for IFR-distribution

b) TTT-transform for exponential distribution

c) TTT-transform for DFR-distribution

F_{TTT} is concave $\Leftrightarrow$ IFR $\qquad$ F_{TTT} is convex $\Leftrightarrow$ DFR

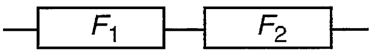
Specific life distributions

Name	f, G, r, μ, σ^2	Properties
Exponential	$f(x)=\lambda e^{-\lambda x},\ x\geq 0$ $G(x)=e^{-\lambda x},\ x\geq 0$ $F_{TTT}(x)=x,\ 0\leq x\leq 1$ $r(x)=\lambda$ $\mu=1/\lambda$ $\sigma^2=1/\lambda^2$	Constant failure rate
Weibull	$f(x)=\beta\lambda^\beta x^{\beta-1}e^{-(\lambda x)^\beta}$ $G(x)=e^{-(\lambda x)^\beta}$ $r(x)=\beta\lambda^\beta x^{\beta-1}$ $\mu=\lambda^{-1}\Gamma(1+1/\beta)$ $\sigma^2=\lambda^{-2}(\Gamma(1+2/\beta)-\Gamma^2(1+1/\beta))$	IFR for $\beta\geq 1$ DFR for $\beta\leq 1$
Lognormal	$f(x)=\dfrac{1}{\beta x\sqrt{2\pi}}e^{-(\ln x-\alpha)^2/2\beta^2}$ $\mu=e^{\alpha+\beta^2/2}$ $\sigma^2=e^{2\alpha+\beta^2}(e^{\beta^2}-1)$	

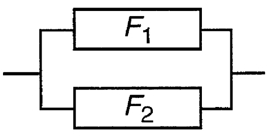
Name	f, G, r, μ, σ^2	Properties
Gamma	$f(x)=\frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x},\ x \geq 0$ For positive integers n $G(x)=\sum_{i=0}^{n-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x}$ $\mu = n/\lambda$ $\sigma^2 = n/\lambda^2$	IFR for $n \geq 1$ DFR for $0 < n \leq 1$
Uniform	$f(x)=1/(b-a),\ a \leq x \leq b$ $G(x)=(b-x)/(b-a)$ $r(x)=1/(b-x)$ $\mu=(a+b)/2$ $\sigma^2=(b-a)^2/12$	IFR

Life distribution for systems

General component life distributions

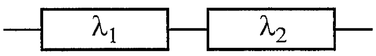


$$F(x)=1-(1-F_1(x))(1-F_2(x))$$
$$G(x)=G_1(x)G_2(x)$$

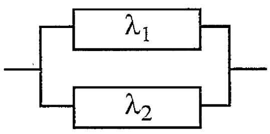


$$F(x)=F_1(x)F_2(x)$$
$$G(x)=G_1(x)+G_2(x)-G_1(x)G_2(x)$$

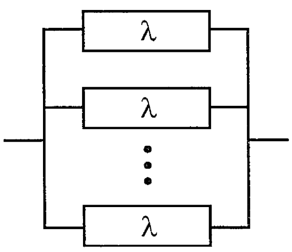
Exponential component life distributions



$$F(x)=1-e^{-(\lambda_1+\lambda_2)x}$$
$$G(x)=e^{-(\lambda_1+\lambda_2)x}$$
$$\mu=1/(\lambda_1+\lambda_2)$$



$$F(x)=(1-e^{-\lambda_1 x})(1-e^{-\lambda_2 x})$$
$$G(x)=e^{-\lambda_1 x}+e^{-\lambda_2 x}-e^{-(\lambda_1+\lambda_2)x}$$
$$\mu=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{1}{\lambda_1+\lambda_2}$$



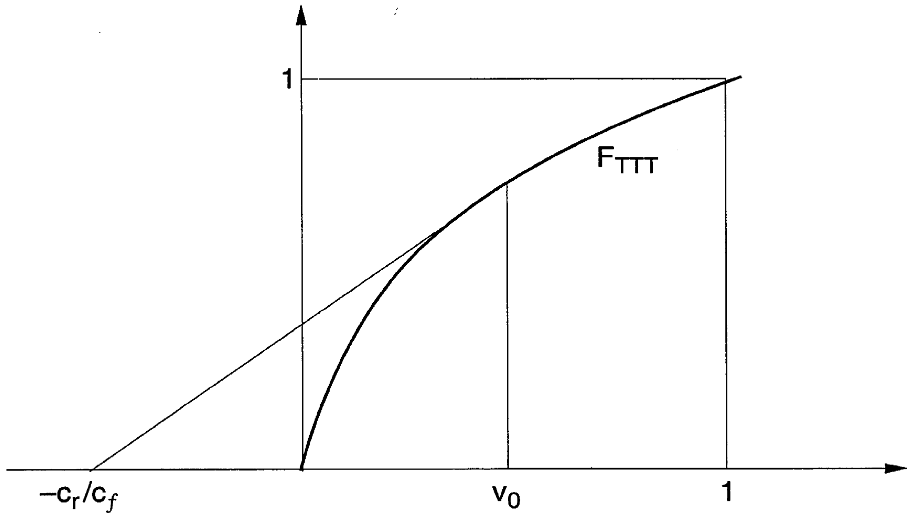
$$F(x)=(1-e^{-\lambda x})^n$$
$$\mu=\frac{1}{\lambda}\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\Big/ i=\frac{1}{\lambda}\sum_{i=1}^{n}\frac{1}{i}$$

Replacement and TTT-transform

A component is replaced after time t_0 or when it fails before time t_0.

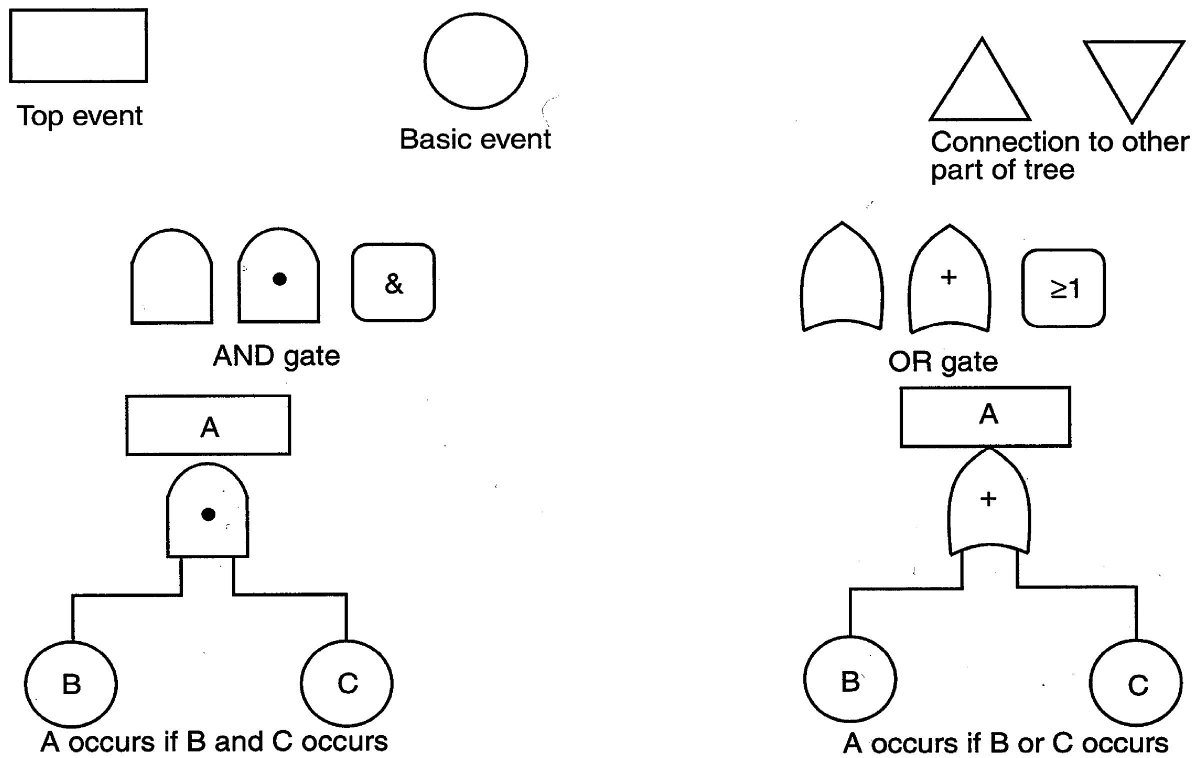
c_r = cost for replacement c_f = additional cost for replacement after failure.

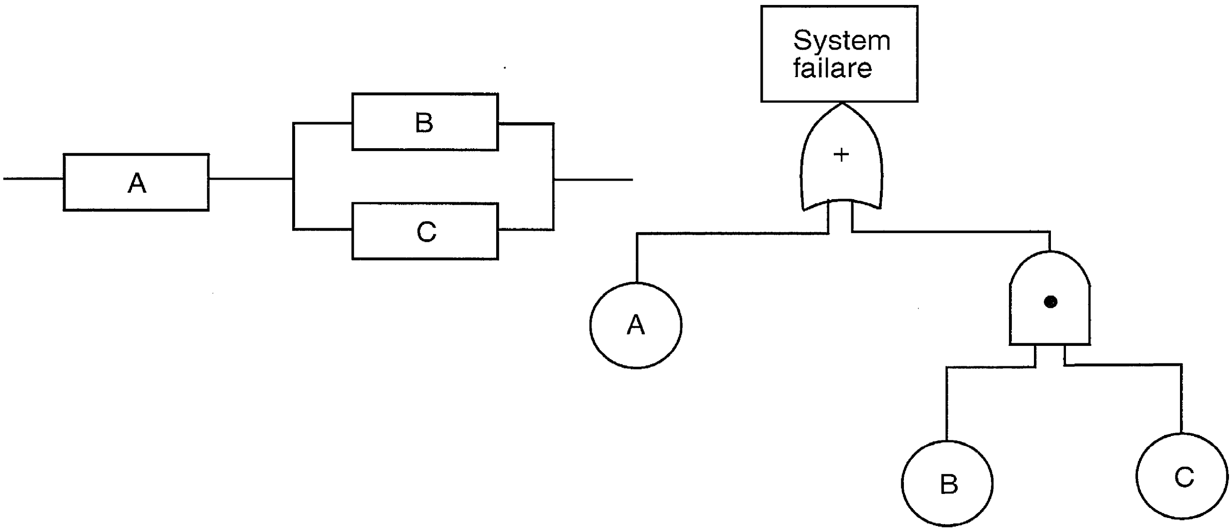
Cost for replacement per unit time is minimized with $t_0 = F^{-1}(v_0)$, where v_0 is found from the TTT-transform according to the following figure. If the TTT-transform is not known the TTT-plot can be used, see 18.10.



Fault tree analysis (FTA)

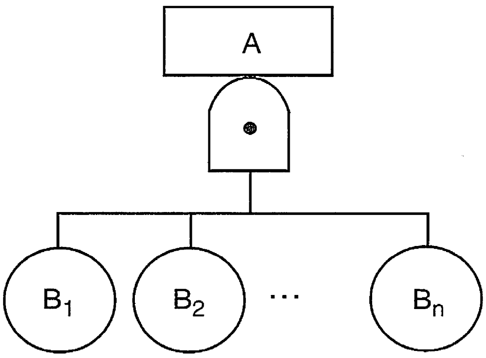
Symbols



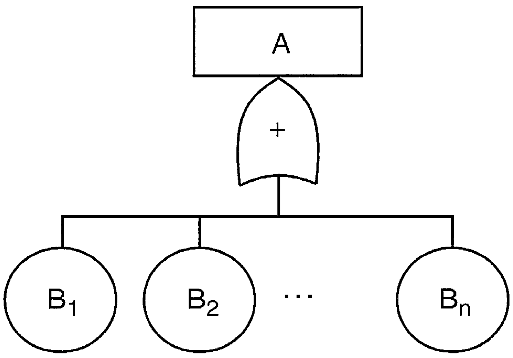


Fault tree for three component series-parallell system

Probability calculations



$P(A) = P(B_1) \cdot P(B_2) \cdot \ldots \cdot P(B_n)$
if $B_1, B_2, \ldots, B_n$ are independent.



$P(A) \approx P(B_1) + P(B_2) + \ldots + P(B_n)$

The probability for the top event for complicated fault trees (systems) can be calculated with especially designed computer programs.

Paths and custs

A *cut set* is a collection of basic events such that if all the events in the cut set occur, the top event will occur.

A *minimal cut set* is a cut set such that if any event is removed the set is no longer a cut set.

A *path set* is a collection of basic events with the property that if none of them occurs the top event will surely occur.

MOCUS (Method for Obtaining Cut Sets) is an algorithm for finding all cut sets of a given fault tree.