44)

$\underline{X}$ = mängd material per bil (kg) med täthet

$$f_{\underline{X}}(x) = \frac{1}{\ln 2} \cdot \frac{1}{x} \quad , \quad 25 \leq x \leq 50$$

a) Verifiera att f är täthetsfunktion för en kontinuerlig stokastisk variabel

Täthet: $\boxed{\int_{-\infty}^{\infty} f_{\underline{X}}(x)\,dx = 1}$

$$\int_{25}^{50} \frac{1}{\ln 2} \cdot \frac{1}{x}\,dx = \frac{1}{\ln 2} \int_{25}^{50} \frac{1}{x}\,dx =$$

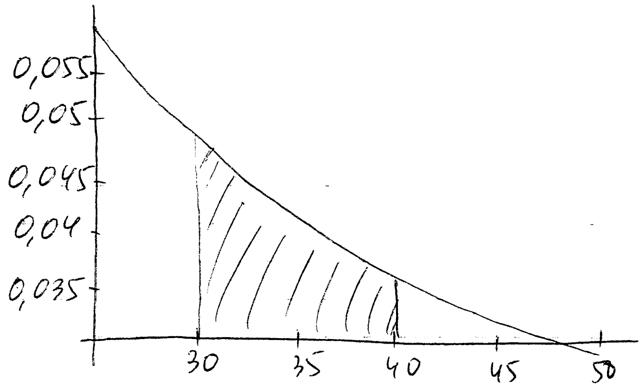
$$= \frac{1}{\ln 2} \cdot \left[\ln x\right]_{x=25}^{50} = \frac{1}{\ln 2}\left[\ln 50 - \ln 25\right] =$$

$$= \frac{1}{\ln 2}\left[\ln \underbrace{\frac{50}{25}}_{=2}\right] = 1$$

b) Sannolikheten att bil innehåller mellan 30 och 40 kg?

$$\int_{30}^{40} \frac{1}{\ln 2} \cdot \frac{1}{x}\,dx = \frac{1}{\ln 2}\left[\ln x\right]_{30}^{40} = \frac{1}{\ln 2} \cdot \ln \frac{4}{3} \approx 0{,}415$$

c)



4.16)

Samma täthet som i uppg. 4.4.

$$f(x) = \frac{1}{\ln(2)\cdot x} \quad , \qquad 25 \le x \le 50$$

Väntevärde, varians, standardavvikelse för X?

$$E(X) = \int_{25}^{50} x \cdot f_X(x)\,dx = \int_{25}^{50} x \cdot \frac{1}{\ln(2)} \cdot \frac{1}{x}\,dx =$$

$$= \frac{1}{\ln(2)} \int_{25}^{50} 1\,dx = \frac{1}{\ln(2)}(50-25) = \frac{25}{\ln(2)} \approx 36{,}07$$

$$E(X^2) = \int_{25}^{50} x^2 \frac{1}{\ln 2} \cdot \frac{1}{x}\,dx = \frac{1}{\ln 2}\int_{25}^{50} x\,dx =$$

$$= \frac{1}{\ln 2}\left[\frac{x^2}{2}\right]_{25}^{50} = \frac{1}{\ln 2}\left(\frac{50^2}{2} - \frac{25^2}{2}\right) \approx 1352{,}5266$$

$$Var(X) = E(X^2) - (E(X))^2 \approx 51{,}67$$

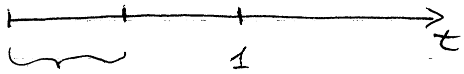
$$\sigma = \sqrt{Var(X)} \approx 7{,}19$$

4.36)

Ljud / knäppningar i sten i gruva.
Inträffar i snitt 3/timme

a) Vad är sannolikheten att inget händer under minst 30 minuter?

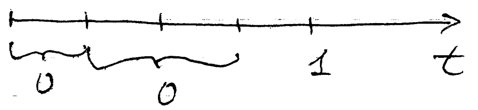
$X_t \sim Poi(3t)$ enhet = h (timmar)



$X_{0,5} = 0.$

$$P(X_{0,5} = 0) = e^{-3 \cdot 0,5} = e^{-3/2} \approx 0,223$$

b) Inget har hänt under 15 minuter
Vad är sannolikheten att 30 minuter till går innan första ljudet inträffar?



$P(W \leq t) = 1 - e^{-3t}$

↖ sats 4.3.3 sid 112

W = tiden tills första händelsen

$$P(W > 0,75 \mid W > 0,25) = \frac{P(W > 0,75)}{P(W > 0,25)} =$$

$$= \frac{e^{-3 \cdot 0,75}}{e^{-3 \cdot 0,25}} = e^{-3 \cdot 0,5} \approx 0,223$$

442

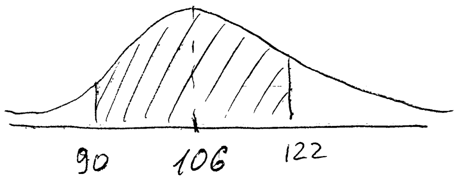
Bland diabetiker

glukosnivån $X \sim N(106, 8)$, enhet = mg/100ml

a) Skissa tätheten för X

Indikera $P(90 \leq X \leq 122)$

$\frac{x-\mu}{\sigma}$



$P(90 \leq X \leq 122) =$

$$= P\left(\frac{90-106}{8} \leq Z \leq \frac{122-106}{8}\right) =$$

$$= P(-2 \leq Z \leq 2) = \Phi(2) - \Phi(-2) =$$

$$= \Phi(2) - (1 - \Phi(2)) = 2\Phi(2) - 1 =$$

ur tabell 5, sid 730

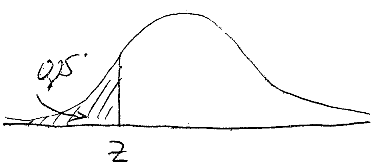
$$= 2 \cdot 0{,}9772 - 1 = 0{,}9544$$

b) $P(X \leq 120) = P\left(Z \leq \frac{120-106}{8}\right) =$

$$= P(Z \leq 1{,}75) = 0{,}9599 \approx 0{,}96$$

c) Hitta nivåvärdet som 25% av diabetikerna ligger under

$P(X \leq x) = 0{,}25 \Rightarrow P\left(Z \leq \frac{x-106}{8}\right) = 0{,}25$



$\Rightarrow Z \approx -0{,}675$ tabell V sid 730

$$\Rightarrow \frac{x-106}{8} = -0{,}675$$

$$\Leftrightarrow x = -8 \cdot 0{,}675 + 106 = 100{,}6$$

d) Om en slumpvis utvald person (diabetiker) har värde ≥ 130 – är det något att oroa sig för?

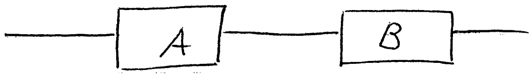
4.42 forts

$$P(\bar{X} \geq 130) = P\left(Z \geq \frac{130-106}{8}\right) = P(Z \geq 3) =$$

$$= 1 - P(Z \leq 3) = 1 - 0{,}9987 = 0{,}0013$$

4.66

2 oberoende komponenter



$T_A \sim \text{Wei}\,(\alpha = 0{,}006,\ \beta = 0{,}5)$

$T_B \sim \text{Exp}\,(\frac{1}{\beta} = 0{,}00004)$

Beräkningar för komponent A

$$F_A(t) = P(T_A \leq t) = \int_0^t \underbrace{\alpha\beta x^{\beta-1} e^{-\alpha x^\beta}}_{f(x)}\, dx =$$

$$= \left[\begin{array}{l|l} z = \alpha x^\beta & x = 0 \Rightarrow z = 0 \\ dz = \alpha\beta x^{\beta-1} dx & x = t \Rightarrow z = \alpha t^\beta \end{array}\right] =$$

$$= \int_0^{\alpha t^\beta} \alpha\beta x^{\beta-1} \cdot e^{-z} \cdot \frac{1}{\alpha\beta x^{\beta-1}}\, dz = \int_0^{\alpha t^\beta} e^{-z} = \left[-e^{-z}\right]_0^{\alpha t^\beta} =$$

$$= 1 - e^{-\alpha t^\beta} \Rightarrow$$

$$\Rightarrow R_A(t) = P(T_A > t) = 1 - F_A(t) = e^{-\alpha t^\beta}$$

Beräkningar för komponent B:

$$F_B(t) = \int_0^t \frac{1}{\beta} e^{-x/\beta}\, dx = \left[\begin{array}{l|l} z = x/\beta & x = 0 \Rightarrow z = 0 \\ dz = \frac{1}{\beta} dx & x = t \Rightarrow z = \frac{t}{\beta} \end{array}\right] =$$

$$= \int_0^{t/\beta} e^{-z}\, dz = 1 - e^{-t/\beta} \Rightarrow R_B(t) = e^{-t/\beta}$$

4.66 forts

Serie

$$R(t) = \prod_{i=1}^{k} R_i(t)$$

Parallell

$$R(t) = 1 - P(\text{alla sannolikheter fungera}) =$$
$$= 1 - \prod_{i=1}^{k} (1 - R_i(t))$$

a) Fungerar 2500 timmar

$$R_S(2500) = R_A(2500) \cdot R_B(2500) =$$
$$= \underbrace{e^{-0{,}006 \cdot 2500^{0{,}5}}}_{=0{,}7408} \cdot \underbrace{e^{-2500/25000}}_{=0{,}9048} = 0{,}67$$

b) Sannolikheten fungera innan 2000 timmar

$$1 - R_S(2000) = 1 - R_A(2000) \cdot R_B(2000) =$$
$$= 1 - \underbrace{e^{-0{,}006 \cdot 2000^{0{,}5}}}_{\approx 0{,}7647} \cdot \underbrace{e^{-2000/25000}}_{\approx 0{,}9231} \approx 1 - 0{,}7059 \approx 0{,}294$$

c) Parallellkopplade komponenter istället – sannolikheten fungerar i 2500 timmar?

$$R_p(2500) = 1 - (1 - R_A(2500))(1 - R_B(2500)) = \ldots = 0{,}9753$$

5.8

$\overline{X}$ = temperatur (c°)

Y = tid att starta motorn (min)

$$f_{\overline{X},Y}(x,y) = c\,(4x+2y+1) \quad ; \quad \begin{cases} 0 \le x \le 40 \\ 0 \le y \le 2 \end{cases}$$

a) Vad är c?

$$\int_0^{40}\int_0^{2} f_{\overline{X},Y}\,dy\,dx = 1 \quad \Rightarrow$$

$$1 = \int_0^{40}\int_0^{2} c\,(4x+2y+1)\,dy\,dx =$$

$$= c\int_0^{40}\left(\int_0^{2} 4x+2y+1)\,dy\right)dx =$$

$$= c\int_0^{40}\left[4xy + \frac{\cancel{2}y^2}{\cancel{2}} + y\right]_{y=0}^{y=2} dx = 8x+4+2 =$$

$$= 8x+6 = c\left[\frac{8x^2}{2} + 6x\right]_0^{40} =$$

$$= c\,(4\cdot 40^2 + 6\cdot 40) = 6640c \quad \Rightarrow \quad c = \frac{1}{6640}$$

b) Sannolikheten lufttemper. över 20°C och minst 1 minut för bilen att starta?

$$\mathbb{P}(\overline{X}>20,\ y\ge 1) = \int_{20}^{40}\int_1^{2}\frac{1}{6640}(4x+2y+1)\,dy\,dx =$$

$$= \frac{1}{6640}\int_{20}^{40}\underbrace{\left[4xy + \frac{\cancel{2}y^2}{\cancel{2}} + y\right]_{y=1}^{y=2}}_{8x+4+2-4x-1-1=4x+4} dx =$$

$$= \frac{1}{6640}\underbrace{\left[\frac{\cancel{4}^{2}x^2}{\cancel{2}} + 4x\right]_{20}^{40}}_{2480} = \frac{2480}{6640} \approx 0{,}373$$

5.8 forts

c) Marginaltätheterna för X och Y:

$$f_X(x) = \int_0^2 \frac{1}{6640}(4x+2y+1)\,dy =$$

$$= \left[\frac{1}{6640}\left(4xy + \frac{2y^2}{2} + y\right)\right]_{y=0}^{y=2} =$$

$$= \frac{1}{6640}(8x+4+2) = \frac{8x+6}{6640} \quad ; \quad 0 \le x \le 40$$

$$f_Y(y) = \int_0^{40} \frac{1}{6640}(4x+2y+1)\,dx =$$

$$= \frac{1}{6640}\left[\frac{4x^2}{2} + 2xy + x\right]_0^{40} =$$

$$= \frac{1}{6640}(2\cdot 40^2 + 2\cdot 40y + 40) =$$

$$= \frac{3240+80y}{6640} = \frac{81+2y}{166} \quad ; \quad 0 \le y \le 2$$

d) Sannolikheten – minst 1 minut att starta bilen?

$$P(Y>1) = \int_1^2 \frac{1}{166}(2y+81)\,dy = \frac{1}{166}\left[\frac{2y^2}{2} + 81y\right]_1^2 =$$

$$= \frac{1}{166}(2^2 + 81\cdot 2 - 1^2 - 81) = \frac{1}{166}\cdot 84 \approx 0{,}506$$

e) Sannolikheten – temperat. högre än 20 C°

$$P(X>20) = \int_{20}^{40} \frac{4x+3}{3320}\,dx = \frac{1}{3320}\left[\frac{4x^2}{2} + 3x\right]_{20}^{40} =$$

$$\approx 0{,}741$$

5.8 forts

f) $\underline{X}$ och Y oberoende?
(Gäller $\mathbb{P}(\underline{X} > 20, Y > 1) = \mathbb{P}(\underline{X} > 20)\,\mathbb{P}(Y > 1)$?)

$$0{,}373 \neq \underbrace{0{,}741 \cdot 0{,}506}_{=0{,}375}$$

Svar: nej, $\underline{X}$, Y är ej oberoende.

5.20) forts. av 5.8: X är temperaturen, Y är tiden det tar tills motorn är klar för start

$$f_{XY}(x,y) = \frac{1}{6640}(4x+2y+1), \quad 0 \le x \le 40, \; 0 \le y \le 2$$

a) Borde $Cov(X,Y)$ vara positiv eller negativ ur fysikalisk synvinkel?

b) $$E(X) = \int_0^{40} x \cdot \underbrace{\frac{4x+3}{3320}}_{f_X(x)} dx = \frac{1}{3320}\int_0^{40}(4x^2+3x)\,dx =$$

$$= \frac{1}{3320}\left[\frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^{40} = \frac{1}{3320}\left(\frac{4\cdot 40^3}{3} + \frac{3\cdot 40^2}{2}\right) =$$

$$= \frac{1}{3320} \cdot 87733{,}3 \approx 26{,}43$$

$$E(Y) = \int_0^2 y \cdot \frac{2y+81}{166}\,dy = \frac{1}{166}\left[\frac{2y^3}{3} + \frac{81y^2}{2}\right]_0^2 =$$

$$= \frac{1}{166}\left(\frac{2\cdot 2^3}{3} + \frac{81\cdot 2^2}{2}\right) \approx 1{,}008$$

$$E(XY) = \int_0^{40}\int_0^2 xy\left(\frac{1}{6640}(4x+2y+1)\right) dy\,dx =$$

$$= \frac{1}{6640}\int_0^{40}\int_0^2 (4x^2y + 2xy^2 + xy)\,dy\,dx =$$

$$= \frac{1}{6640}\int_0^{40}\left[\frac{4x^2y^2}{2} + \frac{2xy^3}{3} + \frac{xy^2}{2}\right]_{y=0}^{y=2} =$$

$$= \frac{1}{6640}\int_0^{40}\left(8x^2 + \frac{16x}{3} + 2x\right)dx =$$

$$= \frac{1}{6640}\left[\frac{8x^3}{3} + \frac{16x^2}{6} + \frac{8x^2}{2}\right]_0^{40} =$$

$$= \frac{1}{6640}\left(\frac{8\cdot 40^3}{3} + \frac{16\cdot 40^2}{6} + 40^2\right) \approx 26{,}586$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 26{,}586 - 1{,}008 \cdot 26{,}43 =$$

$$= -0{,}055$$

5.32

Beräkna $E(X^2)$, $E(Y^2)$, $Var(X)$, $Var(Y)$ och ρ_{XY} för X och Y i uppg. 8 och 20

$$E(X^2) = \int_0^{40} x^2 \frac{4x+3}{3320}\,dx = \frac{1}{3320}\int_0^{40}(4x^3+3x^2)\,dx =$$

$$= \frac{1}{3320}\left[\frac{4x^4}{4} + \frac{3x^3}{3}\right]_0^{40} = \frac{1}{3320}(40^4 + 40^3) \approx 790{,}361$$

$$E(Y^2) = \int_0^2 y^2 \cdot \frac{2y+81}{166}\,dy = \frac{1}{166}\int_0^2 (2y^3 + 81y^2)\,dy =$$

$$= \frac{1}{166}\left[\frac{2y^4}{4} + \frac{81y^3}{3}\right]_0^2 = \frac{1}{166}\left(\frac{2^4}{2} + \frac{81\cdot 2^3}{3}\right) =$$

$$\approx 1{,}349$$

$$Var(X) = E(X^2) - (E(X))^2 = 790{,}361 - 26{,}43^2 =$$

$$= 91{,}816 \approx 92$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = 1{,}349 - 1{,}008^2 \approx 0{,}33$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{-0{,}055}{\sqrt{92 \cdot 0{,}33}} \approx -0{,}00998 \approx -0{,}01$$

5.42)

$$f_{X,Y}(x,y) = \frac{1}{x} \quad ; \quad 0 < y < x < 1$$

a) Hitta regressionskurvan för X på y, $\mu_{X|y}$
Är den linjär?

$$f_{X|y} = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1/x}{f_Y(y)}$$

där $f_Y(y) = \int_y^1 \frac{1}{x}\,dx = [\ln x]_y^1 = -\ln y$

$$\Rightarrow f_{X|y} = -\frac{1}{x \ln y}$$

$$\mu_{X|y} = \int_y^1 x \cdot f_{X|y}\,dx = \int_y^1 -\frac{x}{x \ln y}\,dx =$$

$$= -\int_y^1 \frac{1}{\ln y}\,dx = -\frac{1}{\ln y}[x]_y^1 = -\frac{1}{\ln y}(1-y) =$$

$$= \frac{y-1}{\ln y}$$

$\therefore$ ej linjär

b) $\mu_{X|y=0,5} = \frac{0,5-1}{\ln 0,5} \approx 0,72$

c) $\mu_{Y|X}$ – ?

$$f_X(x) = \int_0^x \frac{1}{x}\,dy = \left[\frac{y}{x}\right]_{y=0}^x = 1$$

$$f_{Y|X} = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1/x}{1} = \frac{1}{x}$$

$$\mu_{Y|x} = \int_0^x y \cdot f_{Y|x}\,dy = \int_0^x \frac{y}{x}\,dy = \frac{1}{x}\int_0^x y\,dy =$$

$$= \frac{1}{x}\left[\frac{y^2}{2}\right]_0^x = \frac{1}{x}\left(\frac{x^2}{2} - 0\right) = \frac{x}{2}$$

$\therefore$ Linjär

5,42 forts

d) $\mu_{Y|X=0,75} = \frac{0,75}{2} = 0,375$

5.56

X = antal 'do'-loopar i Fortranprogram
Y = antal körningar tills nybörjare hittat alla fel

$f_{X,Y}(x,y)$:

x \ y	1	2	3	4
0	0,059	0,1	0,05	0,001
1	0,093	0,12	0,082	0,003
2	0,065	0,102	0,1	0,01
3	0,050	0,075	0,07	0,02

a)

a) Sannolikheten: högst en loop och minst två körningar

$$\mathbb{P}(X \leq 1, Y \geq 2) = \mathbb{P}(X=0, Y=2) + \mathbb{P}(X=0, Y=3) + \mathbb{P}(X=0, Y=4) + \mathbb{P}(X=1, Y=2) + \mathbb{P}(X=1, Y=3) + \mathbb{P}(X=1, Y=4) = \sum_{x=0}^{1} \sum_{y=2}^{4} f_{X,Y}(x,y) = 0,356$$

b)
$$E[XY] = \sum_{x=0}^{3} \sum_{y=1}^{4} f_{X,Y}(x,y) \cdot x \cdot y =$$

$$= 0 \cdot 1 \cdot 0,059 + 1 \cdot 1 \cdot 0,093 + 1 \cdot 2 \cdot 0,065 + 1 \cdot 3 \cdot 0,050 + \ldots =$$
$$= 3,279$$

556 forts)

c) $f_X(x)$ och $f_Y(y)$:

$x \backslash y$	1	2	3	4	$f_X(x)$
0	0,059	0,1	0,05	0,001	0,21
1	0,093	0,12	0,082	0,003	0,298 → d)
2	0,065	0,102	0,1	0,01	0,277
3	0,050	0,075	0,07	0,02	0,215
$f_Y(y)$	0,267	0,397	0,302	0,034	1,0

$$E(X) = \sum_{x=0}^{3} x f_X(x) = 1,497$$

$$E(X^2) = \sum_{x=0}^{3} x^2 f_X(x) = 3,341$$

$$Var(X) = E(X^2) - (E(X))^2 = 3,341 - 1,497^2 =$$
$$= 1,09999 \approx 1,1$$

$$E(Y) = \sum_{y=1}^{4} y f_Y(y) = 2,103$$

$$E(Y^2) = \sum_{y=1}^{4} y^2 f_Y(y) = 5,117$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = 5,117 - 2,103^2 = 0,694$$

d)
$$P(Y \geq 2 \mid X = 1) = \frac{P(Y \geq 2, X = 1)}{P(X = 1)} =$$

$$= \frac{0,12 + 0,082 + 0,003}{0,298} = 0,6879 \approx 0,69$$

e) $Cov(X, Y) = ?$

$\rho_{XY} = ?$

5 56 forts

$$\boxed{Cov(X, Y) = E(XY) - E(X) \cdot E(Y)}$$

$$\boxed{\rho_{XY} = \frac{Cov(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}}}$$

$$Cov(X, Y) = 3,279 - 1,497 \cdot 2,103 \approx 0,131$$

$$\rho_{XY} = \frac{0,131}{\sqrt{1,1 \cdot 0,694}} \approx 0,1497$$