Solution to Homework 1 in TMS115 Probability and Stochastic Processes, Q. 1, 2004/2005

Problem 1. Denote

$$A_i = \{\text{the relay chosen is manufactured by plant } i\}, \quad i = 1, 2, 3.$$

Thus

$$P\{A_1\} = 0.5, \quad P\{A_2\} = 0.3, \quad P\{A_3\} = 0.2.$$

Let

$$C = \{\text{the relay chosen is defective}\}$$

We have

$$P\{C|A_1\} = 0.02, \quad P\{C|A_2\} = 0.05, \quad P\{C|A_3\} = 0.01.$$

a) By the total probability formula,

$$P\{C\} = \sum_{i=1}^{3} P\{C|A_i\}P\{A_i\} = 0.027.$$

b) We have to find

$$P\{A_2^C|C\} = 1 - P\{A_2|C\} = 1 - \frac{P\{C|A_2\}P\{A_2\}}{P\{C\}} = 0.444.$$

Problem 2. Let X denote the number of bits until the third error occurs

- $X \sim$ Negative Binomial (0.1, 3).

  a) $P\{X = 10\} = \binom{9}{2} 0.9^7 \cdot 0.1^3 = 0.172$

  b) The random variable X has a presentation
  $X = X_1 + X_2 + X_3, \quad X_i \sim$ Geometric (0.1).
  Since $E[X_i] = \dfrac{1}{0.1} = 10$, we have $E[X] = 3 * 10 = 30$.

Problem 3. Denote

$$E[X_i] = \mu_i, \quad Var(X_i) = \sigma_i^2, \quad i = 1,\, 2, \quad \rho_{X_1,X_2} = \rho.$$

Since X_1 and X_2 are jointly normal, $X_1 - X_2$ is a normal r. v. with

$$E[X_1 - X_2] = \mu_1 - \mu_2, \quad Var(X_1 - X_2) = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2.$$

Thus

$$P\{X_1 > X_2\} = P\{X_1 - X_2 > 0\} = P\Big\{\frac{X_1 - X_2 - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}} > \frac{-(\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}}\Big\}$$
$$= Q\Big(\frac{\mu_2 - \mu_2}{\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}}\Big).$$

Problem 4. Let X be the number of students which attend the program. We assume that the accepted students decide independently whether to come or not, and that the probability for anyone to come is $p = \frac{45}{135} = \frac{1}{3}$. In this way, $X \sim \mathrm{Bin}(135, 1/3)$. By the CLT,

$$P\{X \geq 40\} = P\{X > 39\} \approx Q\Big(\frac{39 - 45}{\sqrt{135 \cdot 1/3 \cdot 2/3}}\Big) = 1 - Q(1.1) = 0.864.$$

Problem 5. Let $X = X_1 + X_2 + \ldots + X_N$ be the total number of bits generated by the combined source. X is a normal random variable with expected value $m_a = Nm$ and variance $\sigma_a^2 = N\sigma^2$.

a)

$$\begin{aligned} p_{Loss} = P\{X \geq T\} &= \int_T^\infty f_X(x)dx \\ &= \int_{m_a + t\sigma_a}^\infty \frac{1}{\sqrt{2\pi}\sigma_a} \exp\Big(-\frac{(x - m_a)^2}{2\sigma_a^2}\Big)dx \\ &= \int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{u^2}{2}\Big)du \\ &= Q(t). \end{aligned}$$

b) We have

$$X_{Loss} = \begin{cases} X - T, & \text{if} \quad X - T \geq 0, \\ 0, & \text{if} \quad X - T < 0. \end{cases}$$

Then

$$
\begin{aligned}
E[X_{Loss}] &= \int_T^\infty (x-T) f_X(x)dx \\
&= \int_{m_a+t\sigma_a}^\infty x f_X(x)dx - TP\{X \geq m_a + t\sigma_a\} \\
&= \frac{1}{\sqrt{2\pi}} \int_t^\infty (\sigma_a y + m_a) e^{-\frac{y^2}{2}} dy - (m_a + t\sigma_a)Q(t) \\
&= \frac{\sigma_a}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} - t\sigma_a Q(t) = \frac{\sqrt{N}\sigma}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} - t\sqrt{N}\sigma Q(t).
\end{aligned}
$$

Note that when $t \to \infty$ we obtain $E[X_{Loss}] \to 0$, as expected.

c) We need to keep

$$
c = \frac{E[X_{Loss}]}{m_a} = \frac{\frac{\sqrt{N}\sigma}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} - t\sqrt{N}\sigma Q(t)}{Nm}
$$

implying

$$
\frac{cm\sqrt{N}}{\sigma} = \frac{e^{-t^2/2}}{\sqrt{2\pi}} - tQ(t).
$$

This equation can be solved for t by using approximations numerical methods.