Solution to the exam in TMS115 Probability and Stochastic Processes 2005-10-21

Problem 1.

$$P\{\text{system "up"}\} = (1-p)\big[3r(1-r)^2 + (1-r)^3\big].$$

(2)

Problem 2. The inverse transform is $x = e^{-\lambda y},\ y > 0$. Thus

(a)

$$f_Y(y) = f_X(e^{-\lambda y}) \cdot \lambda e^{-\lambda y} = \lambda e^{-\lambda y},\ y > 0.$$

(2)

(b)

$$E[XY] = \int_0^1 x\Big(-\frac{\ln x}{\lambda}\Big)\,dx = \frac{1}{\lambda}\Big[-\frac{x^2}{2}\ln x\Big|_0^1 + \int_0^1 \frac{x}{2}\,dx\Big] = \frac{1}{4\lambda}.$$

(2)

Problem 3. Since X and Y are jointly Gaussian, any non-degenerate linear combination of X and Y is a Gaussian random variable.

(a) $X - Y$ is Gaussian with

$$E[X-Y] = \mu_X - \mu_Y, \quad \sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y - 2\rho\sigma_X\sigma_Y.$$

(2)

(b) X and Y are independent Gaussian random variables with the same mean value and variance, hence $P\{|X| > |Y|\} = P\{|Y| > |X|\}$. Since $P\{|X| > |Y|\} + P\{|Y| > |X|\} = 1$, we get $P\{|X| > |Y|\} = 0.5$. (2)

Problem 4.

(a)

$$P\{\text{input } 1\,|\,Y > T\} = \frac{P\{Y > T\,|\,\text{input } 1\}\,p}{P\{Y > T\,|\,\text{input } 1\}\,p + P\{Y > T\,|\,\text{input } 0\}\,(1-p)},$$
$$= \frac{p\,Q(T-1)}{p\,Q(T-1) + (1-p)\,Q(T)}.$$

When $p = 0.5$ the above gives

$$\frac{Q(-0.5)}{Q(-0.5) + Q(0.5)} = 1 - Q(0.5) \approx 0.692.$$

(2)

(b)

$$\begin{aligned} P_e &= P\{Y > T \,|\, \text{input } 0\}\,(1-p) + P\{Y < T \,|\, \text{input } 1\}\,p \\ &= (1-p)\int_T^{\infty} f_2(t)\,dt + p\int_{-\infty}^{T} f_1(t)\,dt = (1-p)Q(T) + pQ(1-T). \end{aligned}$$

When $p = 0.1$, $T = 0.5 + \ln 9 \approx 2.7,\ 1 - T = -1.7$ and

$$P_e = 0.9\,Q(2.7) + 0.1\,[1 - Q(1.7)] = 0.9 \cdot 0.0034 + 0.1 \cdot 0.044 = 0.099.$$

(2)

Problem 5.

(a) $N(t)$ is not stationary, since the mean-value time function $m_N(t) = E[N(t)] = \lambda t$ is not a constant. (2)

(b)

$$\begin{aligned} &P\{S_1 > 1,\, S_2 > 2 \,|\, N(3) = 3\} \\ &= \frac{P\{S_1 > 1,\, S_2 > 2,\, N(3) = 3\}}{P\{N(3) = 3\}} \\ &= \frac{P\{N(1) = 0\}\,P\{N(2) - N(1) = 1\}\,P\{N(3) - N(2) = 2\}}{P\{N(3) = 3\}} \\ &+ \frac{P\{N(1) = 0\}\,P\{N(2) - N(1) = 0\}\,P\{N(3) - N(2) = 3\}}{P\{N(3) = 3\}} = \frac{4}{27}. \end{aligned}$$

(2)

(c)

$$\begin{aligned} &P\{W_t > x,\, S_k < t < S_k + T_{k+1}\} = \int_0^t P\{T_{k+1} > t - u + x\} f_{S_k}(u)\,du \\ &= \int_0^t e^{-\lambda(t-u+x)} \frac{\lambda^k}{(k-1)!}\, u^{k-1} e^{-\lambda u}\,du = e^{-\lambda(t+x)}\frac{(\lambda\, t)^k}{k!}. \end{aligned}$$

By the total probability formula,

$$P\{W_t > x,\, S_k < t < S_k + T_{k+1}\} = e^{-\lambda(t+x)} \sum_0^{\infty} \frac{(\lambda\, t)^k}{k!} = e^{-\lambda\, x}.$$

(4)

Problem 6.

(a)

$$E[X_{n+k}X_n] = \sum_{i=0}^{p}\sum_{j=0}^{p} \alpha_i\, \alpha_j R_W(k - j + i).$$

If $|k| > p$ we have $|k| - j + i \geq 1$ and $R_W(|k| - j + i) = 0$. Thus $R_X(k) = 0$ for $|k| > p$. For $|k| \leq p$

$$R_X(k) = \sum_{l=|k|}^{p} \alpha_i \, \alpha_{i-k} = \sum_{-\infty}^{\infty} \alpha_i \, \alpha_{i-k},$$

where we have defined $\alpha_i = 0$ for $i < 0$ and $i > p$. (2)

(b)

$$S_X(f) = \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \alpha_i \, \alpha_{i-k} e^{-j\,2\pi\,f\,k} = \sum_{i=-\infty}^{\infty} \alpha_i \, e^{-j\,2\pi\,f\,i} \sum_{m=-\infty}^{\infty} \alpha_m \, e^{j\,2\pi\,fm} = H(f)\,H^*(f)$$

where $H(f) = \sum_{i=-\infty}^{\infty} \alpha_i \, e^{-j\,2\pi\,f\,i}$ is the transfer function of the system. (2)

(c)

$$aR_X(0) + bR_X(1) = R_X(2)$$
$$aR_X(1) + bR_X(0) = R_X(1)$$

(2)

(d)

$$e^2 = R_X(0) - aR_X(2) - bR_X(1).$$

(2)