Solution to the exam in TMS115 Probability and Stochastic Processes 2006-01-13

Problem 1

a) $P\{N=n\} = (1-p)^{n-1}\cdot p, \quad n = 1,2,\ldots$

$E[N] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p = p\Big(-\sum_{n=0}^{\infty}(1-p)^n\Big)'_p = p\Big(-\frac{1}{p}\Big)' = \frac{1}{p}.$ (2)

b) $P\{N=10+k|N>10\} = \dfrac{(1-p)^{10+k-1}\cdot p}{(1-p)^{10}} = (1-p)^{k-1}\cdot p, \quad k=1,2,\ldots.$

$E[N-10\,|N>10\} = \dfrac{1}{p}.$ (2)

Problem 2

(a) $1-e^{-\lambda\cdot 1000} = 0.75, \quad \dfrac{1}{\lambda} = -\dfrac{1000}{\ln 0.25} = 721.348.$ (2)

(b) $\sum_{k=5}^{10}\binom{10}{k}(0.25)^k(0.75)^{10-k} = 0.0781.$ (2)

Problem 3

(a)

$$z_1 = x_1, \quad z_2 = \frac{5}{4}\left[x_2 - \frac{3}{5}x_1\right] = \frac{5}{4}x_2 - \frac{3}{4}x_1,$$

$$J = \det\begin{bmatrix} 1 & 0 \\ -\frac{3}{4} & \frac{5}{4}\end{bmatrix} = \frac{5}{4},$$

$$f_{X_1,X_2}(x_1,x_2) = f_{z_1,z_2}\left(x_1, \frac{5}{4}x_2 - \frac{3}{4}x_1\right)\frac{5}{4}$$

$$= \frac{1}{2\pi\frac{4}{5}}\exp\left\{-\frac{1}{2}\left[x_1^2 + \left(\frac{5}{4}x_2 - \frac{3}{4}x_1\right)^2\right]\right\}$$

$$= \frac{1}{2\pi\sqrt{1-(3/5)^2}}\exp\left\{-\frac{1}{2}\frac{1}{(1-(3/5)^2)}\left[x_1^2 + x_2^2 - 2\cdot\frac{3}{5}x_1\,x_2\right]\right\}.$$

Hense X_1, X_2 have bivariate normal distribution with

$$E[X_1] = E[X_2] = 0, \quad Var(X_1) = Var(X_2) = 1, \quad \rho = 3/5.$$

(2)

(b) $X_2|X_1 = x_1 \sim \mathcal{N}(3x_1/5,\ 16/25)$ (2)

Problem 4

$X =$ the number of participants, $\quad X \sim \text{Bin}(48, 2/3)$.

$$P\{X \le 22\} \approx \Phi\left(\frac{22-32}{\sqrt{48\cdot\frac{1}{3}\cdot\frac{2}{3}}}\right) = Q\left(\frac{10}{\frac{4}{3}\cdot\sqrt{6}}\right) = Q(3.06) = 9.63 * 10^{-4}. \quad (2)$$

Problem 5

(a)

$S_Z(f) = a^2 S_X(f) + b^2 S_Y(f) = (a^2+b^2)S(f)$,

$R_{Z,X}(\tau) = aR(\tau), \quad S_{Z,X}(f) = aS(f).$ (2)

(b)

$Z(t) \sim \mathcal{N}(0, (a^2+b^2)R(0)).$ (2)

Problem 6:

(a)

$$E[X_{n+k}X_n] = \sum_{i=0}^{3}\sum_{j=0}^{3}\alpha_i\,\alpha_j R_W(k-j+i).$$

If $|k| > p$ we have $|k| - j + i \ge 1$ and $R_W(|k|-j+i) = 0$. Thus $R_X(k) = 0$ for $|k| > p$.
For $|k| \le p$

$$R_X(k) = \sum_{l=|k|}^{3}\alpha_i\,\alpha_{i-k} = \sum_{-\infty}^{\infty}\alpha_i\,\alpha_{i-k},$$

where we have defined $\alpha_i = 0$ for $i < 0$ and $i > p$. (2)

(b)

$$S_X(f) = \sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\alpha_i\,\alpha_{i-k}e^{-j\,2\pi\,f\,k}$$

$$= \sum_{i=-\infty}^{\infty}\alpha_i\,e^{-j\,2\pi\,f\,i}\sum_{m=-\infty}^{\infty}\alpha_m\,e^{j\,2\pi\,fm} = H(f)\,H^*(f),$$

where $H(f) = \sum_{i=-\infty}^{\infty}\alpha_i\,e^{-j\,2\pi\,f\,i}$ is the transfer function of the system. (2)

Problem 7

(a) $\hat{X}_n = h_1 X_{n-1} + h_2 X_{n-3}$.

$\hat{X}_n$ – optimal $\Leftrightarrow E[\hat{X}_n X_{n-i}] = E[X_n X_{n-i}], \quad i = 1,\ 2$

$$\begin{bmatrix} R_X(0) & R_X(2) \\ R_X(2) & R_X(0) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} R_X(1) \\ R_X(3) \end{bmatrix} \quad (2)$$

(2)

(b) $e^2 = E[(\hat{X}_n - X_n)(\hat{X}_n - X_n)] = R_X(0) - h_1 R_X(1) - h_2 R_X(3).$ (2)

Problem 8

(a) $X(t) = X(t) - h(t) * X(t) = \underbrace{(\delta(t) - h(t))}_{h_1(t)} * X(t),$

$H_1(t) = \mathcal{F}\{\delta(t) - h(t)\} = 1 - H(f),$

$S_Z(f) = |1 - H(f)|^2 S_X(f).$ (2)

(b) $E[Z^2(t)] = R_Z(0) = \int_{-\infty}^{\infty} S_Z(f)df = \int_{-\infty}^{\infty} |1 - H(f)|^2 S_X(f)\, df.$ (2)

Or:

$R_Z(k) = R_X(k) + R_Y(k) - R_{YX}(k) - R_{XY}(k),$

$S_Z(f) = S_X(f) + |H(t)|^2 S_X(f) - H(f) S_X(f) - H^*(f) S_X(f)$

$= S_X(f)[1 + |H(f)|^2 - H(f) - H^*(f)]$

$= S_X(f)[1 - H(f)][1 - H^*(f)] = S_X(f)|1 - H(f)|^2.$