Chapter 6: Fractional factorial designs

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Review of factorial designs

- Goal of experiment: To find the effect on the response(s) of a set of factors
  - each factor can be set by the experimenter independently of the others
  - each factor is set in the experiment at one of two possible levels (- and +)
- Standard order of factors, 2^n design, calculation of main effects and interaction effects, table of contrasts, standard errors of effect estimates.

Example

- Given 2^3 experimental plan, with factors A, B, C.
- Can we investigate also factor D without increasing number of experiments?
- Possibility: Give D same signs as interaction ABC.
- Then: Main effect of D estimated separately from other main effects.

A	B	C	D=ABC
-	-	-	-
-	-	+	+
-	+	-	+
-	+	+	-
+	-	-	+
+	-	+	-
+	+	-	-
+	+	+	+

Fractional factorial designs

- A design with factors at two levels.
- How to build: Start with *full* factorial design, and then introduce new factors by identifying with interaction effects of the old.
- Notation: A 2^{3-1} design, 2^{4-1} design, 2^{5-2} design, etc
- 2^{n-m}: n is total number of factors, m is number of factors added identified with interaction effects.
- The number of experiments is equal to 2^{n-m}

Example: Biking up a hill

- Goal of experiment: Determine how various factors influence the time it takes to bike up a hill.
- Factors: Seat Up/Down, Dynamo Off/On, Handlebars Up/Down, Gear Low/Medium, Raincoat On/Off, Breakfast On/Off, Tires Hard/Soft.
- The variance of measurements was estimated from separately collected data.
- An 8-run experiment was desired, for initial screening of factors.

Experimental plan: 2^{7-4} experiment

A	B	C	D= AB	E= AC	F= BC	G= ABC
-	-	-	+	+	+	-
+	-	-	-	-	+	+
-	+	-	-	+	-	+
+	+	-	+	-	-	-
-	-	+	+	-	-	+
+	-	+	-	+	-	-
-	+	+	-	-	+	-
+	+	+	+	+	+	+

Conclusions from biking example

- Main effects are computed: Dynamo and gear large
- We saw previously how the standard deviation of the population of effect estimates (the ”standard error” of the effect”) could be estimated as

  $$\sqrt{\sigma^2/4+\sigma^2/4}$$

  where σ^2 is the variance of the population of observations at a setting.
- In this example, repeated runs at some setting had sample standard deviation 3.
- So σ^2 was estimated with 3^2, and the standard error with

$$\sqrt{\sigma^2/4+\sigma^2/4}=\sqrt{3^2/4+3^2/4}=2.1$$

Drawing conclusions

- Fractional factorial experiments are great for screening for factors with effect.
- *Assuming* factors do not interact, a rough impression of the size of effects of factors can be found quickly.
- A quick estimate of the standard error can help interpret the numbers.
- In bicycle example: Effects were 3.5, 12.0, 1.0, 22.5, 0.5, 1.0, 2.5, with st. error 2.1.

Visualization

- A 3D cube can visualize 3 effects: Take average over other factors.

- Alternative: Visualize 4D data by splitting into two cubes, one for + or – of a fourth factor.
- Possible to visualize 5D data, and even 6D data, by splitting into smaller cubes.

What is lost when using fractional designs?

- In a 2^{3-1} design, an interaction between A and B, and an effect of C, will have same effect on data:

A	B	C
-	-	+
-	+	-
+	-	-
+	+	+

	-	+
-	2	3
+	4	14

- Also: The interaction AC is confounded with the effect of B, and the interaction BC is confounced with the effect of A.
- So, how can we keep track of what we can estimate, and what not?

Theory for fractional designs

- Formally define the multiplication AB of factors A and B by multiplying the signs at each experiment.
- The multiplication rule is associative (AB)C=A(BC) and commutative AB=BA
- It has an identity I consisting of + for every experiment: for any A, AI=A.
- For any factor A, we have AA=I
- These rules of calculation can be used to find out which effects and interactions are identified!

Example

- 2^{3-1} design generated by A, B, and C=AB
- We get ABC=I
- Also: AC=B and BC=A
- Conclusions:
  - Main effect A confounded with interaction BC
  - Main effect B confounded with interaction AC
  - Main effect C confounded with interaction AB

Example

- 2^{4-1} design generated by A, B, C, and D=ABC
- We get ABCD=I
- Also: A=BCD, B=ACD, C=ABD, D=ABC, AB=CD, AC=BD, and AD=BC.
- Conclusions:
  - Main effect A confounded with interaction BCD
  - Main effect B confounded with interaction ACD
  - Main effect C confounded with interaction ABD
  - Main effect D confounded with interaction ABC
  - Interactions AB and CD confounded
  - Interactions AC and BD confounded
  - Interactions AD and BC confounded

Example

- 2^{4-1} design generated by A, B, C, and D=BC
- We get BCD=I
- In general: BC=D, BD=C, CD=B, ABC=AD, ABD=AC, ACD=AB, BCD=I, ABCD=A.
- Conclusions:
  - Main effect A confounded with ABCD
  - Main effect B confounded with interaction CD
  - Main effect C confounded with interaction BD
  - Main effect D confounded with interaction BC
  - Interactions ABC and AD confounded
  - Interactions ABD and AC confounded
  - Interactions ACD and AB confounded

Conclusions

- Different designs will have different properties and abilities to detect interactions.
- Choice of design should be made based on the context of the experiment.

Classification of designs.

- The designs above is defined by the ”defining relations”, like ABC=I or ABCD=I.
- The ”resolution” is the smallest set of letters in an equation identifying effects.
- It is denoted with Roman numerals:
- The three examples above can be denoted

$$2^{3-1}_{III} \quad 2^{4-1}_{IV} \quad 2^{4-1}_{III}$$

Example: A 2^{5-2}_{III} design

- Generated by A, B, C, D=AB, E=AC
- Resolution: III
- 16-run
- 2-way interactions confounded by main effects: AB=D, AC=E, AD=B, AE=C, BD=A, CE=A
- 2-way interactions NOT confounded by main effects: BC, BE, CD, DE.

Extending designs

- Factorial expeirments are often part of explorative research.
- Next step is often to extend the experiment in a direction suggested by the data.
- Example:

Example

- A 2^{3-1} design visited before:

A	B	C
-	-	+
-	+	-
+	-	-
+	+	+

A \ B	-	+
-	2	3
+	4	14

- Data indicates either a strong effect of C, or a strong interaction effect AB. Which is it?
- Run 4 more experiments, but with the sign of C switched compared to the first 4 runs.
- C and AB can now be estimated independently.

Foldovers

- You want to know more about a factor X and its interactions
- Repeat all experiments, but with sign of X switched
- You get a new design where new and old data can be analysed jointly
- To get a description of the new design: Rewrite all defining relations so that on ly one of them contains the folded factor X. Then remove that defining relation.
- Result: No interaction containing X is confounded with an interaction not containing X. Two-way interactions with X are confounded at most with other higher-order interactions with X.

Example, cont.

A	B	C	
-	-	+	2
-	+	-	3
+	-	-	4
+	+	+	14
-	-	-	3
-	+	+	10
+	-	+	7
+	+	-	1

- Main effect of C: 5.5
- Main effect of AB: -1
- Note: This is a standard 2^3 design, even if rows are permuted

Repeated fractional factorial designs

- Generally, defeats purpose of fractional design
- When some factors are ”declared inert”, we can get a repeated design by reinterpreting the data.
- Once this is true, we can use some of the extra degrees of freedom to estimate variance, and find standard errors of effect estimates.
- May be better to get variance estimates from separate experiments.