Solutions

(1) $X_i, i=1,\dots,n$ iid $U[-\theta,\theta]$ $\theta>0$

Find MLE and show it is consistent.

$$L(\theta|x^n)=\prod_{i=1}^{n}\frac{1}{2\theta}1\{-\theta\le x_i\le\theta\} \quad \left(\frac{1}{2\theta}\right)^n\underbrace{\prod_{i=1}^{n}1\{|x_i|\le\theta\}}$$

if any of $1\{|x_i|\le\theta\}$

is true – then product is 0

so max if all $1\{|x_i|\le\theta\}$ true

$\Rightarrow$ Need $1\{\max|X_i|\le\theta\}$

$$\Rightarrow L(\theta|x^n)=\left(\frac{1}{2\theta}\right)^n 1\{\max\{|X_1|,\dots,|X_n|\}\le\theta\}$$

$\Rightarrow$ maximized for $\hat\theta=\max\{|X_1|,\dots,|X_n|\}$

[Compare $U[0,\theta]$ problem
Also think of extremes in both directions informative about θ
Compare $U[-\theta_1,\theta_2]$ $\theta_1,\theta_2>0$ then here clearly $\hat\theta_1=\min(X_i)$
$\hat\theta_2=\max(X_i)$]

Consistency $\hat\theta=\max\{|X_1|,\dots,|X_n|\}$

Want to show $P(|\hat\theta-\theta|>\varepsilon)\to 0$ as $n\to\infty$

Now, if $X_i\sim U[-\theta,\theta]$, $|X_i|\sim U[0,\theta]$ and $\hat\theta=\max(|X_i|)$

$=\max(Y_i)$

where $Y_i\sim U[0,\theta]$

$$\Rightarrow P(|\hat\theta-\theta|>\varepsilon)=P(|\max(Y_i)-\theta|>\varepsilon)=P(\max(Y_i)<\theta-\varepsilon)$$

$$=P(Y_i<\theta-\varepsilon \text{ all } i)=\prod_{i=1}^{n}P(Y_i<\theta-\varepsilon)=\left(\frac{\theta-\varepsilon}{\theta}\right)^n=\left(1-\frac{\varepsilon}{\theta}\right)^n\longrightarrow 0 \text{ as } n\to\infty \;\#$$

or can work with $|X_i|$ directly

$$P(|\hat{\theta}-\theta|>\varepsilon) = P(|\max|X_i| - \theta| > \varepsilon) = P(\underbrace{\max|X_i| > \theta+\varepsilon}_{\text{impossible}} \text{ or } \max|X_i| < \theta-\varepsilon)$$

$$= P(\max|X_i| < \theta - \varepsilon) = P(|X_i| < \theta - \varepsilon \text{ all } i)$$

$$= P(-\theta+\varepsilon < X_i < \theta-\varepsilon \text{ all } i) = \left(\frac{(\theta-\varepsilon)-(-\theta+\varepsilon)}{2\theta}\right)^n = \left(\frac{2\theta-2\varepsilon}{2\theta}\right)^n = \left(1-\frac{\varepsilon}{\theta}\right)^n \to 0$$

as $n \to \infty$

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(2) $p_\theta(x) = \frac{1}{\theta} f\left(\frac{x}{\theta}\right)$

$$FI(\theta) = \int \left(\frac{\partial}{\partial\theta} \log\left(\frac{1}{\theta} f\left(\frac{x}{\theta}\right)\right)\right)^2 \frac{1}{\theta} f\left(\frac{x}{\theta}\right) dx$$

$$= \int \left(\frac{1}{1/\theta} \cdot \frac{-1}{\theta^2} + \frac{\partial}{\partial\theta} \log f\left(\frac{x}{\theta}\right)\right)^2 \frac{1}{\theta} f\left(\frac{x}{\theta}\right) dx = \left[\begin{matrix} y = \frac{x}{\theta} \quad dy = \frac{1}{\theta} dx \\ \text{change of variables} \end{matrix}\right]$$

$$= \int \left(-\frac{1}{\theta} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} \log f(y)\right)^2 f(y)\, dy = \left[\frac{\partial y}{\partial\theta} = \frac{\partial\left(\frac{x}{\theta}\right)}{\partial\theta} = \frac{-x}{\theta^2} = -\frac{y}{\theta}\right]$$

$$= \int \left(-\frac{1}{\theta} - \frac{y}{\theta}\frac{f'(y)}{f(y)}\right)^2 f(y)\, dy = \frac{1}{\theta^2}\int \left(1 + \frac{f'(y)}{f(y)}\right)^2 f(y)\, dy$$

$$= \frac{\text{constant}}{\theta^2}$$

(3) $X \sim Be(\theta)$ $\quad \theta = \{\frac{1}{3}, \frac{2}{3}\}$ $\quad$ Find MLE

$$\begin{cases} P(X=1) = \theta \\ P(X=0) = 1-\theta \end{cases} \Rightarrow L(\theta|X) = \theta^X (1-\theta)^{1-X}$$

$$= \begin{cases} \left(\frac{1}{3}\right)^X \left(\frac{2}{3}\right)^{1-X} & \text{if } \theta = \frac{1}{3} \\ \left(\frac{2}{3}\right)^X \left(\frac{1}{3}\right)^{1-X} & \text{if } \theta = \frac{2}{3} \end{cases}$$

Consider cases $X=0$, $X=1$ observed

$$L(\theta|X=0) = \begin{cases} \frac{2}{3} & \text{if } \theta = \frac{1}{3} \\ \frac{1}{3} & \text{if } \theta = \frac{2}{3} \end{cases} \Rightarrow \hat{\theta} = \begin{cases} \frac{1}{3} & \text{if } X=0 \\ \frac{2}{3} & \text{if } X=1 \end{cases}$$

$$L(\theta|X=1) = \begin{cases} \frac{1}{3} & \text{if } \theta = \frac{1}{3} \\ \frac{2}{3} & \text{if } \theta = \frac{2}{3} \end{cases}$$