HW 2 $X \sim \text{Poisson}(\theta)$

$$\begin{cases} \hat{\theta}_{MLE} = X \\ \eta = e^{-\theta} \Rightarrow \text{MLE} \quad \hat{\eta} = e^{-\hat{\theta}} = e^{-X} \end{cases}$$

For n,
$$\begin{cases} \ell(\theta|x^n) = \text{constant} - n\theta + \sum x_i \log\theta \\ \frac{\partial}{\partial\theta}\ell(\theta|x^n) = -n + \frac{\sum x_i}{\theta} = 0 \Rightarrow \hat{\theta} = \bar{x} \\ V(\bar{x}) = \frac{V(X)}{n} = \frac{\theta}{n} = \mathcal{I}_n(\theta)^{-1} \quad \text{since } \frac{\partial^2}{\partial\theta^2}\ell(\theta|x^n) = -\frac{\sum x_i}{\theta^2} \end{cases}$$

$$\mathcal{I}_n(\theta) = \frac{n\theta}{\theta^2} = \frac{n}{\theta}$$

Consider $g(\theta) = e^{-\theta} = \eta$

$$\log p_\psi(x) = \log(-\log\psi) \sum x_i + n\log\psi + \text{constant}$$

$$\frac{\partial}{\partial\psi} \quad \frac{n}{\psi} + \frac{\sum x_i}{\psi\log\psi} = 0 \quad \Rightarrow \log\psi = -\bar{x} \quad \Rightarrow \psi = e^{-\bar{x}} = g(\hat{\theta})$$

UMVUE $n=1$ $\Rightarrow$ $T(X) = \mathbb{1}\{X=0\}$ is an unbiased estimate of $g(\theta)$

since $E[\mathbb{1}\{X=0\}] = P(X=0) = e^{-\theta}$

$$V(T(X)) = e^{-\theta}(1-e^{-\theta})$$

Information bound $\frac{(g'(\theta))^2}{\mathcal{I}(\theta)} = \theta e^{-2\theta} \leq V(T(X))$

~~General~~ n

$$\ell(\theta|x^n) = \text{constant} - n\theta + (\log(\theta)) \underbrace{\sum x_i}_{T(x^n)}$$

For general n

$$S(X^n) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{X_i = 0\}$$ is an unbiased estimator for $g(\theta) = e^{-\theta}$ (2)

but not UMVUE since not a function of $\sum X_i$

$$E(S(X^n) \mid T(X^n)) = E\left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{X_i = 0\} \mid \sum X_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(\mathbb{1}\{X_i = 0\} \mid \sum X_i)$$

Fact $X_i \mid \sum X = \sum x \sim \mathrm{Bin}(\sum x, \frac{1}{n})$ — distribute equally likely across all.

$X \sim$ Poisson

(Note, not indep $X_i \mid \sum X$ of course but we don't need this to compute the expected value)

$$E(S(X^n) \mid T(X^n)) = \frac{1}{n} n \cdot E(\mathbb{1}\{X_i = 0\} \mid \sum X = \sum x)$$

$$= \frac{1}{\cancel{n}} \cdot \cancel{n} \left(1 - \frac{1}{n}\right)^{\sum x} = \left(1 - \frac{1}{n}\right)^{\sum x}$$

$$= \left(1 - \frac{1}{n}\right)^{n\bar{x}}$$

as $n \to \infty$ $\bar{x} \to \mu$ $\left(1 - \frac{1}{n}\right)^n \Rightarrow e^{-1}$ so MLE $e^{-\bar{X}} \approx$ UMVUE for large n.

HW3

(1) $T(X^n) = \bar{X}$ estimate of μ

$\hat{b} = (n-1)(\bar{T} - T)$ where $\bar{T} = \frac{1}{n}\sum T_i$, $T_i = \bar{X}_{-i}$

$$\bar{T} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n-1}\sum_{j\neq i} X_j$$

$$= \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i} X_j = \frac{1}{n(n-1)}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} X_j - X_i\right)$$

$$= \frac{1}{n(n-1)}\left(n\cdot\sum_{j=1}^{n} X_j - \sum_{i=1}^{n} X_i\right) = \frac{1}{n-1}\sum_{j=1} X_j - \frac{1}{n(n-1)}\sum X_i = \frac{n}{n-1}\bar{X} - \frac{n}{n(n-1)}\bar{X}$$

$$= \frac{(n^2-n)\bar{X}}{n(n-1)} = \bar{X}$$

$\hat{b} = (n-1)(\bar{X} - \bar{X}) = 0$ ¥

② 3

- mle $e^{-\bar{x}} = g(\hat{\theta}) = e^{-\hat{\theta}}$
- umvue $\left(1-\frac{1}{n}\right)^{n\bar{x}} = \widehat{e^{-\theta}}$

a) jackknife $\hat{b} = (n-1)(\bar{T} - T)$ here $T = e^{-\bar{x}}$

$$\bar{T} = \frac{1}{n}\sum_{i=1}^{n} T_i \quad , \quad T_i = T(x^{-i}) = e^{-\bar{x}^{-i}}$$

$$\bar{T} = \frac{1}{n}\sum_{i=1}^{n} e^{-\bar{x}^{-i}}$$

$$\hat{b} = (n-1)\left(\frac{1}{n}\sum_{i=1}^{n} e^{-\bar{x}^{-i}} - e^{-\bar{x}}\right)$$

$$= (n-1)\left(\frac{e^{-\frac{n}{n-1}\bar{x}}}{n}\sum_{i=1}^{n} e^{x_i/n-1} - e^{-\bar{x}}\right)$$

Corrected mle $e^{-\bar{x}} - \hat{b} =$

b) see code – bias almost gone!

c) } see code
d) }