MSF100 **Final**

March 8

Do not answer questions with "yes" or "no". Make sure you give all your partial results and fully motivate your answers.

Question 1 (20p)

Let $X_1, \cdots, X_n$ be a sample from the beta distribution $\beta(\theta, 1)$, i.e. $f_\theta(x) = \theta x^{\theta-1}$, $\theta > 0$, $0 < x < 1$.
(a) Find the MLE of $1/\theta$. Is it unbiased? Calculate the information inequality lower bound and check whether the MLE achieves the lower bound?
(b) Find an unbiased estimate of $\theta/(\theta+1)$ (Hint, consider the moments of this beta distribution).
Does the unbiased estimate achieve the information inequality lower bound?

Question 2 (20p)

Let $X_i, i = 1, \cdots, n$ be distributed $f_\theta(x) = \theta e^{-\theta x}$, $\theta > 0$, $x > 0$.
(a) Find the MLE of θ.
(b) Find the asymptotic distribution of $\hat{\theta}(x^n)$.
(c) Let $Z_i = 1\{X_i \leq c\}$ for some known constant c. Find the MLE of θ based on $Z_i, i = 1, \cdots, n$.
(d) Find the asymptotic distribution of $\hat{\theta}(z^n)$.
(e) Compare the asymptotic efficiencies of the estimates and comment on the meaning of this result.

Question 3 (20p)

Let $X_i, i = 1, \cdots, n$ be distributed as $f_\theta(x) = \theta e^{-\theta x}, \theta > 0, x > 0$.
(a) Show that $\hat{\theta} = n \times min(X)$ is an unbiased estimator for θ.
(b) What is the variance of $\hat{\theta}$? Does this estimator achieve the CR lower bound?
(c) Is $\hat{\theta}$ a consistent estimator of θ?

Question 4 (20p)

Let $Y_i, i = 1, \cdots, n$ be $Poisson(\lambda)$ distributed. Let X_i be such that $X_i = Y_i$ if $Y_i \neq 1$. If $Y_i = 1$, X_i is wrongly recorded as 0 with probability θ and 1 otherwise.

(a) Write down the density for X. (Try to simplify this to only depend on the indicators for $Y_i = 1$ and $Y_i = 0$, it will help with b and c).
(b) Write down the loglikelihood. What are the sufficient statistics for θ and λ?
(b) Find the MLE for λ and θ (θ is a bit messy, so start with λ given θ is known).

Question 5 (20p)

Let $X_i, i = 1, \cdots, n$ be distributed as $f_\theta(x) = (\theta + 1)x^\theta$, $0 < x < 1$, $\theta > -1$.
(a) Find the MLE of θ and its asymptotic distribution.
(b) The asymptotic variance of $\hat{\theta}$ in (a) depends on θ. Find a *variance stabilizing* transformation $g(\theta)$ such that the asymptotic distribution of the MLE $g(\theta)$ does not depend on θ.