Solution

(1) a) $\log f_\theta(x) = \log(\theta) + (\theta-1)\log(x)$

$$l(\theta) = n\log(\theta) + (\theta-1)\sum_{i=1}^{n}\log(x_i)$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum \log(x_i) \Rightarrow \boxed{\widehat{\left(\frac{1}{\theta}\right)} = -\frac{\sum \log(x_i)}{n} = -\overline{\log(x)}} \qquad \#$$

$$\frac{\partial^2 l}{\partial \theta^2} = -\frac{n}{\theta^2}, \quad g(\theta) = \frac{1}{\theta}, \quad g'(\theta) = -\frac{1}{\theta^2} \Rightarrow \boxed{\text{Asymptotic variance } (g'(\theta))^2 \cdot \frac{\theta^2}{n} = \frac{1}{n\theta^2}}$$

$$V(-\log(X)) = E((-\log(X))^2) - (E(-\log(X)))^2$$

$$E(-\log(X)) = \int_0^1 -\log x \, \theta x^{\theta-1}\,dx = \left[-\log x \, x^\theta\right]_0^1 - \int_0^1 -\frac{1}{x} x^\theta \,dx = \left[\frac{x^\theta}{\theta}\right]_0^1 = \frac{1}{\theta}$$

unbiased #

$$E((-\log(x_i))^2) = \frac{2}{\theta^2} \Rightarrow V(-\log(x_i)) = \frac{1}{\theta^2} \Rightarrow V(-\overline{\log(x)}) = \frac{1}{n\theta^2}$$

$= \underline{\underline{\text{band}}}$ #

b) $E(X) = \int_0^1 x\,\theta x^{\theta-1}\,dx = \frac{\theta}{\theta+1}$ so $\bar{X}$ is an unbiased estimate of $\frac{\theta}{\theta+1}$ #

$$V(\bar{X}) = \frac{V(X)}{n} = \frac{E(X^2) - (E(X))^2}{n} = \frac{\theta}{n(\theta+1)^2(\theta+2)}$$

$$g(\theta) = \frac{\theta}{\theta+1}, \quad g'(\theta) = \frac{1}{(\theta+1)^2} \Rightarrow \text{band } \frac{\theta^2}{n(\theta+1)^4}$$

$$V(\bar{X}) > \frac{\theta^2}{n(\theta+1)^4}$$

so $\bar{X}$ does not achieve bound for estimate of $\frac{\theta}{\theta+1}$ #

($\bar{X}$ not a linear function of $\log$)

(2) a) $\ell(\theta) = n\log(\theta) - \theta\sum x_i$

$$\frac{\partial \ell}{\partial\theta} = \frac{n}{\theta} - \sum x_i \Rightarrow \hat\theta = \frac{1}{\bar x} \quad \#$$

b) $$\frac{\partial^2\ell}{\partial\theta^2} = -\frac{n}{\theta^2} \Rightarrow \sqrt{n}(\hat\theta-\theta) \xrightarrow{d} N(0,\theta^2) \quad \#$$

c) $Z_i = 1\{X_i \le c\} \Rightarrow P(Z_i=1) = 1-e^{-\theta c}$

$P(Z_i=0)=e^{-\theta c}$ $\qquad Z_i \sim Be(1-e^{-\theta c})$

$$L_Z(\theta)=\prod_{i=1}^n (1-e^{-\theta c})^{z_i}(e^{-\theta c})^{(1-z_i)}$$

$$\ell_Z(\theta) = \left(\sum z_i\right)\log\left(\frac{1-e^{-\theta c}}{e^{-\theta c}}\right) - n\theta c$$

$$\frac{\partial \ell_Z(\theta)}{\partial\theta} = -nc + \left(\sum z_i\right)\frac{c}{1-e^{-\theta c}} \Rightarrow \hat\theta = -\frac{1}{c}\log(1-\bar z) \quad //$$

d) $$\frac{\partial^2 \ell_Z(\theta)}{\partial\theta^2} = \frac{-c^2e^{-\theta c}}{(1-e^{-\theta c})^2}\sum z_i, \quad E\left(-\frac{\partial^2\ell_Z(\theta)}{\partial\theta^2}\right) = \frac{c^2e^{-\theta c}\; nE(Z)}{(1-e^{-\theta c})^2}$$

$$= \frac{nc^2e^{-\theta c}}{(1-e^{-\theta c})}$$

$$\Rightarrow \sqrt n(\hat\theta_Z-\theta)\xrightarrow{d} N\left(0, \frac{1-e^{-\theta c}}{c^2e^{-\theta c}}\right)$$ (compare with θ^2

e) Is $\dfrac{1-e^{-\theta c}}{c^2e^{-\theta c}} > \theta^2$? Since θc appears everywhere in this expression, can consider just one value for θ, e.g. 1

$$\frac{1-e^{-c}}{c^2e^{-c}} > 1 ? \Rightarrow c^2 < e^c - 1 ?$$ Yes, always

<u>Meaning</u>: $AsVar(\hat\theta_Z) > AsVar(\hat\theta_X)$, so rounding off X to $Z \Rightarrow$ loss of information.

③ a) Distribution of n·min X ?

$$P(\min(X) > t) = P(\text{all } X > t) = (e^{-t/\theta})^n = e^{-nt/\theta}$$

$$P(n \cdot \min(X) > t) = P(\min(X) > \tfrac{t}{n}) = e^{-t/\theta}$$

So $n \min(X) =^d X$

$\Rightarrow E(n\min(X)) = \theta = E(X)$

b) $V(n\min(X)) = V(X) = \theta^2$

c) Since $V(n\min(X))$ indep of n, $\hat{\theta}$ cannot be consistent.

④ a)

$$P_{\lambda,\theta}(x) = \begin{cases} \dfrac{\lambda^x e^{-\lambda}}{x!} & x > 1 \\ \lambda e^{-\lambda}(1-\theta) & x = 1 \\ e^{-\lambda} + \theta\lambda e^{-\lambda} & x = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\lambda^x e^{-\lambda}}{x!} & x > 1 \\ \dfrac{\lambda^x e^{-\lambda}}{x!}(1-\theta) & x = 1 \\ \dfrac{\lambda^x e^{-\lambda}}{x!}(1+\theta\lambda) & x = 0 \end{cases}$$

$$P_{\lambda,\theta}(x) = \frac{\lambda^x e^{-\lambda}}{x!}(1+\theta\lambda)^{1\{x=0\}}(1-\theta)^{1\{x=1\}}$$

b) $\log P_{\lambda,\theta}(x) = x\log(\lambda) - \log x! - \lambda + 1\{x=0\}\log(1+\theta\lambda) + 1\{x_i=1\}\log(1-\theta)$

$$\ell(\theta) = (\textstyle\sum x_i)\log(\lambda) - n\lambda + \log(1+\theta\lambda)\sum 1\{x_i=0\} + \log(1-\theta)\sum 1\{x_i=1\} + \text{constant}$$

sufficient statistics

c) $$\begin{cases} \dfrac{\partial l}{\partial \lambda} = \dfrac{\sum x_i}{\lambda} - n + \dfrac{\theta}{1+\theta\lambda} \sum 1\{x_i = 0\} = 0 & (1) \\ \dfrac{\partial l}{\partial \theta} = \dfrac{\lambda}{1+\theta\lambda} \sum 1\{x_i = 0\} - \dfrac{1}{1-\theta} \sum 1\{x_i = 1\} = 0 & (2) \end{cases}$$

Denote $\frac{1}{n}\sum 1(x_i = 0) = p_0$

$\frac{1}{n}\sum 1\{x_i = 1\} = p_1$

Take $\lambda(1) - \theta(2)$ $\Rightarrow$ $\bar{x} - \lambda + \dfrac{\theta}{1-\theta} p_1 = 0$

$$\Rightarrow \boxed{\hat{\lambda} = \bar{x} + \frac{\hat{\theta}}{1-\hat{\theta}} p_1}$$

So, can solve for λ given θ. Now plug in $\lambda = \lambda(\theta)$ into (2)

$$\Rightarrow \frac{\lambda}{1+\theta\lambda} p_0 = \frac{1}{1-\theta} p_1 \Rightarrow (p_0 + p_1)\theta\lambda + p_1 - p_0\lambda = 0$$

$$\Rightarrow \theta^2\left((p_1 - \bar{x})(p_0 + p_1)\right) + \theta\left((2p_0 + p_1)\bar{x} - p_1(1+p_0)\right) + (p_1 - p_0\bar{x}) = 0$$

$\rightarrow$ solve quadratic equation for θ.

Then solve for λ

(5) a) $\log f_\theta(x) = \theta \log(x) + \log(\theta+1)$

$l(\theta) = \theta \sum \log(x_i) + n \log(\theta+1)$

$$\frac{\partial l}{\partial \theta} = \sum \log(x_i) + \frac{n}{\theta+1} = 0$$

$$\Rightarrow \hat{\theta} = -\left(1 + \frac{1}{\overline{\log(x)}}\right) \quad \#$$

$$\frac{\partial^2 l}{\partial \theta^2} = \frac{-2n}{(\theta+1)^2} \Rightarrow \text{As. Variance } \frac{(\theta+1)^2}{2n} \quad \#$$

b) Want $g(\theta)$ s.t. $g'(\theta) \frac{(\theta+1)^2}{2n} g'(\theta) = \text{constant}$

$\Rightarrow g'(\theta) = \frac{1}{\theta+1}$ works

$\Rightarrow g(\theta) = \log(\theta+1)$ #