Probability and Random Processes

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Abstract

Lecture notes based on the book Probability and Random Processes by Geoffrey Grimmett and David Stirzaker. In this text the formula labels $(*)$, $(**)$ operate locally.

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Contents

1 Random events and random variables

1.1 Probability space

A random experiment is modeled in terms of a *probability space* $(\Omega, \mathcal{F}, \mathrm{P})$

- the *sample space* Ω is the set of all possible outcomes of the experiment,
- the *σ-field* (or sigma-algebra) $\mathcal{F}$ is a collection of measurable subsets $A \subset \Omega$ (which are called *random events*) satisfying
  1. $\emptyset \in \mathcal{F}$,
  2. if $A_i \in \mathcal{F}$, $0 = 1, 2, \ldots$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$, countable unions,
  3. if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, *complementary event*,

- the probability measure P is a function on $\mathcal{F}$ satisfying three probability axioms
    1. if $A \in \mathcal{F}$, then $\mathrm{P}(A) \geq 0$,
    2. $\mathrm{P}(\Omega) = 1$,
    3. if $A_i \in \mathcal{F}$, $0 = 1, 2, \ldots$ are all disjoint, then $\mathrm{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathrm{P}(A_i)$.

De Morgan laws

$$\Big(\bigcap_i A_i\Big)^c = \bigcup_i A_i^c, \qquad \Big(\bigcup_i A_i\Big)^c = \bigcap_i A_i^c.$$

Properties derived from the axioms

$$\begin{aligned}
\mathrm{P}(\emptyset) &= 0,\\
\mathrm{P}(A^c) &= 1 - \mathrm{P}(A),\\
\mathrm{P}(A \cup B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B).
\end{aligned}$$

Inclusion-exclusion rule

$$\begin{aligned}
\mathrm{P}(A_1 \cup \ldots \cup A_n) = \sum_i \mathrm{P}(A_i) - \sum_{i<j} \mathrm{P}(A_i \cap A_j) + \sum_{i<j<k} \mathrm{P}(A_i \cap A_j \cap A_k) - \ldots\\
+ (-1)^{n+1} \mathrm{P}(A_1 \cap \ldots \cap A_n).
\end{aligned}$$

Continuity of the probability measure

- if $A_1 \subset A_2 \subset \ldots$ and $A = \cup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i$, then $\mathrm{P}(A) = \lim_{i \to \infty} \mathrm{P}(A_i)$,
- if $B_1 \supset B_2 \supset \ldots$ and $B = \cap_{i=1}^{\infty} B_i = \lim_{i \to \infty} B_i$, then $\mathrm{P}(B) = \lim_{i \to \infty} \mathrm{P}(B_i)$.

1.2 Conditional probability and independence

If $\mathrm{P}(B) > 0$, then the conditional probability of A given B is

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

The law of total probability and the Bayes formula. Let $B_1, \ldots, B_n$ be a partition of Ω, then

$$\begin{aligned}
\mathrm{P}(A) &= \sum_{i=1}^{n} \mathrm{P}(A|B_i)\mathrm{P}(B_i),\\
\mathrm{P}(B_j|A) &= \frac{\mathrm{P}(A|B_j)\mathrm{P}(B_j)}{\sum_{i=1}^{n} \mathrm{P}(A|B_i)\mathrm{P}(B_i)}.
\end{aligned}$$

Definition 1.1 Events $A_1, \ldots, A_n$ are independent, if for any subset of events $(A_{i_1}, \ldots, A_{i_k})$

$$\mathrm{P}(A_{i_1} \cap \ldots \cap A_{i_k}) = \mathrm{P}(A_{i_1}) \ldots \mathrm{P}(A_{i_k}).$$

Example 1.2 Pairwise independence does not imply independence of three events. Toss two coins and consider three events

- $A =$\{heads on the first coin\},
- $B =$\{tails on the first coin\},
- $C =$\{one head and one tail\}.

Clearly, $\mathrm{P}(A|C) = \mathrm{P}(A)$ and $\mathrm{P}(B|C) = \mathrm{P}(B)$ but $\mathrm{P}(A \cap B|C) = 0$.

1.3 Random variables

A real random variable is a measurable function $X : \Omega \to \mathbb{R}$ so that different outcomes $\omega \in \Omega$ can give different values $X(\omega)$. Measurability of $X(\omega)$:

$$\{\omega : X(\omega) \le x\} \in \mathcal{F} \text{ for any real number } x.$$

Probability distribution $\mathrm{P}_X(B) = \mathrm{P}(X \in B)$ defines a new probability space $(\mathbb{R}, \mathcal{B}, \mathrm{P}_X)$, where $\mathcal{B} = \sigma$(all open intervals) is the Borel sigma-algebra.

Definition 1.3 Distribution function (cumulative distribution function)

$$F(x) = F_X(x) = \mathrm{P}_X\{(-\infty, x]\} = \mathrm{P}(X \le x).$$

In terms of the distribution function we get

$$\begin{aligned}
\mathrm{P}(a < X \le b) &= F(b) - F(a), \\
\mathrm{P}(X < x) &= F(x-), \\
\mathrm{P}(X = x) &= F(x) - F(x-).
\end{aligned}$$

Any monotone right-continuous function with

$$\lim_{x \to -\infty} F(x) = 0 \text{ and } \lim_{x \to \infty} F(x) = 1$$

can be a distribution function.

Definition 1.4 The random variable X is called discrete, if for some countable set of possible values

$$\mathrm{P}(X \in \{x_1, x_2, \ldots\}) = 1.$$

Its distribution is described by the probability mass function $f(x) = \mathrm{P}(X = x)$.

The random variable X is called (absolutely) continuous, if its distribution has a probability density function $f(x)$:

$$F(x) = \int_{-\infty}^{x} f(y)dy, \quad \text{for all } x,$$

so that $f(x) = F'(x)$ almost everywhere.

Example 1.5 The indicator of a random event $1_A = 1_{\{\omega \in A\}}$ with $p = \mathrm{P}(A)$ has a Bernoulli distribution

$$\mathrm{P}(1_A = 1) = p, \quad \mathrm{P}(1_A = 0) = 1 - p.$$

For several events $S_n = \sum_{i=1}^{n} 1_{A_i}$ counts the number of events that occurred. If independent events $A_1, A_2, \ldots$ have the same probability $p = \mathrm{P}(A_i)$, then S_n has a binomial distribution $\mathrm{Bin}(n, p)$

$$\mathrm{P}(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

Example 1.6 Consider a sequence of Bernoulli trials, that is independent experiments with two possible outcomes: success with probability p and failure with probability $q = 1 - p$. Let X be the number of failures until the first success. Then X has a geometric distribution $\mathrm{Geom}(p)$:

$$\mathrm{P}(X = k) = q^k p, \quad k = 0, 1, \ldots$$

The number of trials $Y = X + 1$ until the first success has a shifted geometric distribution $\mathrm{Geom}_1(p)$:

$$\mathrm{P}(Y = k) = q^{k-1} p, \quad k = 1, 2, \ldots$$

Example 1.7 (Cantor distribution) Consider $(\Omega, \mathcal{F}, \mathrm{P})$ with $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$, and

$$\begin{aligned}
&\mathrm{P}([0, 1]) = 1 \\
&\mathrm{P}([0, 1/3]) = \mathrm{P}([2/3, 1]) = 2^{-1} \\
&\mathrm{P}([0, 1/9]) = \mathrm{P}([2/9, 1/3]) = \mathrm{P}([2/3, 7/9]) = \mathrm{P}([8/9, 1]) = 2^{-2}
\end{aligned}$$

and so on. Put $X(\omega) = \omega$, its distribution, called the Cantor distribution, is neither discrete nor continuous. Its distribution function, called the Cantor function, is continuous but not absolutely continuous.

1.4 Random vectors

Definition 1.8 The joint distribution of a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is the function

$$F_{\mathbf{X}}(x_1, \ldots, x_n) = \mathrm{P}(\{X_1 \le x_1\} \cap \ldots \cap \{X_n \le x_n\}).$$

Marginal distributions

$$\begin{aligned} F_{X_1}(x) &= F_{\mathbf{X}}(x, \infty, \ldots, \infty), \\ F_{X_2}(x) &= F_{\mathbf{X}}(\infty, x, \infty, \ldots, \infty), \\ &\ldots \\ F_{X_n}(x) &= F_{\mathbf{X}}(\infty, \ldots, \infty, x). \end{aligned}$$

The existence of the joint probability density function $f(x_1, \ldots, x_n)$ means that the distribution function

$$F_{\mathbf{X}}(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \ldots \int_{-\infty}^{x_n} f(y_1, \ldots, y_n) dy_1 \ldots dy_n, \quad \text{for all } (x_1, \ldots, x_n),$$

is absolutely continuous, so that $f(x_1, \ldots, x_n) = \frac{\partial^n F(x_1, \ldots, x_n)}{\partial x_1 \ldots \partial x_n}$ almost everywhere.

Definition 1.9 Random variables $(X_1, \ldots, X_n)$ are called independent if for any $(x_1, \ldots, x_n)$

$$\mathrm{P}(X_1 \le x_1, \ldots, X_n \le x_n) = \mathrm{P}(X_1 \le x_1) \ldots \mathrm{P}(X_n \le x_n).$$

In the jointly continuous case this equivalent to

$$f(x_1, \ldots, x_n) = f_{X_1}(x_1) \ldots f_{X_n}(x_n).$$

Example 1.10 In general, the joint distribution can not be recovered form the marginal distributions. If

$$F_{X,Y}(x, y) = xy 1_{\{(x,y) \in [0,1]^2\}},$$

then vectors (X, Y) and (X, X) have the same marginal distributions.

Example 1.11 Consider

$$F(x, y) = \begin{cases} 1 - e^{-x} - xe^{-y} & \text{if } 0 \le x \le y, \\ 1 - e^{-y} - ye^{-y} & \text{if } 0 \le y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $F(x, y)$ is the joint distribution function of some pair (X, Y). Find the marginal distribution functions and densities.

Solution. Three properties should be satisfied for $F(x, y)$ to be the joint distribution function of some pair (X, Y):

1. $F(x, y)$ is non-decreasing on both variables,
2. $F(x, y) \to 0$ as $x \to -\infty$ and $y \to -\infty$,
3. $F(x, y) \to 1$ as $x \to \infty$ and $y \to \infty$.

Observe that

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} = e^{-y} 1_{\{0 \le x \le y\}}$$

is always non-negative. Thus the first property follows from the integral representation:

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) du dv,$$

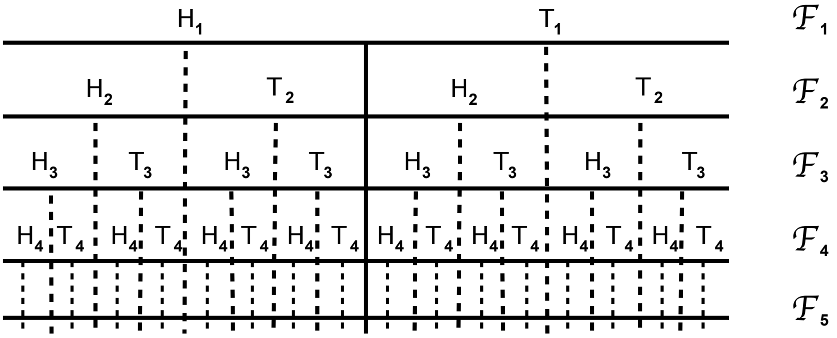


Figure 1: Filtration for four consecutive coin tossings.

which, for $0 \leq x \leq y$, is verifies as

$$\int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)dudv = \int_{0}^{x}\Big(\int_{u}^{y} e^{-v}dv\Big)du = 1 - e^{-x} - xe^{-y},$$

and for $0 \leq y \leq x$ as

$$\int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)dudv = \int_{0}^{y}\Big(\int_{u}^{y} e^{-v}dv\Big)du = 1 - e^{-y} - ye^{-y}.$$

The second and third properties are straightforward. We have shown also that $f(x,y)$ is the joint density.

For $x \geq 0$ and $y \geq 0$ we obtain the marginal distributions as limits

$$F_X(x) = \lim_{y\to\infty} F(x,y) = 1 - e^{-x}, \qquad f_X(x) = e^{-x},$$
$$F_Y(y) = \lim_{x\to\infty} F(x,y) = 1 - e^{-y} - ye^{-y}, \qquad f_Y(y) = ye^{-y}.$$

$X \sim \text{Exp}(1)$ and $Y \sim \text{Gamma}(2,1)$.

1.5 Filtration

Definition 1.12 A sequence of sigma-fields $\{\mathcal{F}_n\}_{n=1}^{\infty}$ such that

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_n \subset \ldots, \quad \mathcal{F}_n \subset \mathcal{F} \text{ for all } n$$

is called a filtration.

To illustrate this definition use an infinite sequence of Bernoulli trials. Let S_n be the number of heads in n independent tosses of a fair coin. Figure 1 shows embedded partitions $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \mathcal{F}_4 \subset \mathcal{F}_5$ of the sample space generated by S_1, S_2, S_3, S_4, S_5.

The events representing our knowledge of the first three tosses is given by $\mathcal{F}_3$. From the perspective of $\mathcal{F}_3$ we can not say exactly the value of S_4. Clearly, there is dependence between S_3 and S_4. The joint distribution of S_3 and S_4:

	$S_4 = 0$	$S_4 = 1$	$S_4 = 2$	$S_4 = 3$	$S_4 = 4$	Total
$S_3 = 0$	1/16	1/16	0	0	0	1/8
$S_3 = 1$	0	3/16	3/16	0	0	3/8
$S_3 = 2$	0	0	3/16	3/16	0	3/8
$S_3 = 3$	0	0	0	1/16	1/16	1/8
Total	1/16	1/4	3/8	1/4	1/16	1

The conditional expectation

$$\mathrm{E}(S_4|S_3) = S_3 + 0.5$$

is a discrete random variable with values $0.5, 1.5, 2.5, 3.5$ and probabilities $1/8, 3/8, 3/8, 1/8$.

For finite n the picture is straightforward. For $n = \infty$ it is a non-trivial task to define an overall $(\Omega, \mathcal{F}, \mathrm{P})$ with $\Omega = (0, 1]$. One can use the Lebesgue measure $\mathrm{P}(dx) = dx$ and the sigma-field $\mathcal{F}$ of Lebesgue measurable subsets of $(0, 1]$. Not all subsets of $(0, 1]$ are Lebesgue measurable.

2 Expectation and conditional expectation

2.1 Expectation

The expected value of X is

$$\mathrm{E}(X) = \int_\Omega X(\omega)\mathrm{P}(d\omega).$$

A discrete r.v. X with a finite number of possible values is a simple r.v. in that

$$X = \sum_{i=1}^n x_i 1_{A_i}$$

for some partition $A_1, \ldots, A_n$ of Ω. In this case the meaning of the expectation is obvious

$$\mathrm{E}(X) = \sum_{i=1}^n x_i \mathrm{P}(A_i).$$

For any non-negative r.v. X there are simple r.v. such that $X_n(\omega) \nearrow X(\omega)$ for all $\omega \in \Omega$, and the expectation is defined as a possibly infinite limit $\mathrm{E}(X) = \lim_{n\to\infty} \mathrm{E}(X_n)$.

Any r.v. X can be written as a difference of two non-negative r.v. $X_1 = X \wedge 0$ and $X_2 = -X \wedge 0$. If at least one of $\mathrm{E}(X_1)$ and $\mathrm{E}(X_2)$ is finite, then $\mathrm{E}(X) = \mathrm{E}(X_1) - \mathrm{E}(X_2)$, otherwise $\mathrm{E}(X)$ does not exist.

Example 2.1 A discrete r.v. with the probability mass function $f(k) = \frac{1}{2k(k-1)}$ for $k = -1, \pm 2, \pm 3, \ldots$ has no expectation.

For a discrete r.v. X with mass function f and any function g

$$\mathrm{E}(g(X)) = \sum_x g(x)f(x).$$

For a continuous r.v. X with density f and any measurable function g

$$\mathrm{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

In general

$$\mathrm{E}(X) = \int_\Omega X(\omega)\mathrm{P}(d\omega) = \int_{-\infty}^{\infty} x\mathrm{P}_X(dx) = \int_{-\infty}^{\infty} xdF(x).$$

Example 2.2 Turn to the example of $(\Omega, \mathcal{F}, \mathrm{P})$ with $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$, and a random variable $X(\omega) = \omega$ having the Cantor distribution. A sequence of simple r.v. monotonely converging to X is $X_n(\omega) = k3^{-n}$ for $\omega \in [(k-1)3^{-n}, k3^{-n})$, $k = 1, \ldots, 3^n$ and $X_n(1) = 1$.

$$\begin{aligned}
&X_1(\omega) = 0, \quad \mathrm{E}(X_1) = 0, \\
&X_2(\omega) = (1/3) \cdot 1_{\{\omega\in[1/3,2/3)\}} + (2/3) \cdot 1_{\{\omega\in[2/3,1]\}}, \quad \mathrm{E}(X_2) = (2/3) \cdot (1/2) = 1/3, \\
&\mathrm{E}(X_3) = (2/9) \cdot (1/4) + (2/3) \cdot (1/4) + (8/9) \cdot (1/4) = 4/9,
\end{aligned}$$

and so on, gives $\mathrm{E}(X_n) \nearrow 1/2 = \mathrm{E}(X)$.

Lemma 2.3 *Cauchy-Schwartz inequality. For r.v. X and Y we have*

$$\big(\mathrm{E}(XY)\big)^2 \le \mathrm{E}(X^2)\mathrm{E}(Y^2)$$

with equality if only if $aX = bY$ a.s. for some non-trivial pair of constants (a, b).

Definition 2.4 Variance, standard deviation, covariance and correlation

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}\big(X - \mathrm{E}X\big)^2 = \mathrm{E}(X^2) - (\mathrm{E}X)^2, \quad \sigma_X = \sqrt{\mathrm{Var}(X)},\\ \mathrm{Cov}(X,Y) &= \mathrm{E}\big(X - \mathrm{E}X\big)\big(Y - \mathrm{E}Y\big) = \mathrm{E}(XY) - (\mathrm{E}X)(\mathrm{E}Y),\\ \rho(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}.\end{aligned}$$

The covariance matrix of a random vector $(X_1, \ldots, X_n)$ with means $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$

$$\mathbf{V} = \mathrm{E}\big(\mathbf{X} - \boldsymbol{\mu}\big)^{\mathrm{t}}\big(\mathbf{X} - \boldsymbol{\mu}\big) = \|\mathrm{Cov}(X_i, X_j)\|$$

is symmetric and nonnegative-definite. For any vector $\mathbf{a} = (a_1, \ldots, a_n)$ the r.v. $a_1X_1 + \ldots + a_nX_n$ has mean $\mathbf{a}\boldsymbol{\mu}^{\mathrm{t}}$ and variance

$$\mathrm{Var}(a_1X_1 + \ldots + a_nX_n) = \mathrm{E}\big(\mathbf{a}\mathbf{X}^{\mathrm{t}} - \mathbf{a}\boldsymbol{\mu}^{\mathrm{t}}\big)\big(\mathbf{X}\mathbf{a}^{\mathrm{t}} - \boldsymbol{\mu}\mathbf{a}^{\mathrm{t}}\big) = \mathbf{a}\mathbf{V}\mathbf{a}^{\mathrm{t}}.$$

If $(X_1, \ldots, X_n)$ are independent, then they are uncorrelated: $\mathrm{Cov}(X_i, X_j) = 0$.

2.2 Conditional expectation and prediction

Definition 2.5 For a pair of discrete random variables (X, Y) the conditional expectation $\mathrm{E}(Y|X)$ is defined as $\psi(X)$, where

$$\psi(x) = \sum_y y\mathrm{P}(Y = y|X = x).$$

Definition 2.6 Consider a pair of random variables (X, Y) with joint density $f(x, y)$, marginal densities

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy, \quad f_2(x) = \int_{-\infty}^{\infty} f(x,y)dx,$$

and conditional densities

$$f_1(x|y) = \frac{f(x,y)}{f_2(y)}, \qquad f_2(y|x) = \frac{f(x,y)}{f_1(x)}.$$

The conditional expectation $\mathrm{E}(Y|X)$ is defined as $\psi(X)$, where

$$\psi(x) = \int_{-\infty}^{\infty} yf_2(y|x)dy.$$

Similarly, one can define a conditional expectation $\mathrm{E}(Y|X,Z)$ of a random variable Y conditioned on a random vector (X, Z).

Properties of conditional expectations:
(i) linearity: $\mathrm{E}(aY + bZ|X) = a\mathrm{E}(Y|X) + b\mathrm{E}(Z|X)$ for any constants (a, b),
(ii) pull-through property: $\mathrm{E}(Yg(X)|X) = g(X)\mathrm{E}(Y|X)$ for any measurable function $g(x)$,
(iii) $\mathrm{E}(Y1_G) = \mathrm{E}(\psi(X)1_G)$ for $G = \{\omega : X(\omega) \in B\}$, where $B \in \mathcal{R}$,
(iv) tower property: $\mathrm{E}(\mathrm{E}(Y|X,Z)|X) = \mathrm{E}(Y|X)$,
(v) total expectation: $\mathrm{E}(\mathrm{E}(Y|X)) = \mathrm{E}(Y)$,
(vi) total variance: $\mathrm{Var}(Y) = \mathrm{Var}(\mathrm{E}(Y|X)) + \mathrm{E}(\mathrm{Var}(Y|X))$.

Proof of (ii) in the discrete case:

$$\begin{aligned}\mathrm{E}(Yg(X)|X) &= \sum_{x,y} yg(x)\mathrm{P}(Y = y, X = x) = \sum_{x,y} g(x)\mathrm{P}(X = x)y\mathrm{P}(Y = y|X = x)\\ &= \sum_x g(x)\psi(x)\mathrm{P}(X = x) = g(X)\mathrm{E}(Y|X).\end{aligned}$$

Definition 2.7 General definition. Let Y be a r.v. on $(\Omega, \mathcal{F}, \mathrm{P})$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. If there exists a $\mathcal{G}$-measurable r.v. Z such that

$$\mathrm{E}((Y-Z)1_G) = 0 \text{ for all } G \in \mathcal{G},$$

then Z is called the conditional expectation of Y given $\mathcal{G}$ and is written $Z = \mathrm{E}(Y|\mathcal{G})$.

Properties of conditional expectations:
(vii) if $\mathrm{E}(Y|\mathcal{G})$ exists, then it is a.s. unique,
(viii) if $\mathrm{E}|Y| < \infty$, then $\mathrm{E}(Y|\mathcal{G})$ exists due to the Radon-Nikodym theorem,
(ix) if $\mathcal{G} = \sigma(X)$, then $\mathrm{E}(Y|X) := \mathrm{E}(Y|\mathcal{G})$ and $\mathrm{E}(Y|X) = \psi(X)$ for some measurable function ψ.

Proof of (viii). Consider the probability space $(\Omega, \mathcal{G}, \mathrm{P})$ and define a finite signed measure $\mathrm{P}_1(G) = \mathrm{E}(Y1_G) = \int_G Y(\omega)\mathrm{P}(d\omega)$ which is absolutely continuous with respect to P. Thus $\mathrm{P}_1(G) = \int_G Z(\omega)\mathrm{P}(d\omega)$ with $Z = \partial \mathrm{P}_1/\partial \mathrm{P}$ being the Radon-Nikodym derivative.

Definition 2.8 Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathrm{P})$ such that $\mathrm{E}(Y^2) < \infty$. The best predictor of Y given the knowledge of X is the function $\hat{Y} = h(X)$ that minimizes $\mathrm{E}((Y - \hat{Y})^2)$.

Let $L^2(\Omega, \mathcal{F}, \mathrm{P})$ be the set of random variables Z on $(\Omega, \mathcal{F}, \mathrm{P})$ such that $\mathrm{E}(Z^2) < \infty$. Define a scalar product on the linear space $L^2(\Omega, \mathcal{F}, \mathrm{P})$ by $\langle U, V\rangle = \mathrm{E}(UV)$ leading to the norm

$$\|Z\| = \langle Z, Z\rangle^{1/2} = (\mathrm{E}(Z^2))^{1/2}.$$

Let H be the subspace of $L^2(\Omega, \mathcal{F}, \mathrm{P})$ of all functions of X having finite second moment

$$H = \{h(X) : \mathrm{E}(h(X)^2) < \infty\}.$$

Geometrically, the best predictor of Y given X is the projection $\hat{Y}$ of Y on H so that

$$\mathrm{E}((Y - \hat{Y})Z) = 0, \text{ for all } Z \in H. \qquad (*)$$

Theorem 2.9 *Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathrm{P})$ such that $\mathrm{E}(Y^2) < \infty$. The best predictor of Y given X is the conditional expectation $\hat{Y} = \mathrm{E}(Y|X)$.*

Proof. Put $\hat{Y} = \mathrm{E}(Y|X)$. We have due to the Jensen inequality $\hat{Y}^2 \le \mathrm{E}(Y^2|X)$ and therefore

$$\mathrm{E}(\hat{Y}^2) \le \mathrm{E}(\mathrm{E}(Y^2|X)) = \mathrm{E}(Y^2) < \infty,$$

implying $\hat{Y} \in H$. To verify (*) we observe that

$$\mathrm{E}((Y - \hat{Y})Z) = \mathrm{E}(\mathrm{E}((Y - \hat{Y})Z|Z)) = \mathrm{E}(\mathrm{E}(Y|X)Z - \hat{Y}Z) = 0.$$

To prove uniqueness assume that there is another predictor $\bar{Y}$ with $\mathrm{E}((Y - \bar{Y})^2) = \mathrm{E}((Y - \hat{Y})^2) = d^2$. Then $\mathrm{E}((Y - \frac{\hat{Y}+\bar{Y}}{2})^2) \ge d^2$ and according to the parallelogram rule

$$2\Big(\|Y - \hat{Y}\|^2 + \|Y - \bar{Y}\|^2\Big) = 4\|Y - \frac{\hat{Y}+\bar{Y}}{2}\|^2 + \|\bar{Y} - \hat{Y}\|^2$$

we have

$$\|\bar{Y} - \hat{Y}\|^2 \le 2\Big(\|Y - \hat{Y}\|^2 + \|Y - \bar{Y}\|^2\Big) - 4d^2 = 0.$$

2.3 Multinomial distribution

De Moivre trials: each trial has r possible outcomes with probabilities $(p_1, \ldots, p_r)$. Consider n such independent trials and let $(X_1, \ldots, X_r)$ be the counts of different outcomes. Multinomial distribution $\mathrm{Mn}(n, p_1, \ldots, p_r)$

$$\mathrm{P}(X_1 = k_1, \ldots, X_r = k_r) = \frac{n!}{k_1! \ldots k_r!} p_1^{k_1} \ldots p_r^{k_r}.$$

Marginal distributions $X_i \sim \text{Bin}(n, p_i)$, also

$$(X_1 + X_2, X_3 \ldots, X_r) \sim \text{Mn}(n, p_1 + p_2, p_3, \ldots, p_r).$$

Conditionally on X_1

$$(X_2, \ldots, X_r) \sim \text{Mn}(n - X_1, \frac{p_2}{1-p_1}, \ldots, \frac{p_r}{1-p_1}),$$

so that $(X_i|X_j) \sim \text{Bin}(n - X_j, \frac{p_i}{1-p_j})$ and $\text{E}(X_i|X_j) = (n - X_j)\frac{p_i}{1-p_j}$. It follows

$$\begin{aligned}\text{E}(X_iX_j) &= \text{E}(\text{E}(X_iX_j|X_j)) \\ &= \text{E}(X_j\text{E}(X_i|X_j)) = \text{E}(nX_j - X_j^2)\frac{p_i}{1-p_j} \\ &= (n^2p_j - np_j(1-p_j) + n^2p_j^2)\frac{p_i}{1-p_j} = n(n-1)p_ip_j\end{aligned}$$

and $\text{Cov}(X_i, X_j) = -np_ip_j$ so that

$$\rho(X_i, X_j) = -\sqrt{\frac{p_ip_j}{(1-p_i)(1-p_j)}}.$$

2.4 Multivariate normal distribution

Bivariate normal distribution with parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{(\frac{x-\mu_1}{\sigma_1})^2 - 2\rho(\frac{x-\mu_1}{\sigma_1})(\frac{y-\mu_2}{\sigma_2}) + (\frac{y-\mu_2}{\sigma_2})^2}{2(1-\rho^2)}\right\}.$$

Marginal distributions

$$f_1(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}, \quad f_2(y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}},$$

and conditional distributions

$$f_1(x|y) = \frac{f(x, y)}{f_2(y)} = \frac{1}{\sigma_1\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{(x - \mu_1 - \frac{\rho\sigma_1}{\sigma_2}(y - \mu_2))^2}{2\sigma_1^2(1-\rho^2)}\right\},$$

$$f_2(y|x) = \frac{f(x, y)}{f_1(x)} = \frac{1}{\sigma_2\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{(y - \mu_2 - \frac{\rho\sigma_2}{\sigma_1}(x - \mu_1))^2}{2\sigma_2^2(1-\rho^2)}\right\}.$$

Exercise: check the total variance formula for this example.

A multivariate normal distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and covariance matrix $\mathbf{V}$ has density

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det\mathbf{V}}} e^{-(\mathbf{x}-\boldsymbol{\mu})\mathbf{V}^{-1}(\mathbf{x}-\boldsymbol{\mu})^t}.$$

For any vector $(a_1, \ldots, a_n)$ the r.v. $a_1X_1 + \ldots + a_nX_n$ is normally distributed. Application in statistics: in the IID case: $\boldsymbol{\mu} = (\mu, \ldots, \mu)$ and $\mathbf{V} = \text{diag}\{\sigma^2, \ldots, \sigma^2\}$ the sample mean and sample variance

$$\bar{X} = \frac{X_1 + \ldots + X_n}{n}, \quad s^2 = \frac{(X_1 - \bar{X})^2 + \ldots + (X_n - \bar{X})^2}{n-1}$$

are independent and $\frac{\sqrt{n}(\bar{X}-\mu)}{s}$ has a t-distribution with $n-1$ degrees of freedom.

If Y and Z are independent r.v. with standard normal distribution, their ratio $X = Y/Z$ has a Cauchy distribution with density

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

In the Cauchy distribution case the mean is undefined and $\bar{X} \stackrel{d}{=} X$. Cauchy and normal distributions are examples of stable distributions. The Cauchy distribution provides with a counterexample for the law of large numbers.

2.5 Sampling from a distribution

Computers generate pseudo-random numbers $U_1, U_2, \ldots$ which we consider as IID r.v. with $\mathrm{U}_{[0,1]}$ distribution.

Inverse transform sampling: if F is a cdf and $U \sim \mathrm{U}_{[0,1]}$, then $X = F_{-1}(U)$ has cdf F. It follows from

$$\{F_{-1}(U) \le x\} = \{U \le F(x)\}.$$

Example 2.10 Examples of the inverse transform sampling.
(i) Bernoulli distribution $X = 1_{\{U \le p\}}$,
(ii) Binomial sampling: $S_n = X_1 + \ldots + X_n$, $X_k = 1_{\{U_k \le p\}}$,
(iii) Exponential distribution $X = -\log(U)/\lambda$,
(iv) Gamma sampling: $S_n = X_1 + \ldots + X_n$, $X_k = -\log(U_k)/\lambda$.

Lemma 2.11 *Rejection sampling. Suppose that we know how to sample from density $g(x)$ but we want to sample from density $f(x)$ such that $f(x) \le ag(x)$ for some $a > 0$. Algorithm*
step 1: sample x from $g(x)$ and u from $U_{[0,1]}$,
step 2: if $u \le \frac{f(x)}{ag(x)}$, accept x as a realization of sampling from $f(x)$,
step 3: if not, reject the value of x and repeat the sampling step.

Proof. Let Z and U be independent, Z has density $g(x)$ and $U \sim \mathrm{U}_{[0,1]}$. Then

$$\mathrm{P}\Big(Z \le x \Big| U \le \frac{f(Z)}{ag(Z)}\Big) = \frac{\int_{-\infty}^{x} \mathrm{P}\Big(U \le \frac{f(y)}{ag(y)}\Big)g(y)dy}{\int_{-\infty}^{\infty} \mathrm{P}\Big(U \le \frac{f(y)}{ag(y)}\Big)g(y)dy} = \int_{-\infty}^{x} f(y)dy.$$

2.6 Probability generating, moment generating, and characteristic functions

Definition 2.12 If X takes values $k = 0, 1, 2, \ldots$ with probabilities p_k and $\sum_{k=0}^{\infty} p_k = 1$, then the distribution of X is fully described by its probability generating function

$$G(s) = \mathrm{E}(s^X) = \sum_{k=0}^{\infty} p_k s^k.$$

Properties of probability generating functions:
(i) $p_0 = G(0), \quad p_k = \frac{1}{k!}\frac{d^k G(s)}{ds^k}|_{s=0}$,
(ii) $\mathrm{E}(X) = G'(1), \quad \mathrm{E}(X(X-1)) = G''(1)$.
(iii) if X and Y are independent, then $G_{X+Y}(s) = G_X(s)G_Y(s)$,

Example 2.13 Examples of probability generating functions.
(i) Bernoulli distribution $G(s) = q + ps$.
(ii) Binomial distribution $G(s) = (q + ps)^n$.
(iii) Geometric distribution $G(s) = \frac{p}{1-(1-p)s}$. Shifted geometric distribution $G(s) = \frac{ps}{1-(1-p)s}$.
(iv) Poisson distribution $G(s) = e^{\lambda(s-1)}$.

Definition 2.14 Moment generating function of X is $M(\theta) = \mathrm{E}(e^{\theta X})$. In the continuous case $M(\theta) = \int e^{\theta x} f(x)dx$. Moments $\mathrm{E}(X) = M'(0)$, $\mathrm{E}(X^k) = M^{(k)}(0)$.

Example 2.15 Examples of moment generating functions
(i) Normal distribution $M(\theta) = e^{\theta\mu + \frac{1}{2}\theta^2\sigma^2}$,
(ii) Exponential distribution $M(\theta) = \frac{\lambda}{\lambda-\theta}$ for $\theta < \lambda$,
(ii) Gamma(α, λ) distribution has density $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$ and $M(\theta) = \Big(\frac{\lambda}{\lambda-\theta}\Big)^\alpha$ for $\theta < \lambda$, it follows that the sum of k exponentials with parameter λ has a Gamma(k, λ) distribution,
(iii) Cauchy distribution $M(0) = 1$, $M(t) = \infty$ for $t \neq 0$.

Definition 2.16 The characteristic function of X is complex valued $\phi(\theta) = \mathrm{E}(e^{i\theta X})$. The joint characteristic function for $\mathbf{X} = (X_1, \ldots, X_n)$ is $\phi(\boldsymbol{\theta}) = \mathrm{E}(e^{i\boldsymbol{\theta}\mathbf{X}^{\mathrm{t}}})$.

Example 2.17 Examples of characteristic functions
(i) Normal distribution $\phi(\theta) = e^{i\theta\mu - \frac{1}{2}\theta^2\sigma^2}$,
(ii) Gamma distribution $\phi(\theta) = \Big(\frac{\lambda}{\lambda - i\theta}\Big)^{\alpha}$,
(iii) Cauchy distribution $\phi(\theta) = e^{-|\theta|}$,
(iv) Multinomial distribution $\phi(\boldsymbol{\theta}) = \Big(\sum_{j=1}^{r} p_j e^{i\theta_j}\Big)^n$.
(v) Multivariate normal distribution $\phi(\boldsymbol{\theta}) = e^{i\boldsymbol{\theta}\boldsymbol{\mu}^{\mathrm{t}} - \frac{1}{2}\boldsymbol{\theta}\mathbf{V}\boldsymbol{\theta}^{\mathrm{t}}}$.

Example 2.18 Given a vector $\mathbf{X} = (X_1, \ldots, X_n)$ with a multivariate normal distribution any linear combination $\mathbf{a}\mathbf{X}^{\mathrm{t}} = a_1X_1 + \ldots + a_nX_n$ is normally distributed since

$$\mathrm{E}(e^{\theta\mathbf{a}\mathbf{X}^{\mathrm{t}}}) = \phi(\theta\mathbf{a}) = e^{i\theta\mu - \frac{1}{2}\theta^2\sigma^2}, \quad \mu = \mathbf{a}\boldsymbol{\mu}^{\mathrm{t}}, \quad \sigma^2 = \mathbf{a}\mathbf{V}\mathbf{a}^{\mathrm{t}}.$$

3 Convergence of random variables

3.1 Borel-Cantelli lemmas

Given a sequence of random events $A_1, A_2, \ldots$ define new events

$$\sup_n A_n = \bigcup_n A_n, \qquad \inf_n A_n = \bigcap_n A_n,$$
$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m, \qquad \operatorname{liminf}_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m.$$

These relations are explained in terms of the indicator random variables. For example,

$$\sup_n 1_{\{\omega\in A_n\}} = \begin{cases} 1, & \text{if } \omega \in A_i \text{ for some } i \\ 0, & \text{otherwise} \end{cases} = 1_{\{\omega\in\bigcup_n A_n\}}.$$

We will write $\{A_n \text{ i.o.}\} = \{\text{infinitely many of events } A_1, A_2, \ldots \text{ occur}\}$. If $A = \{A_n \text{ i.o.}\}$, then

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m = \{\forall n\ \exists m \ge n \text{ such that } A_m \text{ occurs}\} = A,$$
$$A^c = \operatorname{liminf}_{n\to\infty} A_n^c = \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m^c = \{\exists n \text{ such that } A_m^c \text{ occur } \forall m \ge n\} = \{\text{events } A_n \text{ occur finitely often}\}.$$

Theorem 3.1 *Borel-Cantelli lemmas.*

1. *If* $\sum_{n=1}^{\infty} \mathrm{P}(A_n) < \infty$, *then* $P(A_n \text{ i.o.}) = 0$,
2. *If* $\sum_{n=1}^{\infty} \mathrm{P}(A_n) = \infty$ *and events* $A_1, A_2, \ldots$ *are independent, then* $\mathrm{P}(A_n \text{ i.o.}) = 1$.

Proof. (1) Put $A = \{A_n \text{ i.o.}\}$. We have $A \subset \cup_{m\ge n} A_m$, and so $\mathrm{P}(A) \le \sum_{m\ge n} \mathrm{P}(A_m) \to 0$ as $n \to \infty$.
(2) By independence

$$\mathrm{P}(\bigcap_{m\ge n} A_m^c) = \lim_{N\to\infty} \mathrm{P}(\bigcap_{m=n}^{N} A_m^c) = \prod_{m\ge n}(1 - \mathrm{P}(A_m)) \le \exp\Big(-\sum_{m\ge n}\mathrm{P}(A_m)\Big) = 0.$$

It follows $\mathrm{P}(A^c) \le \sum_n \mathrm{P}(\cap_{m\ge n} A_m^c) = 0$ which gives $\mathrm{P}(A) = 1$.

If events $A_1, A_2, \ldots$ are independent, then either $P(A_n \text{ i.o.}) = 0$ or $P(A_n \text{ i.o.}) = 1$. This is an example of a general zero-one law.

Definition 3.2 Let $X_1, X_2, \ldots$ be a sequence of random variables defined on the same probability space and $\mathcal{H}_n = \sigma(X_{n+1}, X_{n+2}, \ldots)$. Then $\mathcal{H}_n \supset \mathcal{H}_{n+1} \supset \ldots$, and we define the tail σ-algebra as $\mathcal{T} = \cap_n \mathcal{H}_n$.

Event H is in the tail σ-algebra if and only if changing the values of $X_1, \ldots, X_N$ does not affect the occurrence of H for any finite N.

Theorem 3.3 *Kolmogorov zero-one law. Let $X_1, X_2, \ldots$ be independent random variables defined on the same probability space. For all events $H \in \mathcal{T}$ from the tail σ-algebra we have either $\mathrm{P}(H) = 0$ or $\mathrm{P}(H) = 1$.*

Proof. A standard result of measure theory asserts that for any $H \in \mathcal{H}_1$ there exists a sequence of events $C_n = \sigma(X_1, \ldots, X_n)$ such that $\mathrm{P}(H \Delta C_n) \to 0$ as $n \to \infty$. If $H \in \mathcal{T}$, then by independence

$$\mathrm{P}(H \cap C_n) = \mathrm{P}(H)\mathrm{P}(C_n) \to \mathrm{P}(H)^2$$

implying $\mathrm{P}(H) = \mathrm{P}(H)^2$.

Example 3.4 Examples of events belonging to the tail σ-algebra $\mathcal{T}$:

$$\{X_n > 0 \text{ i.o.}\}, \quad \{\limsup_{n\to\infty} X_n = \infty\}, \quad \{\sum_n X_n \text{ converges}\}.$$

These events are not affected by $X_1, \ldots, X_N$ for any fixed N.

Example 3.5 An example of an event $A \notin \mathcal{T}$ not belonging to the tail σ-algebra: suppose X_n may take only two values 1 and -1, and consider

$$A = \{S_n = 0 \text{ i.o.}\}, \text{ where } S_n = X_1 + \ldots + X_n.$$

Whether A occurs or not depends on the value of X_1. Indeed, if $X_2 = X_4 = \ldots = 1$ and $X_3 = X_5 = \ldots = -1$, then A occurs if $X_1 = -1$ and does not occur if $X_1 = 1$. If (X_n) are independent and identically distributed, then by the so-called Hewitt-Savage zero-one law, the event A has probability either 0 or 1.

3.2 Inequalities

Jensen inequality. Given a convex function $J(x)$ and a random variable X we have

$$J(\mathrm{E}(X)) \le \mathrm{E}(J(X)).$$

Proof. Put $\mu = \mathrm{E}(X)$. Due to convexity there is λ such that $J(x) \ge J(\mu) + \lambda(x - \mu)$ for all x. Thus

$$\mathrm{E}(J(X)) \ge \mathrm{E}(J(\mu) + \lambda(X - \mu)) = J(\mu).$$

Markov inequality. For any random variable X and $a > 0$

$$\mathrm{P}(|X| \ge a) \le \frac{\mathrm{E}|X|}{a}.$$

Proof. Using truncation,

$$\mathrm{E}|X| \ge \mathrm{E}(|X| 1_{\{|X|\ge a\}}) \ge a\mathrm{E}(1_{\{|X|\ge a\}}) = a\mathrm{P}(|X| \ge a).$$

Chebyshev inequality. Given a random variable X with mean μ and variance σ^2 for any $\epsilon > 0$ we have

$$\mathrm{P}(|X - \mu| \ge \epsilon) \le \frac{\sigma^2}{\epsilon^2}.$$

Proof. By Markov inequality,

$$\mathrm{P}(|X - \mu| \ge \epsilon) = \mathrm{P}((X - \mu)^2 \ge \epsilon^2) \le \frac{\mathrm{E}((X - \mu)^2)}{\epsilon^2}.$$

Cauchy-Schwartz inequality. The following inequality holds and it becomes an equality if and only if $aX = bY$ a.s. for a pair of constants $(a, b) \ne (0, 0)$:

$$\big(\mathrm{E}(XY)\big)^2 \le \mathrm{E}(X^2)\mathrm{E}(Y^2).$$

Exercise 3.6 For a random variable X define its cumulant generating function by $\Lambda(t) = \log M(t)$, where $M(t) = \mathrm{E}(e^{tX})$ is the moment generating function assumed to be finite on an interval $t \in [0, z)$. Show that $\Lambda(0) = 0$, $\Lambda'(0) = \mu$, and

$$\Lambda''(t) = \frac{\mathrm{E}(e^{tX})\mathrm{E}(X^2 e^{tX}) - (\mathrm{E}(Xe^{tX}))^2}{M^2(t)}.$$

Deduce that $\Lambda(t)$ is convex on $[0, z)$.

Hölder inequality. If $p, q > 1$ and $p^{-1} + q^{-1} = 1$, then

$$\mathrm{E}|XY| \le \big(\mathrm{E}|X^p|\big)^{1/p}\big(\mathrm{E}|Y^q|\big)^{1/q}.$$

Lyapunov inequality. If $0 < s < r$, then

$$\big(\mathrm{E}|X^s|\big)^{1/s} \le \big(\mathrm{E}|X^r|\big)^{1/r}.$$

Proof. Using Hölder inequality with $p = r/s$ and $q = (1 - s/r)^{-1}$ we obtain

$$\mathrm{E}\big(|X^s| \cdot 1\big) \le \big(\mathrm{E}|X^s|^p\big)^{1/p} = \big(\mathrm{E}|X^r|\big)^{s/r}.$$

Minkowski inequality. If $p \ge 1$, then the following triangle inequality holds

$$\big(\mathrm{E}|X + Y|^p\big)^{1/p} \le \big(\mathrm{E}|X^p|\big)^{1/p} + \big(\mathrm{E}|Y^p|\big)^{1/p}.$$

Kolmogorov inequality. Let $\{X_n\}$ be iid with zero means and variances σ_n^2. Then for any $\epsilon > 0$

$$\mathrm{P}(\max_{1\le i\le n} |X_1 + \ldots + X_i| \ge \epsilon) \le \frac{\sigma_1^2 + \ldots + \sigma_n^2}{\epsilon^2}.$$

3.3 Modes of convergence

Theorem 3.7 *If $X_1, X_2, \ldots$ are random variables defined on the same probability space, then so are*

$$\inf_n X_n, \quad \sup_n X_n, \quad \operatorname*{liminf}_n X_n, \quad \limsup_n X_n.$$

Proof. For any real x

$$\{\omega : \inf_n X_n \le x\} = \bigcup_n \{\omega : X_n \le x\} \in \mathcal{F}, \quad \{\omega : \sup_n X_n \le x\} = \bigcap_n \{\omega : X_n \le x\} \in \mathcal{F}.$$

It remains to observe that

$$\operatorname*{liminf}_n X_n = \sup_n \inf_{m\ge n} X_m, \qquad \limsup_n X_n = \inf_n \sup_{m\ge n} X_m.$$

Definition 3.8 Let $X, X_1, X_2, \ldots$ be random variables on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Define four modes of convergence of random variables

(i) almost sure convergence $X_n \overset{\text{a.s.}}{\to} X$, if $\mathrm{P}(\omega : \lim X_n = X) = 1$,

(ii) convergence in r-th mean $X_n \overset{L^r}{\to} X$ for a given $r \ge 1$, if $\mathrm{E}|X_n^r| < \infty$ for all n and

$$\mathrm{E}(|X_n - X|^r) \to 0,$$

(iii) convergence in probability $X_n \overset{\mathrm{P}}{\to} X$, if $\mathrm{P}(|X_n - X| > \epsilon) \to 0$ for all positive ϵ,

(iv) convergence in distribution $X_n \overset{d}{\to} X$ (does not require a common probability space), if

$$\mathrm{P}(X_n \le x) \to \mathrm{P}(X \le x) \text{ for all } x \text{ such that } \mathrm{P}(X = x) = 0,$$

or equivalently, $\mathrm{E}f(X_n) \to \mathrm{E}f(X)$ for all bounded uniformly continuous $f : R \to R$.

Theorem 3.9 *Let $q \geq r \geq 1$. The following implications hold*

$$X_n \overset{L^q}{\to} X \Rightarrow_{\text{(i)}} \begin{array}{l} X_n \overset{\text{a.s.}}{\to} X \Rightarrow_{\text{(ii)}} \\ X_n \overset{L^r}{\to} X \Rightarrow_{\text{(iii)}} \end{array} X_n \overset{\text{P}}{\to} X \Rightarrow_{\text{(iv)}} \quad X_n \overset{d}{\to} X.$$

Proof. The implication (i) is due to the Lyapunov inequality, while (iii) follows from the Markov inequality. For (ii) observe that the a.s. convergence is equivalent to $\text{P}(\cup_{n \geq m}\{|X_n - X| > \epsilon\}) \to 0$, $m \to \infty$ for any $\epsilon > 0$. To prove (iv) use

$$\text{P}(X_n \leq x) = \text{P}(X_n \leq x, X \leq x + \epsilon) + \text{P}(X_n \leq x, X > x + \epsilon) \leq \text{P}(X \leq x + \epsilon) + \text{P}(|X - X_n| > \epsilon),$$
$$\text{P}(X \leq x - \epsilon) = \text{P}(X_n \leq x, X \leq x - \epsilon) + \text{P}(X_n > x, X \leq x - \epsilon) \leq \text{P}(X_n \leq x) + \text{P}(|X - X_n| > \epsilon).$$

Theorem 3.10 *Reverse implications:*

(i) $X_n \overset{d}{\to} c \Rightarrow X_n \overset{\text{P}}{\to} c$ *for a constant limit,*

(ii) $X_n \overset{\text{P}}{\to} X \Rightarrow X_n \overset{L^r}{\to} X$, *if* $\text{P}(|X_n| \leq a) = 1$ *for all n and some positive constant a,*

(iii) $X_n \overset{\text{P}}{\to} X \Rightarrow X_n \overset{\text{a.s.}}{\to} X$ *if* $\text{P}(|X_n - X| > \epsilon) \to 0$ *so fast that*

$$\sum_n \text{P}(|X_n - X| > \epsilon) < \infty \text{ for any } \epsilon > 0. \qquad (*)$$

(iv) $X_n \overset{\text{P}}{\to} X \Rightarrow X_{n'} \overset{\text{a.s.}}{\to} X$ *along a subsequence.*

Proof. To prove (i) use

$$\text{P}(|X_n - c| > \epsilon) = \text{P}(X_n < c - \epsilon) + \text{P}(X_n > c + \epsilon).$$

To prove (ii) use $\text{P}(|X| \leq a) = 1$ and

$$|X_n - X|^r \leq \epsilon^r 1_{\{|X_n - X| \leq \epsilon\}} + (2a)^r 1_{\{|X_n - X| > \epsilon\}}.$$

The implication (iii) follows from the first Borel-Cantelli lemma. Indeed, put $B_m = \{|X_n - X| > m^{-1} \text{ i.o.}\}$. Due to the Borel-Cantelli lemma, condition $(*)$ implies $\text{P}(B_m) = 0$ for any natural m. Remains to observe that $\{\omega : \lim X_n \neq X\} = \sup_m B_m$. The implication (iv) follows from (iii).

Example 3.11 Let $\Omega = [0, 2]$ and put $A_n = [a_n, a_{n+1}]$, where a_n is the fractional part of $1 + 1/2 + \ldots + 1/n$.

- The random variables 1_{A_n} converge to zero in mean (and therefore in probability) but not a.s.
- The random variables $n1_{A_n}$ converge to zero in probability but not in mean and not a.s.
- $\text{P}(A_n \text{ i.o.}) = 0.5$ and $\sum_n \text{P}(A_n) = \infty$.

Example 3.12 Let $\Omega = [0, 1]$ and put $B_n = [0, 1/n]$.

- The random variables $n \cdot 1_{B_n}$ converge to zero a.s but not in mean.
- Put $X_{2n} = 1_{B_2}$ and $X_{2n+1} = 1 - X_{2n}$. The random variables X_n converge to 1_{B_2} in distribution but not in probability. The random variables X_n converge in distribution to $1 - 1_{B_2}$ as well.
- Both $X_{2n} = 1_{B_2}$ and $X_{2n+1} = 1 - X_{2n}$ converge to 1_{B_2} in distribution but their sum does not converge to $2 \cdot 1_{B_2}$.

Exercise 3.13 Suppose $X_n \overset{L^r}{\to} X$, where $r \geq 1$. Show that $\text{E}(|X_n|^r) \to \text{E}(|X|^r)$.
Hint: use Minkowski inequality two times as you need two estimate $\overline{\lim}\, \text{E}(|X_n|^r)$ from above and $\underline{\lim}\, \text{E}(|X_n|^r)$ from below.

Solution. By Minkowski inequality,

$$(\text{E}|X_n^r|)^{1/r} \leq (\text{E}|X^r|)^{1/r} + (\text{E}|X_n - X|^r)^{1/r},$$

implying $\overline{\lim}\, \text{E}(|X_n|^r) \leq \text{E}(|X|^r)$. On the other hand, $\underline{\lim}\, \text{E}(|X_n|^r) \geq \text{E}(|X|^r)$, since

$$(\text{E}|X^r|)^{1/r} \leq (\text{E}|X_n^r|)^{1/r} + (\text{E}|X_n - X|^r)^{1/r}.$$

Exercise 3.14 Suppose $X_n \xrightarrow{L^1} X$. Show that $\mathrm{E}(X_n) \to \mathrm{E}(X)$. (Hint: use Jensen inequality.) Is the converse true?

Exercise 3.15 Suppose $X_n \xrightarrow{L^2} X$. Show that $\mathrm{Var}(X_n) \to \mathrm{Var}(X)$.
Hint: use the previous two exercises.

3.4 Continuity of expectation

Theorem 3.16 *Bounded convergence. Suppose $|X_n| \le M$ almost surely and $X_n \xrightarrow{\mathrm{P}} X$. Then $\mathrm{E}(X) = \lim \mathrm{E}(X_n)$.*

Proof. Let $\epsilon > 0$ and use

$$\begin{aligned}|\mathrm{E}(X_n) - \mathrm{E}(X)| &\le \mathrm{E}|X_n - X| = \mathrm{E}\big(|X_n - X| \cdot 1_{\{|X_n - X| \le \epsilon\}}\big) + \mathrm{E}\big(|X_n - X| \cdot 1_{\{|X_n - X| > \epsilon\}}\big)\\ &\le \epsilon + M\mathrm{P}(|X_n - X| > \epsilon).\end{aligned}$$

Lemma 3.17 *Fatou lemma. If almost surely $X_n \ge 0$, then* $\mathrm{liminf}\, \mathrm{E}(X_n) \ge \mathrm{E}(\mathrm{liminf}\, X_n)$*. In particular, applying this to $X_n = 1_{\{A_n\}}$ and $X_n = 1 - 1_{\{A_n\}}$ we get*

$$\mathrm{P}(\mathrm{liminf}\, A_n) \le \mathrm{liminf}\, \mathrm{P}(A_n) \le \limsup \mathrm{P}(A_n) \le \mathrm{P}(\limsup A_n).$$

Proof. Put $Y_n = \inf_{m \ge n} X_n$. We have $Y_n \le X_n$ and $Y_n \nearrow Y = \mathrm{liminf}\, X_n$. It suffices to show that $\mathrm{liminf}\, \mathrm{E}(Y_n) \ge \mathrm{E}(Y)$. Since, $|Y_n \wedge M| \le M$, the bounded convergence theorem implies

$$\underset{n\to\infty}{\mathrm{liminf}}\, \mathrm{E}(Y_n) \ge \underset{n\to\infty}{\mathrm{liminf}}\, \mathrm{E}(Y_n \wedge M) = \mathrm{E}(Y \wedge M).$$

The convergence $\mathrm{E}(Y \wedge M) \to \mathrm{E}(Y)$ as $M \to \infty$ can be shown using the definition of expectation.

Theorem 3.18 *Monotone convergence. If $0 \le X_n(\omega) \le X_{n+1}(\omega)$ for all n and ω, then, clearly, for all ω, there exists a limit (possibly infinite) for the sequence $X_n(\omega)$. In this case* $\mathrm{E}(\lim X_n) = \lim \mathrm{E}(X_n)$.

Proof. From $\mathrm{E}(X_n) \le \mathrm{E}(\lim X_n)$ we have $\limsup \mathrm{E}(X_n) \le \mathrm{E}(\lim X_n)$. Now use Fatou lemma.

Lemma 3.19 *Let X be a non-negative random variable with finite mean. Show that*

$$\mathrm{E}(X) = \int_0^\infty \mathrm{P}(X > x)dx.$$

Proof. Put $F(x) = P(X \le x)$. Integrating by parts we get

$$\mathrm{E}(X) = \int_0^\infty x dF(x) = \int_0^\infty x d(F(x) - 1) = x(F(x) - 1)|_0^\infty - \int_0^\infty (F(x) - 1)dx.$$

Thus it suffices to prove that $n(1 - F(n)) \to 0$ as $n \to \infty$. This follows from the monotone convergence theorem since

$$n(1 - F(n)) \le \mathrm{E}(X 1_{X>n}) \to 0.$$

Exercise 3.20 Let X be a non-negative random variable. Show that

$$\mathrm{E}(X^r) = r \int_0^\infty x^{r-1} \mathrm{P}(X > x)dx$$

for any $r \ge 1$ for which the expectation is finite.

Theorem 3.21 *Dominated convergence. If $X_n \xrightarrow{\text{a.s.}} X$ and almost surely $|X_n| \le Y$ and $\mathrm{E}(Y) < \infty$, then $\mathrm{E}|X| < \infty$ and $\mathrm{E}(X_n) \to \mathrm{E}(X)$.*

Proof. Apply Fatou lemma twice.

Definition 3.22 A sequence X_n of random variables is said to be uniformly integrable if

$$\sup_n \mathrm{E}(|X_n|; |X_n| > a) \to 0, \quad a \to \infty.$$

Exercise 3.23 Put $Z = \sup_n |X_n|$. If $\mathrm{E}(Z) < \infty$, then the sequence X_n is uniformly integrable.
Hint: $\mathrm{E}(|X_n| 1_A) \le \mathrm{E}(Z 1_A)$ for any event A.

Lemma 3.24 *A sequence X_n of random variables is uniformly integrable if and only if both of the following hold:*
(i) $\sup_n \mathrm{E}|X_n| < \infty$,
(ii) for all $\epsilon > 0$, there is $\delta > 0$ such that, for any event A such that $\mathrm{P}(A) < \delta$,

$$\sup_n \mathrm{E}(|X_n| 1_A) < \epsilon.$$

Proof. Step 1. Assume that (X_n) is uniformly integrable. For any positive a,

$$\sup_n \mathrm{E}|X_n| = \sup_n \Big(\mathrm{E}(|X_n|; |X_n| \le a) + \mathrm{E}(|X_n|; |X_n| > a)\Big) \le a + \sup_n \mathrm{E}(|X_n|; |X_n| > a).$$

Thus $\sup_n \mathrm{E}|X_n| < \infty$.
Step 2. Assume that (X_n) is uniformly integrable. Pick a positive a such that

$$\sup_n \mathrm{E}(|X_n| 1_{\{B_n\}}) < \epsilon/2, \qquad B_n = \{|X_n| > a\},$$

and put $\delta = \frac{\epsilon}{2a}$. If event A is such that $\mathrm{P}(A) < \delta$, then for all n,

$$\mathrm{E}(|X_n| 1_A) = \mathrm{E}(|X_n| 1_{\{A \cap B_n\}}) + \mathrm{E}(|X_n| 1_{\{A \cap B_n^c\}}) \le \mathrm{E}(|X_n| 1_{\{B_n\}}) + a\mathrm{P}(A) < \epsilon.$$

Step 3. Assume that (i) and (ii) hold. Let $\epsilon > 0$ and pick δ according to (ii). To prove that (X_n) is uniformly integrable it suffices to verify, see (ii), that

$$\sup_n \mathrm{P}(|X_n| > a) < \delta$$

for sufficiently large a such that $a > \delta^{-1} \sup_n \mathrm{E}|X_n|$. But this is an easy consequence of the Markov inequality

$$\sup_n \mathrm{P}(|X_n| > a) \le a^{-1} \sup_n \mathrm{E}|X_n| < \delta.$$

Theorem 3.25 *Let $X_n \xrightarrow{\mathrm{P}} X$. The following three statements are equivalent to one another.*
(a) The sequence X_n is uniformly integrable.
(b) $\mathrm{E}|X_n| < \infty$ *for all n,* $\mathrm{E}|X| < \infty$*, and* $X_n \xrightarrow{L^1} X$.
(c) $\mathrm{E}|X_n| < \infty$ *for all n,* $\mathrm{E}|X| < \infty$*, and* $\mathrm{E}|X_n| \to \mathrm{E}|X|$.

Theorem 3.26 *Let $r \ge 1$. If $X_n \xrightarrow{\mathrm{P}} X$ and the sequence $|X_n^r|$ is uniformly integrable, then $X_n \xrightarrow{L^r} X$.*

Proof. Step 1. Show that $\mathrm{E}|X^r| < \infty$. There is a subsequence $(X_{n'})$ which converges to X almost surely. By Fatou lemma and Lemma 3.24,

$$\mathrm{E}|X^r| = \mathrm{E}\Big(\liminf_{n' \to \infty} |X_{n'}^r|\Big) \le \liminf_{n' \to \infty} \mathrm{E}|X_{n'}^r| \le \sup_n \mathrm{E}|X_n^r| < \infty.$$

Step 2. Fix an arbitrary $\epsilon > 0$ and put $B_n = \{|X_n - X| > \epsilon\}$. We have $\mathrm{P}(B_n) \to 0$ as $n \to 0$, and

$$\mathrm{E}|X_n - X|^r \le \epsilon^r + \mathrm{E}\Big(|X_n - X|^r 1_{B_n}\Big).$$

By the Minkowski inequality

$$\Big[\mathrm{E}\Big(|X_n - X|^r 1_{B_n}\Big)\Big]^{1/r} \le \Big[\mathrm{E}\Big(|X_n|^r 1_{B_n}\Big)\Big]^{1/r} + \Big[\mathrm{E}\Big(|X|^r 1_{B_n}\Big)\Big]^{1/r}.$$

It remains to see that the last two expectations both go to 0 as $n \to \infty$: the first by Lemma 3.24 and the second by step 1.

4 Limit theorems for the sums if IID random variables

4.1 Weak law of large numbers

Definition 4.1 Convergence in distribution $X_n \overset{d}{\to} X$ means

$$\mathrm{P}(X_n \le x) \to \mathrm{P}(X \le x) \text{ for all } x \text{ such that } \mathrm{P}(X = x) = 0.$$

This is equvalent to the weak convergence $F_n \overset{d}{\to} F$ of distribution functions when $F_n(x) \to F(x)$ at each point x where F is continuous.

Theorem 4.2 *Weak convergence and convergence of characteristic functions:*
(i) two r.v. have the same characteristic function iff they have the same distribution function,
(ii) if $X_n \overset{d}{\to} X$, then $\phi_n(t) \to \phi(t)$ for all t,
(iii) conversely, if $\phi(t) = \lim \phi_n(t)$ exists and continuous at $t = 0$, then ϕ is cf of some F, and $F_n \overset{d}{\to} F$.

Theorem 4.3 *If $X_1, X_2, \ldots$ are iid with finite mean μ and $S_n = X_1 + \ldots + X_n$, then*

$$S_n/n \overset{d}{\to} \mu, \quad n \to \infty.$$

PROOF. Let F_n and ϕ_n be the df and cf of $n^{-1}S_n$. To prove $F_n(x) \overset{d}{\to} 1_{\{x \ge \mu\}}$ we have to see that $\phi_n(t) \to e^{it\mu}$ which is obtained using a Taylor expansion

$$\phi_n(t) = \Big(\phi_1(tn^{-1})\Big)^n = \Big(1 + i\mu t n^{-1} + o(n^{-1})\Big)^n \to e^{it\mu}.$$

Example 4.4 Statistical application: the sample mean is a consistent estimate of the population mean. Counterexample: if $X_1, X_2, \ldots$ are iid with the Cauchy distribution, then $S_n/n \overset{d}{=} X_1$ since $\phi_n(t) = \phi_1(t)$.

4.2 Central limit theorem

The LLN says that $|S_n - n\mu|$ is much smaller than n. The CLT says that this difference is of order $\sqrt{n}$.

Theorem 4.5 *If $X_1, X_2, \ldots$ are iid with finite mean μ and positive finite variance σ^2, then for any x*

$$\mathrm{P}\Big(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\Big) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad n \to \infty.$$

PROOF. Let ψ_n be the cf of $\frac{S_n - n\mu}{\sigma\sqrt{n}}$. Using a Taylor expansion we obtain

$$\psi_n(t) = \Big(1 - \frac{t^2}{2n} + o(n^{-1})\Big)^n \to e^{-t^2/2}.$$

Example 4.6 Important example: simple random walks. 280 years ago de Moivre (1733) obtained the first CLT in the symmetric case with $p = 1/2$.

Statistical application: the standardized sample mean has the sampling distribution which is approximately $\mathrm{N}(0,1)$. Approximate 95% confidence interval formula for the mean $\bar{X} \pm 1.96\frac{s}{\sqrt{n}}$.

Theorem 4.7 *Let $(X_1^n, \ldots, X_r^n)$ have the multinomial distribution $Mn(n, p_1, \ldots, p_r)$. Then the normalized vector $\Big(\frac{X_1^n - np_1}{\sqrt{n}}, \ldots, \frac{X_r^n - np_r}{\sqrt{n}}\Big)$ converges in distribution to the multivariate normal distribution with zero means and the covariance matrix*

$$\mathbf{V} = \begin{pmatrix} p_1(1-p_1) & -p_1p_2 & -p_1p_3 & \ldots & -p_1p_r \\ -p_2p_1 & p_2(1-p_2) & -p_2p_3 & \ldots & -p_2p_r \\ -p_3p_1 & -p_3p_2 & p_3(1-p_3) & \ldots & -p_3p_r \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -p_rp_1 & -p_rp_2 & -p_rp_3 & \ldots & p_r(1-p_r) \end{pmatrix}.$$

Proof. To apply the continuity property of the multivariate characteristic functions consider

$$\mathrm{E}\exp\Big(i\theta_1\frac{X_1^n-np_1}{\sqrt{n}}+\ldots+i\theta_r\frac{X_r^n-np_r}{\sqrt{n}}\Big)=\Big(\sum_{j=1}^r p_j e^{i\tilde\theta_j/\sqrt{n}}\Big)^n,$$

where $\tilde\theta_j=\theta_j-(\theta_1p_1+\ldots+\theta_rp_r)$. Similarly to the classical case we have

$$\Big(\sum_{j=1}^r p_j e^{i\tilde\theta_j/\sqrt{n}}\Big)^n=\Big(1-\frac{1}{2n}\sum_{j=1}^r p_j\tilde\theta_j^2+o(n^{-1})\Big)^n\to e^{-\frac12\sum_{j=1}^r p_j\tilde\theta_j^2}=e^{-\frac12(\sum_{j=1}^r p_j\theta_j^2-(\sum_{j=1}^r p_j\theta_j)^2)}.$$

It remains to see that the right hand side equals $e^{-\frac12\boldsymbol{\theta}\mathbf{V}\boldsymbol{\theta}^{\mathrm{t}}}$ which follows from the representation

$$\mathbf{V}=\begin{pmatrix}p_1 & & 0\\ & \ddots & \\ 0 & & p_r\end{pmatrix}-\begin{pmatrix}p_1\\ \vdots\\ p_r\end{pmatrix}\Big(p_1,\ldots,p_r\Big).$$

4.3 Strong LLN

Theorem 4.8 *Let $X_1,X_2,\ldots$ be iid random variables defined on the same probability space with mean μ and finite second moment. Then*

$$\frac{X_1+\ldots+X_n}{n}\xrightarrow{L^2}\mu.$$

Proof. Since $\sigma^2:=\mathrm{E}(X_1^2)-\mu^2<\infty$, we have

$$\mathrm{E}\left(\left(\frac{X_1+\ldots+X_n}{n}-\mu\right)^2\right)=\mathrm{Var}\left(\frac{X_1+\ldots+X_n}{n}\right)=\frac{n\sigma^2}{n^2}\to 0.$$

Theorem 4.9 *Strong LLN. Let $X_1,X_2,\ldots$ be iid random variables defined on the same probability space. Then*

$$\frac{X_1+\ldots+X_n}{n}\xrightarrow{\text{a.s.}}\mu$$

for some constant μ iff $\mathrm{E}|X_1|<\infty$. In this case $\mu=\mathrm{E}X_1$ and $\frac{X_1+\ldots+X_n}{n}\xrightarrow{L^1}\mu$.

There are cases when convergence in probability holds but not a.s. In those cases of course $\mathrm{E}|X_1|=\infty$.

Theorem 4.10 *The law of the iterated logarithm. Let $X_1,X_2,\ldots$ be iid random variables with mean 0 and variance 1. Then*

$$\mathrm{P}\left(\limsup_{n\to\infty}\frac{X_1+\ldots+X_n}{\sqrt{2n\log\log n}}=1\right)=1$$

and

$$\mathrm{P}\left(\liminf_{n\to\infty}\frac{X_1+\ldots+X_n}{\sqrt{2n\log\log n}}=-1\right)=1.$$

Proof sketch. The second assertion of Theorem 4.10 follows from the first one after applying it to $-X_i$. The Proof of the first part is difficult. One has to show that the events

$$A_n(c)=\{X_1+\ldots+X_n\ge c\sqrt{2n\log\log n}\}$$

occur for infinitely many values of n if $c<1$ and for only finitely many values of n if $c>1$.

4.4 Large deviations

Let $X_1, X_2, \ldots$ be iid random variables with mean μ and variance σ^2. Let $\Lambda(t)$ be the cumulant generating function for a typical X_i, see Exercise 3.6. For the convex function $\Lambda(t)$ we define the Fenchel-Legendre transform by

$$\Lambda^*(a) = \sup_{t\in\mathbb{R}}\{at - \Lambda(t)\}, \qquad a \in \mathbb{R}.$$

Observe that $\Lambda^*(a) > 0$ for $a > \mu$. This follows from the representation

$$at - \Lambda(t) = \log \frac{e^{at}}{M(t)} = \log \frac{1 + at + o(t)}{1 + \mu t + \frac{1}{2}\sigma^2 t^2 + o(t^2)}$$

showing that $at - \Lambda(t)$ is positive for sufficiently small $t > 0$.

Theorem 4.11 *Let $S_n = X_1 + \ldots + X_n$, where $X_1, X_2, \ldots$ are iid random variables with mean μ, and suppose that their moment generating function $M(t) = \mathrm{E}(e^{tX})$ is finite in some neighborhood of the origin $t = 0$.*

(i) If $a > \mu$ and the following regularity condition hold

$$\Lambda^*(a) = a\tau - \Lambda(\tau), \quad \text{for some } \tau > 0,\ M(\tau) < \infty,$$

then $\Lambda^*(a) > 0$ and

$$\frac{1}{n}\log \mathrm{P}(S_n < na) \to -\Lambda^*(a), \quad n \to \infty.$$

(ii) If $a < \mu$ and the following regularity condition hold

$$\Lambda^*(a) = a\tau - \Lambda(\tau), \quad \text{for some } \tau < 0,\ M(\tau) < \infty,$$

then $\Lambda^*(a) > 0$ and

$$\frac{1}{n}\log \mathrm{P}(S_n > na) \to -\Lambda^*(a), \quad n \to \infty.$$

Proof. Without loss of generality assume $\mu = 0$ and take $a > 0$. The upper bound for (i) is obtained in the form

$$\frac{1}{n}\log \mathrm{P}(S_n > na) \le -\Lambda^*(a).$$

Indeed, for $t > 0$,

$$e^{tS_n} > e^{nat}1_{\{S_n > na\}},$$

so that

$$\mathrm{P}(S_n > na) \le e^{-nat}\mathrm{E}e^{tS_n} = (e^{-at}M(t))^n = e^{-n(at-\Lambda(t))}.$$

With $\mu = 0$ and $a > 0$ we have

$$\Lambda^*(a) = \sup_{t>0}\{at - \Lambda(t)\},$$

implying

$$\frac{1}{n}\log \mathrm{P}(S_n > na) \le -\sup_{t>0}\{at - \Lambda(t)\} = -\Lambda^*(a).$$

Next we obtain the lower bound for (i)

$$\liminf_{n\to\infty} \frac{1}{n}\log \mathrm{P}(S_n > na) \ge -\Lambda^*(a).$$

Let F be the common distribution function of the X_i. For a "tilted distribution"

$$\tilde{F}(y) = \frac{1}{M(\tau)}\int_{-\infty}^{y} e^{\tau u}dF(u),$$

we have

$$\tilde{M}(t) = \int_{-\infty}^{\infty} e^{ty}d\tilde{F}(y) = \frac{1}{M(\tau)}\int_{-\infty}^{\infty} e^{ty}e^{\tau y}dF(y) = \frac{M(t+\tau)}{M(\tau)}.$$

Consider $\tilde{S}_n = \tilde{X}_1 + \ldots + \tilde{X}_n$, where iid r.v. $(\tilde{X}_n)$ have common distribution $\tilde{F}$. Denote

$$F_n(x) = \mathrm{P}(S_n \le x), \qquad \tilde{F}_n(x) = \mathrm{P}(\tilde{S}_n \le x).$$

From

$$\int_{-\infty}^{\infty} e^{ty} d\tilde{F}_n(y) = \Big(\frac{M(t+\tau)}{M(\tau)}\Big)^n = \frac{1}{M^n(\tau)} \int_{-\infty}^{\infty} e^{(t+\tau)y} dF_n(y)$$

we see that

$$d\tilde{F}_n(y) = M^{-n}(\tau) e^{\tau y} dF_n(y).$$

Let $b > a$. It follows,

$$\begin{aligned}\mathrm{P}(S_n > na) &= \int_{na}^{\infty} dF_n(y) = M^n(\tau) \int_{na}^{\infty} e^{-\tau y} d\tilde{F}_n(y) \\ &\ge M^n(\tau) e^{-\tau nb} \int_{na}^{nb} d\tilde{F}_n(y) = e^{-n(\tau b - \Lambda(\tau))} \mathrm{P}(na < \tilde{S}_n \le nb).\end{aligned}$$

Since

$$\mathrm{E}\tilde{X}_i = \tilde{M}'(0) = \Lambda'(\tau) = a, \qquad \mathrm{Var}\tilde{X}_i = \tilde{M}''(0) - \tilde{M}'(0)^2 = \Lambda''(\tau) \in (0, \infty),$$

we can apply the CLT which gives $\mathrm{P}(na < \tilde{S}_n \le nb) \to 1/2$. We conclude that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}(S_n > na) \ge -(\tau b - \Lambda(\tau)),$$

and to finish the proof for (i), it remains to send $b \searrow a$.

To derive (ii) from (i) we replace X_i by $-\bar{X}_i$ and put $\bar{a} = -a$. Then $\{S_n < na\} = \{\bar{S}_n > n\bar{a}\}$ and $\bar{a} > \bar{\mu}$. Moreover, $\bar{\Lambda}(t) = \Lambda(-t)$ and therefore, $\bar{\Lambda}^*(a) = \Lambda^*(-a)$. Thus, according to (i),

$$\frac{1}{n} \log \mathrm{P}(S_n < na) = \frac{1}{n} \log \mathrm{P}(\bar{S}_n > n\bar{a}) \to -\bar{\Lambda}^*(\bar{a}) = -\Lambda^*(a), \quad n \to \infty.$$

Example 4.12 For the normally distributed X_i with a common density $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, the normalised partial sums S_n/n are also normally distributed with mean μ and variance σ^2/n so that

$$\mathrm{P}(S_n > na) = \frac{1}{\sqrt{2\pi}} \int_{\frac{(a-\mu)\sqrt{n}}{\sigma}}^{\infty} e^{-\frac{y^2}{2}} dy.$$

For $x > 0$, we have

$$(x^{-1} - x^{-3}) e^{-\frac{x^2}{2}} \le \int_x^{\infty} e^{-\frac{y^2}{2}} dy \le x^{-1} e^{-\frac{x^2}{2}}.$$

Thus for $a > \mu$ we obtain

$$\mathrm{P}(S_n > na) \sim \frac{\sigma}{\sqrt{2\pi n}(a-\mu)} e^{-\frac{(a-\mu)^2}{2\sigma^2} \cdot n}.$$

It follows,

$$\frac{1}{n} \log \mathrm{P}(S_n < na) \to -\frac{(a-\mu)^2}{2\sigma^2}, \quad n \to \infty.$$

From $M(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$, we find $\Lambda(t) = t\mu + \frac{1}{2}t^2\sigma^2$ and

$$at - \Lambda(t) = t(a-\mu) - \frac{1}{2}t^2\sigma^2 = \frac{(a-\mu)^2}{2\sigma^2} - \frac{1}{2}\Big(\frac{a-\mu}{\sigma} - t\sigma\Big)^2.$$

Clearly, this quadratic function reaches its maximum at $t = \frac{a-\mu}{\sigma^2}$. The maximum is

$$\Lambda^*(a) = \frac{(a-\mu)^2}{2\sigma^2}.$$

Example 4.13 For X_i with Ber(p) distribution, the partial sum S_n has a binomial distribution with parameters (n, p)

$$\mathrm{P}(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Put

$$H(a) = a \log \frac{a}{p} + (1-a) \log \frac{1-a}{1-p}, \qquad a \in (p, 1).$$

Since $H(p) = 0$ and $H'(a) = \log \frac{a(1-p)}{p(1-a)}$, we conclude that $H(a) > 0$. Using the Stirling formula

$$\sqrt{2\pi n}\; n^n e^{-n} e^{\frac{1}{12n+1}} \le n! \le \sqrt{2\pi n}\; n^n e^{-n} e^{\frac{1}{12n}},$$

one can show that

$$\mathrm{P}(S_n > na) \sim \frac{a(1-p)}{a-p} \frac{1}{\sqrt{2\pi a(1-a)n}} e^{-nH(a)}.$$

On the other hand $M(t) = pe^t + 1 - p$ and

$$f(t) = at - \Lambda(t) = at - \log(pe^t + 1 - p)$$

has the following derivatives

$$f'(t) = a - \frac{pe^t}{pe^t + 1 - p} = a - 1 + \frac{1-p}{pe^t + 1 - p}, \qquad f''(t) = -\frac{(1-p)pe^t}{(pe^t + 1 - p)^2}.$$

Thus the maximum of $f(t)$ is achieved at t satisfying

$$pe^t + 1 - p = \frac{1-p}{1-a}, \qquad t = \log \frac{a(1-p)}{p(1-a)},$$

and the maximum equals

$$\Lambda^*(a) = a \log \frac{a(1-p)}{p(1-a)} - \log \frac{1-p}{1-a} = H(a).$$

5 Markov chains

5.1 Simple random walks

Let $S_n = a + X_1 + \ldots + X_n$ where $X_1, X_2, \ldots$ are IID r.v. taking values 1 and -1 with probabilities p and $q = 1 - p$. This Markov chain is homogeneous both in space and time. We have $S_n = 2Z_n - n$, with $Z_n \sim \mathrm{Bin}(n, p)$. Symmetric random walk if $p = 0.5$. Drift upwards $p > 0.5$ or downwards $p < 0.5$ (like in casino).

(i) The ruin probability $p_k = p_k(N)$: your starting capital k against casino $N - k$. The difference equation

$$p_k = p \cdot p_{k+1} + q \cdot p_{k-1}, \quad p_N = 0, \quad p_0 = 1$$

gives

$$p_k(N) = \begin{cases} \frac{(q/p)^N - (q/p)^k}{(q/p)^N - 1}, & \text{if } p \ne 0.5, \\ \frac{N-k}{N}, & \text{if } p = 0.5. \end{cases}$$

Start from zero and let τ_b be the first hitting time of b, then for $b > 0$

$$\mathrm{P}(\tau_{-b} < \infty) = \lim_{N \to \infty} p_b(N) = \begin{cases} 1, & \text{if } p \le 0.5, \\ (q/p)^b, & \text{if } p > 0.5, \end{cases}$$

and

$$\mathrm{P}(\tau_b < \infty) = \begin{cases} 1, & \text{if } p \ge 0.5, \\ (p/q)^b, & \text{if } p < 0.5. \end{cases}$$

(ii) The mean number $D_k = D_k(N)$ of steps before hitting either 0 or N. The difference equation

$$D_k = p \cdot (1 + D_{k+1}) + q \cdot (1 + D_{k-1}), \quad D_0 = D_N = 0$$

gives

$$D_k(N) = \begin{cases} \frac{1}{q-p}\left[k - N \cdot \frac{1-(q/p)^k}{1-(q/p)^N}\right], & \text{if } p \neq 0.5, \\ k(N-k), & \text{if } p = 0.5. \end{cases}$$

If $p < 0.5$, then the expected ruin time is computed as $D_k(N) \to \frac{k}{q-p}$ as $N \to \infty$.

(iii) There are

$$N_n(a,b) = \binom{n}{k}, \quad k = \frac{n+b-a}{2}$$

paths from a to b in n steps. Each path has probability $p^k q^{n-k}$. Thus

$$\mathrm{P}(S_n = b|S_0 = a) = \binom{n}{k} p^k q^{n-k}, \quad k = \frac{n+b-a}{2}.$$

In particular, $\mathrm{P}(S_{2n} = a|S_0 = a) = \binom{2n}{n}(pq)^n$. Reflection principle: the number of n-paths visiting r is

$$N_n^r(a,b) = N_n(2r-a,b), \quad a \geq r, b \geq r,$$
$$N_n^r(a,b) = N_n(a,2r-b), \quad a < r, b < r.$$

(iv) Ballot theorem: if $b > 0$, then the number of n-paths $0 \to b$ not revisiting zero is

$$\begin{aligned} N_{n-1}(1,b) - N_{n-1}^0(1,b) &= N_{n-1}(1,b) - N_{n-1}(-1,b) \\ &= \binom{n-1}{\frac{n+b}{2}-1} - \binom{n-1}{\frac{n+b}{2}} = (b/n)N_n(0,b). \end{aligned}$$

Thus (by default we will assume $S_0 = 0$)

$$\begin{aligned} \mathrm{P}(S_1 > 0, \ldots S_{n-1} > 0|S_n = b) &= \frac{b}{n}, \quad b > 0, \\ \mathrm{P}(S_1 \neq 0, \ldots S_{n-1} \neq 0, S_n = b) &= \frac{|b|}{n}\mathrm{P}(S_n = b), \\ \mathrm{P}(S_1 \neq 0, \ldots S_n \neq 0) &= n^{-1}\mathrm{E}|S_n|. \end{aligned}$$

It follows that

$$\mathrm{P}(S_1 \neq 0, \ldots S_{2n} \neq 0) = \mathrm{P}(S_{2n} = 0) \text{ for } p = 0.5. \qquad (*)$$

Indeed,

$$\begin{aligned} \mathrm{P}(S_1 \neq 0, \ldots S_{2n} \neq 0) &= 2\sum_{k=1}^{n} \frac{2k}{2n}\mathrm{P}(S_{2n} = 2k) = 2\sum_{k=1}^{n} \frac{2k}{2n}\binom{2n}{n+k}2^{-2n} \\ &= 2^{-2n+1}\sum_{k=1}^{n}\binom{2n-1}{n+k-1} - \binom{2n-1}{n+k} = 2^{-2n+1}\binom{2n-1}{n} = \mathrm{P}(S_{2n} = 0). \end{aligned}$$

(v) For the maximum $M_n = \max\{S_0, \ldots, S_n\}$ using $N_n^r(0,b) = N_n(0,2r-b)$ for $r > b$ and $r > 0$ we get

$$\mathrm{P}(M_n \geq r, S_n = b) = (q/p)^{r-b}\mathrm{P}(S_n = 2r-b),$$

implying for $b > 0$

$$\mathrm{P}(S_1 < b, \ldots S_{n-1} < b, S_n = b) = \frac{b}{n}\mathrm{P}(S_n = b).$$

The obtained equality

$$\mathrm{P}(S_1 > 0, \ldots S_{n-1} > 0, S_n = b) = \mathrm{P}(S_1 < b, \ldots S_{n-1} < b, S_n = b)$$

can be explained in terms of the reversed walk also starting at zero: the initial walk comes to b without revisiting zero means that the reversed walk reaches its maximum on the final step.

(vi) The first hitting time τ_b has distribution

$$\mathrm{P}(\tau_b = n) = \frac{|b|}{n}\mathrm{P}(S_n = b), \quad n > 0.$$

The mean number of visits of $b \neq 0$ before revisiting zero

$$\mathrm{E}\sum_{n=1}^{\infty} 1_{\{S_1 \neq 0, \ldots S_{n-1} \neq 0, S_n = b\}} = \sum_{n=1}^{\infty} \mathrm{P}(\tau_b = n) = \mathrm{P}(\tau_b < \infty).$$

Theorem 5.1 *Arcsine law for the last visit to the origin. Let $p = 0.5$, $S_0 = 0$, and T_{2n} be the time of the last visit to zero up to time $2n$. Then*

$$\mathrm{P}(T_{2n} \leq 2xn) \to \int_0^x \frac{dy}{\pi\sqrt{y(1-y)}} = \frac{2}{\pi}\arcsin\sqrt{x}, \quad n \to \infty.$$

Proof sketch. Using $(*)$ we get

$$\begin{aligned}\mathrm{P}(T_{2n} = 2k) &= \mathrm{P}(S_{2k} = 0)\mathrm{P}(S_{2k+1} \neq 0, \ldots, S_{2n} \neq 0 | S_{2k} = 0) \\ &= \mathrm{P}(S_{2k} = 0)\mathrm{P}(S_{2(n-k)} = 0),\end{aligned}$$

and it remains to apply Stirling formula.

Theorem 5.2 *Arcsine law for sojourn times. Let $p = 0.5$, $S_0 = 0$, and T_{2n}^+ be the number of time intervals spent on the positive side up to time $2n$. Then $T_{2n}^+ \overset{d}{=} T_{2n}$.*

Proof sketch. First using

$$\mathrm{P}(S_1 > 0, \ldots, S_{2n} > 0) = \mathrm{P}(S_1 = 1, S_2 \geq 1, \ldots, S_{2n} \geq 1) = \frac{1}{2}\mathrm{P}(T_{2n}^+ = 2n)$$

and then $(*)$, we obtain

$$\mathrm{P}(T_{2n}^+ = 0) = \mathrm{P}(T_{2n}^+ = 2n) = \mathrm{P}(S_{2n} = 0).$$

Then by induction over n one can show that

$$\mathrm{P}(T_{2n}^+ = 2k) = \mathrm{P}(S_{2k} = 0)\mathrm{P}(S_{2(n-k)} = 0)$$

for $k = 1, \ldots, n-1$, applying the following useful relation

$$\mathrm{P}(S_{2n} = 0) = \sum_{k=1}^{n} \mathrm{P}(S_{2(n-k)} = 0)\mathrm{P}(\tau_0 = 2k),$$

where τ_0 is the time of first return to zero.

5.2 Markov chains

Conditional on the present value, the future of the system is independent of the past. A Markov chain $\{X_n\}_{n=0}^{\infty}$ with countably many states and transition matrix $\mathbf{P}$ with elements p_{ij}

$$\mathrm{P}(X_n = j | X_{n-1} = i, X_{n-2} = i_{n-2}, \ldots, X_0 = i_0) = p_{ij}.$$

The n-step transition matrix with elements $p_{ij}^{(n)} = \mathrm{P}(X_{n+m} = j | X_m = i)$ equals $\mathbf{P}^n$. Given the initial distribution $\mathbf{a}$ as the vector with components $a_i = \mathrm{P}(X_0 = i)$, the distribution of X_n is given by the vector $\mathbf{aP}^n$ since

$$\mathrm{P}(X_n = j) = \sum_{i=-\infty}^{\infty} \mathrm{P}(X_n = j | X_0 = i)\mathrm{P}(X_0 = i) = \sum_{i=-\infty}^{\infty} a_i p_{ij}^{(n)}.$$

Example 5.3 Examples of Markov chains:

- IID chain has transition probabilities $p_{ij} = p_j$,
- simple random walk has transition probabilities $p_{ij} = p1_{\{j=i+1\}} + q1_{\{j=i-1\}}$,
- Bernoulli process has transition probabilities $p_{ij} = p1_{\{j=i+1\}} + q1_{\{j=i\}}$ and state space $S = \{0, 1, 2, \ldots\}$.

Lemma 5.4 *Hitting times. Let* $T_i = \min\{n \geq 1 : X_n = i\}$ *and put* $f_{ij}^{(n)} = \mathrm{P}(T_j = n|X_0 = i)$. *Define the generating functions*

$$P_{ij}(s) = \sum_{n=0}^{\infty} s^n p_{ij}^{(n)}, \qquad F_{ij}(s) = \sum_{n=1}^{\infty} s^n f_{ij}^{(n)}.$$

It is not difficult to see that

$$P_{ij}(s) = 1_{\{j=i\}} + F_{ij}(s)P_{jj}(s)$$
$$P_{ii}(s) = \frac{1}{1 - F_{ii}(s)}.$$

Definition 5.5 Classification of states

- state i is called recurrent (persistent), if $\mathrm{P}(T_i < \infty|X_0 = i) = 1$,
- a non-recurrent state is called a transient state,
- a recurrent state i is called null-recurrent, if $\mathrm{E}(T_i|X_0 = i) = \infty$,
- state i is called positive-recurrent, if $\mathrm{E}(T_i|X_0 = i) < \infty$.

Theorem 5.6 *A state i is recurrent iff* $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$. *A recurrent state i is null-recurrent iff* $p_{ii}^{(n)} \to 0$. *In the latter case* $p_{ij}^{(n)} \to 0$ *for all* j.

Proof sketch. To prove the first claim observe that $F_{ii}(1) = \mathrm{P}(T_i < \infty|X_0 = i)$. Using the previous lemma we conclude that state i is recurrent iff the expected number of visits of the state is infinite $P_{ii}(1) = \infty$. The second claim in the aperiodic case follows from the ergodic Theorem 5.14. The periodic case requires an extra argument.

Definition 5.7 The period $d(i)$ of state i is the greatest common divisor of n such that $p_{ii}^{(n)} > 0$. We call i periodic if $d(i) \geq 2$ and aperiodic if $d(i) = 1$.

If two states i and j communicate with each other, then

- i and j have the same period,
- i is transient iff j is transient,
- i is null-recurrent iff j is null-recurrent.

Example 5.8 For a simple random walk, we have

$$\mathrm{P}(S_{2n} = i|S_0 = i) = \binom{2n}{n}(pq)^n.$$

Notice that it is a periodic chain with period 2. Using the Stirling formula $n! \sim n^n e^{-n}\sqrt{2\pi n}$ we get

$$p_{ii}^{(2n)} \sim \frac{(4pq)^n}{\sqrt{\pi n}}, \qquad n \to \infty.$$

Criterium of recurrence $\sum p_{ii}^{(n)} = \infty$ holds only if $p = 0.5$ when $p_{ii}^{(2n)} \sim \frac{1}{\sqrt{\pi n}}$. The one and two-dimensional symmetric simple random walks are null-recurrent but the three-dimensional walk is transient!

Definition 5.9 A chain is called irreducible if all states communicate with each other.

All states in an irreducible chain have the same period d. It is called the period of the chain. Example: a simple random walk is periodic with period 2. Irreducible chains are classified as transient, recurrent, positively recurrent, or null-recurrent.

Definition 5.10 State i is absorbing if $p_{ii} = 1$. More generally, C is called a closed set of states, if $p_{ij} = 0$ for all $i \in C$ and $j \notin C$.

The state space S can be partitioned uniquely as

$$S = T \cup C_1 \cup C_2 \cup \dots,$$

where T is the set of transient states, and the C_i are irreducible closed sets of recurrent states. If S is finite, then at least one state is recurrent and all recurrent states are positively recurrent.

5.3 Stationary distributions

A vector of probabilities $\boldsymbol{\pi} = (\pi_j, j \in S)$ is a stationary distribution for the Markov chain X_n, if given X_0 has distribution $\boldsymbol{\pi}$, X_n has the same distribution $\boldsymbol{\pi}$ for any n, or in other words $\boldsymbol{\pi}$ is a left eigenvector of the transition matrix

$$\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}.$$

Theorem 5.11 *An irreducible chain (aperiodic or periodic) has a stationary distribution $\boldsymbol{\pi}$ iff the chain is positively recurrent; in this case $\boldsymbol{\pi}$ is the unique stationary distribution and is given by $\pi_i = 1/\mu_i$, where $\mu_i = \mathrm{E}(T_i|X_0 = i)$ and T_i is the time of first return to i.*

Proof sketch. Let $\boldsymbol{\rho}(k) = (\rho_j(k), j \in S)$ where $\rho_k(k) = 1$ and

$$\rho_j(k) = \sum_{n=1}^{\infty} \mathrm{P}(X_n = j, T_k \geq n | X_0 = k)$$

is the mean number of visits of the chain to the state j between two consecutive visits to state k. Then

$$\begin{aligned}\sum_{j \in S} \rho_j(k) &= \sum_{j \in S} \sum_{n=1}^{\infty} \mathrm{P}(X_n = j, T_k \geq n | X_0 = k) \\ &= \sum_{n=1}^{\infty} \mathrm{P}(T_k \geq n | X_0 = k) = \mathrm{E}(T_k | X_0 = k) = \mu_k.\end{aligned}$$

If the chain is irreducible recurrent, then $\rho_j(k) < \infty$ for any k and j, and furthermore, $\boldsymbol{\rho}(k)\mathbf{P} = \boldsymbol{\rho}(k)$. Thus there exists a positive root $\mathbf{x}$ of the equation $\mathbf{x}\mathbf{P} = \mathbf{x}$, which is unique up to a multiplicative constant; the chain is positively recurrent iff $\sum_{j \in S} x_j < \infty$.

Theorem 5.12 *Let s be any state of an irreducible chain. The chain is transient iff there exists a non-zero bounded solution $(y_j : j \neq s)$ satisfying $|y_j| \leq 1$ for all j to the equations*

$$y_i = \sum_{j \in S \backslash \{s\}} p_{ij} y_j, \quad i \in S \backslash \{s\}. \qquad (*)$$

Proof sketch. Main step. Let τ_j be the probability of no visit to s ever for a chain started at state j. Then the vector $(\tau_j : j \neq s)$ satisfies $(*)$.

Example 5.13 Random walk with retaining barrier. Transition probabilities

$$p_{00} = q, \quad p_{i-1,i} = p, \quad p_{i,i-1} = q, \quad i \geq 1.$$

Let $\rho = p/q$.

- If $q < p$, take $s = 0$ to see that $y_j = 1 - \rho^{-j}$ satisfies $(*)$. The chain is transient.

- Solve the equation $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ to find that there exists a stationary distribution, with $\pi_j = \rho^j(1-\rho)$, if and only if $q > p$.
- If $q > p$, the chain is positively recurrent, and if $q = p = 1/2$, the chain is null recurrent.

Theorem 5.14 *Ergodic theorem. For an irreducible aperiodic chain we have that*

$$p_{ij}^{(n)} \to \frac{1}{\mu_j} \text{ as } n \to \infty \text{ for all } (i,j).$$

More generally, for an aperiodic state j and any state i we have that $p_{ij}^{(n)} \to \frac{f_{ij}}{\mu_j}$, where f_{ij} is the probability that the chain ever visits j starting at i.

5.4 Reversibility

Theorem 5.15 *Put $Y_n = X_{N-n}$ for $0 \le n \le N$ where X_n is a stationary Markov chain. Then Y_n is a Markov chain with*

$$\mathrm{P}(Y_{n+1} = j | Y_n = i) = \frac{\pi_j p_{ji}}{\pi_i}.$$

The chain Y_n is called the time-reversal of X_n. If $\boldsymbol{\pi}$ exists and $\frac{\pi_j p_{ji}}{\pi_i} = p_{ij}$, the chain X_n is called reversible (in equilibrium). The detailed balance equations

$$\pi_i p_{ij} = \pi_j p_{ji} \text{ for all } (i,j). \qquad (*)$$

Theorem 5.16 *Consider an irreducible chain and suppose there exists a distribution $\boldsymbol{\pi}$ such that $(*)$ holds. Then $\boldsymbol{\pi}$ is a stationary distribution of the chain. Furthermore, the chain is reversible.*

Proof. Using $(*)$ we obtain

$$\sum_i \pi_i p_{ij} = \sum_i \pi_j p_{ji} = \pi_j.$$

Example 5.17 Ehrenfest model of diffusion: flow of m particles between two connected chambers. Pick a particle at random and move it to another chamber. Let X_n be the number of particles in the first chamber. State space $S = \{0, 1, \dots, m\}$ and transition probabilities

$$p_{i,i+1} = \frac{m-i}{m}, \qquad p_{i,i-1} = \frac{i}{m}.$$

The detailed balance equations

$$\pi_i \frac{m-i}{m} = \pi_{i+1} \frac{i+1}{m}$$

imply

$$\pi_i = \frac{m-i+1}{i} \pi_{i-1} = \binom{m}{i} \pi_0.$$

Using $\sum_i \pi_i = 1$ we find that the stationary distribution $\pi_i = \binom{m}{i} 2^{-n}$ is a symmetric binomial.

5.5 Branching process

We introduce here a basic branching process (Z_n) called the Galton-Watson process. It models the sizes of a population of particles which reproduce independently of each other. Suppose the population stems from a single particle: $Z_0 = 1$. Denote by X the offspring number for the ancestor and assume that all particles arising in this process reproduce independently with the numbers of offspring having the same distribution as X. Let $\mu = \mathrm{E}(X)$ and always assume that

$$\mathrm{P}(X = 1) < 1.$$

Consider the number of particles Z_n in generation n. Clearly, (Z_n) is a Markov chain with the state space $\{0, 1, 2, \ldots\}$ where 0 is an absorbing state. The sequence $Q_n = \mathrm{P}(Z_n = 0)$ never decreases. Its limit

$$\eta = \lim_{n\to\infty} \mathrm{P}(Z_n = 0)$$

is called the extinction probability of the Galton-Watson process. Using the branching property

$$Z_{n+1} = X_1^{(n)} + \ldots + X_{Z_n}^{(n)},$$

where the $X_j^{(n)}$ are independent copies of X, we obtain $\mathrm{E}(Z_n) = \mu^n$.

Definition 5.18 The Galton-Watson process is called subcritical if $\mu < 1$. It is called critical if $\mu = 1$, and supercritical if $\mu > 1$.

Theorem 5.19 *Put $h(s) = \mathrm{E}(s^X)$. If $\mu \le 1$, then $\eta = 1$. If $\mu > 1$, then the extinction probability η is the unique solution of the equation $h(x) = x$ in the interval $x \in [0, 1)$.*

Proof. Let $Z_n^{(j)}$, $j = 1, \ldots, X$ be the branching processes stemming from the offspring of the progenitor particle. Then

$$Z_{n+1} = Z_n^{(1)} + \ldots + Z_n^{(X)},$$

and

$$Q_{n+1} = \mathrm{P}(Z_n^{(1)} = 0, \ldots, Z_n^{(X)} = 0) = \mathrm{E}(Q_n^X) = h(Q_n).$$

Letting $n \to \infty$ in the relation $Q_{n+1} = h(Q_n)$ we get $Q = h(Q)$.

Now, if $\mu \le 1$, then there is only one root of $h(x) = x$ in the interval $x \in [0, 1]$ which is $x = 1$. Thus $\eta = 1$. If $\mu > 1$, then there are two roots of $h(x) = x$ in the interval $x \in [0, 1]$: one of them is $x = 1$ and the other $x = x_0$ is less than 1. To see that $\eta = x_0$ is the smaller root, it is enough to observe that

$$Q_1 = \mathrm{P}(X = 0) = h(0) < x_0,$$

so that, by induction,

$$Q_{n+1} = h(Q_n) \le h(x_0) < x_0.$$

Theorem 5.20 *If σ^2 stands for the variance of the offspring number X, then*

$$\mathrm{Var}(Z_n) = \begin{cases} \frac{\sigma^2\mu^{n-1}(1-\mu^n)}{1-\mu} & \text{if } \mu < 1, \\ n\sigma^2 & \text{if } \mu = 1, \\ \frac{\sigma^2\mu^{n-1}(\mu^n-1)}{\mu-1} & \text{if } \mu > 1. \end{cases}$$

Proof. The variance of the generation size $x_n = \mathrm{Var}(Z_n)$ satisfies iteration

$$x_{n+1} = \mu^n\sigma^2 + \mu^2 x_n, \ x_1 = \sigma^2,$$

because

$$\begin{aligned} \mathbb{V}ar(Z_{n+1}) &= \mathrm{E}(\mathrm{Var}(Z_{n+1}|Z_n)) + \mathrm{Var}(\mathrm{E}(Z_{n+1})|Z_n)) \\ &= \mathrm{E}(Z_n\sigma^2) + \mathrm{Var}(\mu Z_n). \end{aligned}$$

Solving this iteration we arrive to the asserted formula.

5.6 Poisson process and continuous-time Markov chains

Definition 5.21 A pure birth process $X(t)$ with intensities $\{\lambda_i\}_{i=0}^\infty$

(i) holds at state i an exponential time with parameter λ_i,

(ii) after the holding time it jumps up from i to $i + 1$.

Exponential holding times has no memory and therefore imply the Markov property in the continuous time setting.

Example 5.22 A Poisson process $N(t)$ with intensity λ is the number of events observed up to time t given that the inter-arrival times are independent exponentials with parameter λ:

$$\mathrm{P}(N(t)=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t},\quad k\geq 0.$$

To check the last formula observe that

$$\mathrm{P}(N(t)=k)=\mathrm{P}(T_1+\ldots+T_k\leq t)-\mathrm{P}(T_1+\ldots+T_{k+1}\leq t),$$

where $T_1+\ldots+T_k$ has a gamma distribution with parameters (k,λ) so that

$$\begin{aligned}\mathrm{P}(T_1+\ldots+T_k\leq t)&=\int_0^t\frac{\lambda^k}{(k-1)!}x^{k-1}e^{-\lambda x}dx=\frac{(\lambda t)^k}{(k-1)!}\int_0^1 y^{k-1}e^{-\lambda ty}dy\\&=\frac{(\lambda t)^k}{k!}e^{-\lambda t}+\frac{(\lambda t)^{k+1}}{k!}\int_0^1 y^k e^{-\lambda ty}dy=\frac{(\lambda t)^k}{k!}e^{-\lambda t}+\mathrm{P}(T_1+\ldots+T_{k+1}\leq t).\end{aligned}$$

Explosion: $\mathrm{P}(X(t)=\infty)>0$ for a finite t. It is possible iff $\sum 1/\lambda_i<\infty$.

Definition 5.23 A continuous-time process $X(t)$ with a countable state space S satisfies the Markov property if

$$\mathrm{P}(X(t_n)=j|X(t_1)=i_1,\ldots,X(t_{n-1})=i_{n-1})=\mathrm{P}(X(t_n)=j|X(t_{n-1})=i_{n-1})$$

for any states $j,i_1,\ldots,i_{n-1}\in S$ and any times $t_1<\ldots<t_n$.

In the time homogeneous case compared to the discrete time case instead of transition matrices $\mathbf{P}^n$ with elements $p_{ij}^{(n)}$ we have transition matrices $\mathbf{P}_t$ with elements

$$p_{ij}(t)=\mathrm{P}(X(u+t)=j|X(u)=i).$$

Chapman-Kolmogorov: $\mathbf{P}_{t+s}=\mathbf{P}_t\mathbf{P}_s$ for all $t\geq 0$ and $s\geq 0$. Here $\mathbf{P}_0=\mathbf{I}$ is the identity matrix.

Example 5.24 For the Poisson process we have $p_{ij}(t)=\frac{(\lambda t)^{j-i}}{(j-i)!}e^{-\lambda t}$, and

$$\sum_k p_{ik}(t)p_{kj}(s)=\sum_k\frac{(\lambda t)^{k-i}}{(k-i)!}e^{-\lambda t}\frac{(\lambda s)^{j-k}}{(j-k)!}e^{-\lambda s}=\frac{(\lambda t+\lambda s)^{j-i}}{(j-i)!}e^{-\lambda(t+s)}=p_{ij}(t+s).$$

5.7 The generator of a continuous-time Markov chain

A generator $\mathbf{G}=(g_{ij})$ is a matrix with non-negative off-diagonal elements such that $\sum_j g_{ij}=0$. A Markov chain $X(t)$ with generator $\mathbf{G}$

- holds at state i an exponential time with parameter $\lambda_i=-g_{ii}$,
- after the holding time it jumps from i to $j\neq i$ with probability $h_{ij}=\frac{g_{ij}}{\lambda_i}$.

The embedded discrete Markov chain is governed by transition matrix $\mathbf{H}=(h_{ij})$ satisfying $h_{ii}=0$. A continuous-time MC is a discrete MC plus holding intensities (λ_i).

Example 5.25 The Poisson process and birth process have the same embedded MC with $h_{i,i+1}=1$. For the birth process $g_{ii}=-\lambda_i$, $g_{i,i+1}=\lambda_i$ and all other $g_{ij}=0$.

Kolmogorov equations. Forward equation: for any $i,j\in S$

$$p'_{ij}(t)=\sum_k p_{ik}(t)g_{kj}$$

or in the matrix form $\mathbf{P}'_t=\mathbf{P}_t\mathbf{G}$. It is obtained from $\mathbf{P}_{t+\epsilon}-\mathbf{P}_t=\mathbf{P}_t(\mathbf{P}_\epsilon-\mathbf{P}_0)$ watching for the last change. Backward equation $\mathbf{P}'_t=\mathbf{G}\mathbf{P}_t$ is obtained from $\mathbf{P}_{t+\epsilon}-\mathbf{P}_t=(\mathbf{P}_\epsilon-\mathbf{P}_0)\mathbf{P}_t$ watching for the initial change. These equations often have a unique solution

$$\mathbf{P}_t=e^{t\mathbf{G}}:=\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathbf{G}^n.$$

Theorem 5.26 *Stationary distribution: $\boldsymbol{\pi}\mathbf{P}_t = \boldsymbol{\pi}$ for all t iff $\boldsymbol{\pi}\mathbf{G} = \mathbf{0}$.*

PROOF:

$$\boldsymbol{\pi}\mathbf{P}_t \overset{\forall t}{=} \boldsymbol{\pi} \quad \Leftrightarrow \quad \sum_{n=0}^{\infty} \frac{t^n}{n!}\boldsymbol{\pi}\mathbf{G}^n \overset{\forall t}{=} \boldsymbol{\pi} \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{t^n}{n!}\boldsymbol{\pi}\mathbf{G}^n \overset{\forall t}{=} \mathbf{0} \quad \Leftrightarrow \quad \boldsymbol{\pi}\mathbf{G}^n \overset{\forall n}{=} \mathbf{0}.$$

Example 5.27 Check that the birth process has no stationary distribution.

Theorem 5.28 *Let $X(t)$ be irreducible with generator $\mathbf{G}$. If there exists a stationary distribution $\boldsymbol{\pi}$, then it is unique and for all (i, j)*

$$p_{ij}(t) \to \pi_j, \quad t \to \infty.$$

If there is no stationary distribution, then $p_{ij}(t) \to 0$ as $t \to \infty$.

Example 5.29 Poisson process holding times $\lambda_i = \lambda$. Then $\mathbf{G} = \lambda(\mathbf{H} - \mathbf{I})$ and $\mathbf{P}_t = e^{\lambda t(\mathbf{H}-\mathbf{I})}$.

6 Stationary processes

6.1 Weakly and strongly stationary processes

Definition 6.1 A real-valued process $\{X(t), t \geq 0\}$ is called strongly stationary if the vectors $(X(t_1), \ldots, X(t_n))$ and $(X(t_1 + h), \ldots, X(t_n + h))$ have the same joint distribution for all $t_1, \ldots, t_n$ and $h > 0$.

Definition 6.2 A real-valued process $\{X(t), t \geq 0\}$ is called weakly stationary with mean μ and autocovariance function $c(h)$ if for all $t \geq 0$ and $h \geq 0$

$$\mathrm{E}(X(t)) = \mu, \qquad \mathrm{Cov}(X(t), X(t + h)) = c(h) \in (-\infty, \infty).$$

Given that $c(0) > 0$, the ratio $\rho(h) = \frac{c(h)}{c(0)}$ is called the autocorrelation function of the weakly stationary process.

Example 6.3 Consider an irreducible Markov chain $\{X(t), t \geq 0\}$ with countably many states and a stationary distribution $\boldsymbol{\pi}$ as the initial distribution. This is a strongly stationary process since

$$\mathrm{P}(X(h + t_1) = i_1, X(h + t_1 + t_2) = i_2, \ldots, X(h + t_1 + \ldots + t_n) = i_n) = \pi_{i_1} p_{i_1,i_2}(t_2) \ldots p_{i_{n-1},i_n}(t_n).$$

More specifically, look at Example 5.13.

Example 6.4 A sequence $\{X_n, n = 1, 2, \ldots\}$ of independent random variables with the Cauchy distribution is an example of a strongly stationary process, which is not weakly stationary.

6.2 Linear prediction

Knowing the past $(X_r, X_{r-1}, \ldots, X_{r-s})$ we would like to predict a future value X_{r+k} by a linear combination

$$\hat{X}_{r+k} = a + a_0 X_r + a_1 X_{r-1} + \ldots + a_s X_{r-s}.$$

We will call $\hat{X}_{r+k}$ the best linear predictor if $a = a(k, s), a_j = a_j(k, s), 0 \leq j \leq s$ are chosen to minimise the mean-square error size $\mathrm{E}(X_{r+k} - \hat{X}_{r+k})^2$.

Theorem 6.5 *For a weakly stationary sequence with mean μ and autocovariance $c(m)$, the best linear predictor is given by*

$$a = \mu(1 - a_0 - \ldots - a_s),$$

and the vector $(a_0, \ldots, a_s)$ satisfying the system of linear equations

$$\sum_{j=0}^{s} a_j c(|j - m|) = c(k + m), \quad 0 \leq m \leq s.$$

Proof. By subtracting the mean μ we reduce the problem to the zero-mean case. Geometrically, in the zero-mean case, the best linear predictor $\hat{X}_{r+k}$ makes an error, $X_{r+k} - \hat{X}_{r+k}$, which is *orthogonal* to the past $(X_r, X_{r-1}, \ldots, X_{r-s})$, in that the covariances are equal to zero:

$$\mathrm{E}((X_{r+k} - \hat{X}_{r+k})X_{r-m}) = 0, \quad m = 0, \ldots, s.$$

Plugging $\hat{X}_{r+k} = a_0X_r + a_1X_{r-1} + \ldots + a_sX_{r-s}$ into the last relation, we arrive at the claimed equations.

To justify the geometric intuition let H be a linear space generated by $(X_r, X_{r-1}, \ldots, X_{r-s})$. In the zero-mean case, if $W \in H$, then for any real a,

$$\mathrm{E}(X_{r+k} - a - W)^2 = a^2 + \mathrm{E}(X_{r+k} - W)^2 \geq \mathrm{E}(X_{r+k} - W)^2,$$

showing that the best linear predictor belongs to H. We have to show that if $W \in H$ is such that for all $Z \in H$,

$$\mathrm{E}(X_{r+k} - W)^2 \leq \mathrm{E}(X_{r+k} - Z)^2,$$

then

$$\mathrm{E}((X_{r+k} - W)X_{r-m}) = 0, \quad m = 0, \ldots, s.$$

Indeed, suppose to the contrary that for some m,

$$\mathrm{E}((X_{r+k} - W)X_{r-m}) = c(0)d, \qquad d > 0.$$

Then writing $W' = W + d \cdot X_{r-m}$ we arrive at a contradiction

$$\begin{aligned}\mathrm{E}(X_{r+k} - W')^2 &= \mathrm{E}(X_{r+k} - W - d \cdot X_{r-m})^2 \\ &= \mathrm{E}(X_{r+k} - W)^2 - 2d \cdot \mathrm{E}(X_{r+k} - W)X_{r-m} + d^2c(0) \\ &= \mathrm{E}(X_{r+k} - W)^2 - d^2c(0).\end{aligned}$$

Example 6.6 AR(1) process Y_n satisfies

$$Y_n = \alpha Y_{n-1} + Z_n, \quad -\infty < n < \infty,$$

where Z_n are independent r.v. with zero means and unit variance. If $|\alpha| < 1$, then $Y_n = \sum_{m\geq 0} \alpha^m Z_{n-m}$ is weakly stationary with zero mean and autocovariance for $m \geq 0$,

$$\begin{aligned}c(m) = \mathrm{E}(Y_nY_{n+m}) &= \mathrm{E}(Z_n + Z_{n-1}\alpha + Z_{n-2}\alpha^2 + \ldots)(Z_{n+m} + Z_{n+m-1}\alpha + Z_{n+m-2}\alpha^2 + \ldots) \\ &= \mathrm{E}(Z_n^2)\alpha^m + \mathrm{E}(Z_{n-1}^2)\alpha\alpha^{m+1} + \mathrm{E}(Z_{n-2}^2)\alpha^2\alpha^{m+2} + \ldots \\ &= \alpha^m + \alpha^{m+2} + \alpha^{m+4} + \ldots = \frac{\alpha^m}{1 - \alpha^2}.\end{aligned}$$

The best linear predictor is $\hat{Y}_{r+k} = \alpha^k Y_r$. This follows from the equations

$$\begin{aligned}a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_s\alpha^s &= \alpha^k, \\ a_0\alpha + a_1 + a_2\alpha + \ldots + a_s\alpha^{s-1} &= \alpha^{k+1}.\end{aligned}$$

The mean squared error of the best prediction is

$$\mathrm{E}(\hat{Y}_{r+k} - Y_{r+k})^2 = \mathrm{E}(\alpha^k Y_r - Y_{r+k})^2 = \alpha^{2k}c(0) + c(0) - 2\alpha^k c(k) = \frac{1 - \alpha^{2k}}{1 - \alpha^2}.$$

Example 6.7 Let $X_n = (-1)^n X_0$, where X_0 is -1 or 1 equally likely. The best linear predictor is $\hat{X}_{r+k} = (-1)^k X_r$. The mean squared error of prediction is zero.

6.3 Linear combination of sinusoids

Example 6.8 For a sequence of fixed frequencies $0 \leq \lambda_1 < \ldots < \lambda_k < \infty$ define a continuous time stochastic process by

$$X(t) = \sum_{j=1}^{k} (A_j \cos(\lambda_j t) + B_j \sin(\lambda_j t)),$$

where $A_1, B_1, \ldots, A_k, B_k$ are uncorrelated r.v. with zero means and $\mathrm{Var}(A_j) = \mathrm{Var}(B_j) = \sigma_j^2$. Then

$$\begin{aligned}\mathrm{Cov}(X(t), X(s)) = \mathrm{E}(X(t)X(s)) &= \sum_{j=1}^{k} \mathrm{E}(A_j^2 \cos(\lambda_j t)\cos(\lambda_j s) + B_j^2 \sin(\lambda_j t)\sin(\lambda_j s)) \\ &= \sum_{j=1}^{k} \sigma_j^2 \cos(\lambda_j (s-t)), \\ \mathrm{Var}(X(t)) &= \sum_{j=1}^{k} \sigma_j^2.\end{aligned}$$

Thus $X(t)$ is weakly stationary with $\mu = 0$ and

$$\begin{aligned} c(t) &= \sum_{j=1}^{k} \sigma_j^2 \cos(\lambda_j t), \quad c(0) = \sum_{j=1}^{k} \sigma_j^2, \\ \rho(t) &= \frac{c(t)}{c(0)} = \sum_{j=1}^{k} g_j \cos(\lambda_j t) = \int_0^\infty \cos(\lambda t) dG(\lambda),\end{aligned}$$

where G is a distribution function over $[0, \infty)$ defined as

$$g_j = \frac{\sigma_j^2}{\sigma_1^2 + \ldots + \sigma_k^2}, \qquad G(\lambda) = \sum_{j:\lambda_j \leq \lambda} g_j.$$

We can write

$$X(t) = \int_0^\infty \cos(t\lambda) dU(\lambda) + \int_0^\infty \sin(t\lambda) dV(\lambda),$$

where

$$U(\lambda) = \sum_{j:\lambda_j \leq \lambda} A_j, \qquad V(\lambda) = \sum_{j:\lambda_j \leq \lambda} B_j.$$

Example 6.9 Let specialize further and put $k = 1$, $\lambda_1 = \frac{\pi}{4}$, assuming that A_1 and B_1 are iid with

$$\mathrm{P}(A_1 = \frac{1}{\sqrt{2}}) = \mathrm{P}(A_1 = -\frac{1}{\sqrt{2}}) = \frac{1}{2}.$$

Then $X(t) = \cos(\frac{\pi}{4}(t + \tau))$ with

$$\mathrm{P}(\tau = 1) = \mathrm{P}(\tau = -1) = \mathrm{P}(\tau = 3) = \mathrm{P}(\tau = -3) = \frac{1}{4}.$$

This stochastic process has only four possible trajectories. This is not a strongly stationary process since

$$\mathrm{E}(X^4(t)) = \frac{1}{2}\Big(\cos^4\left(\frac{\pi}{4}t + \frac{\pi}{4}\right) + \sin^4\left(\frac{\pi}{4}t + \frac{\pi}{4}\right)\Big) = \frac{1}{4}\Big(2 - \sin^2\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)\Big) = \frac{1 + \sin^2(\frac{\pi}{2}t)}{2}.$$

Example 6.10 Put

$$X(t) = \cos(t + Y) = \cos(t)\cos(Y) - \sin(t)\sin(Y),$$

where Y is uniformly distributed over $[0, 2\pi]$. In this case $k = 1, \lambda = 1, \sigma_1^2 = \frac{1}{2}$. It is a strongly stationary process, since $X(t + h) = \cos(t + Y')$, where $Y' = Y + h$ is uniformly distributed over $[0, 2\pi]$ modulo 2π.

To find the distribution of $X(t)$ observe that for an arbitrary bounded measurable function $\phi(x)$,

$$\begin{aligned}\mathrm{E}(\phi(X(t))) = \mathrm{E}(\phi(\cos(t+Y))) &= \frac{1}{2\pi}\int_0^{2\pi}\phi(\cos(t+y))dy = \frac{1}{2\pi}\int_t^{t+2\pi}\phi(\cos(z))dz = \frac{1}{2\pi}\int_0^{2\pi}\phi(\cos(z))dz \\ &= \frac{1}{2\pi}\Big(\int_0^{\pi}\phi(\cos(z))dz + \int_\pi^{2\pi}\phi(\cos(z))dz\Big) = \frac{1}{2\pi}\Big(\int_0^{\pi}\phi(\cos(\pi-y))dy + \int_0^{\pi}\phi(\cos(\pi+y))dy\Big) \\ &= \frac{1}{\pi}\int_0^{\pi}\phi(-\cos(y))dy.\end{aligned}$$

The change of variables $x = -\cos(y)$ yields $dx = \sin(y)dy = \sqrt{1-x^2}dy$, hence

$$\mathrm{E}(\phi(X)) = \frac{1}{\pi}\int_{-1}^{1}\frac{\phi(x)dx}{\sqrt{1-x^2}}.$$

Thus $X(t)$ has the so-called arcsine density $f(x) = \frac{1}{\pi\sqrt{1-x^2}}$ over the interval $[-1,1]$. Notice that $Z = \frac{X+1}{2}$ has a Beta$(\frac{1}{2},\frac{1}{2})$ distribution, since

$$\mathrm{E}(\phi(Z)) = \frac{1}{\pi}\int_{-1}^{1}\frac{\phi(\frac{x+1}{2})dx}{\sqrt{1-x^2}} = \frac{1}{\pi}\int_0^1\frac{\phi(z)dz}{\sqrt{z(1-z)}}.$$

Example 6.11 In the *discrete time setting* for $n \in \mathbb{Z}$ put

$$X_n = \sum_{j=1}^{k}(A_j\cos(\lambda_j n) + B_j\sin(\lambda_j n)),$$

where $0 \le \lambda_1 < \ldots < \lambda_k \le \pi$ is a set of fixed frequencies, and again, $A_1, B_1, \ldots, A_k, B_k$ are uncorrelated r.v. with zero means and $\mathrm{Var}(A_j) = \mathrm{Var}(B_j) = \sigma_j^2$. Similarly to the continuous time case we get

$$\mathrm{E}(X_n) = 0, \quad c(n) = \sum_{j=1}^{k}\sigma_j^2\cos(\lambda_j n), \quad \rho(n) = \int_0^{\pi}\cos(\lambda n)dG(\lambda),$$

$$X_n = \int_0^{\pi}\cos(n\lambda)dU(\lambda) + \int_0^{\pi}\sin(n\lambda)dV(\lambda).$$

6.4 Spectral representation

It turns out that any weakly stationary process $\{X(t) : -\infty < t < \infty\}$ with zero mean can be approximated by a linear combination of uncorrelated sinusoids.

The autocovariance $c(t)$ of a stationary process is a non-negative definite function in that

$$\sum_{j=1}^{n}\sum_{k=1}^{n}c(t_k - t_j)z_j z_k = \mathrm{Var}\Big(\sum_{k=1}^{n}z_k X(t_k)\Big) \ge 0$$

for any $t_1, \ldots, t_n$ and $z_1, \ldots, z_n$. Thus due to the Bochner theorem, given that $c(t)$ is continuous at zero, there is a probability distribution function G on $[0,\infty)$ such that

$$\rho(t) = \int_0^{\infty}\cos(t\lambda)dG(\lambda).$$

In the discrete time case there is a probability distribution function G on $[0,\pi]$ such that

$$\rho(n) = \int_0^{\pi}\cos(n\lambda)dG(\lambda).$$

Definition 6.12 The function G is called the spectral distribution function of the corresponding stationary random process, and the set of real numbers λ such that

$$G(\lambda+\epsilon) - G(\lambda-\epsilon) > 0 \text{ for all } \epsilon > 0$$

is called the spectrum of the random process. If G has density g it is called the spectral density function.

In the discrete time case, if

$$\sum_{n=-\infty}^{\infty} |\rho(n)| < \infty,$$

then the spectral density exists and is given by

$$g(\lambda) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \rho(n) \cos(n\lambda), \quad 0 \le \lambda \le \pi.$$

Exercise 6.13 Find the autocorrelation function of a stationary process with the spectral density function
(a) $f(\lambda) = \sqrt{2/\pi} e^{-\lambda^2/2}$ for $\lambda \ge 0$,
(b) $f(\lambda) = e^{-\lambda}$ for $\lambda \ge 0$.
Discuss the differences in the dependence patterns for these two cases.

Solution. (a) To compute

$$\rho(t) = \sqrt{2/\pi} \int_0^\infty \cos(\lambda t) e^{-\lambda^2/2} \mathrm{d}\lambda = \sqrt{1/(2\pi)} \int_{-\infty}^\infty \cos(\lambda t) e^{-\lambda^2/2} \mathrm{d}\lambda,$$

we can use the formula for the characteristic function of the standard normal distribution

$$\sqrt{1/(2\pi)} \int_{-\infty}^\infty e^{it\lambda} e^{-\lambda^2/2} \mathrm{d}\lambda = e^{-t^2/2}.$$

We get immediately that $\rho(t) = e^{-t^2/2}$. The correlation between two values of the process separate by t units of time very quickly decreases with increasing t.

(b) To compute

$$\rho(t) = \int_0^\infty \cos(\lambda t) e^{-\lambda} \mathrm{d}\lambda,$$

we can use the formula for the characteristic function of the exponential distribution

$$\int_{-\infty}^\infty e^{it\lambda} e^{-\lambda} \mathrm{d}\lambda = \frac{1}{1-it} = \frac{1+it}{1+t^2}.$$

We derive $\rho(t) = \frac{1}{1+t^2}$. Correlation decreases much slower than in the previous case.

Example 6.14 Consider an irreducible continuous time Markov chain $\{X(t), t \ge 0\}$ with two states $\{1, 2\}$ and generator

$$\mathbf{G} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

Its stationary distribution is $\boldsymbol{\pi} = (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$ will be taken as the initial distribution. From

$$\begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} = e^{t\mathbf{G}} = \sum_{n=0}^\infty \frac{t^n}{n!} \mathbf{G}^n = \mathbf{I} + \mathbf{G} \sum_{n=1}^\infty \frac{t^n}{n!} (-\alpha - \beta)^{n-1} = \mathbf{I} + (\alpha+\beta)^{-1} (1 - e^{-t(\alpha+\beta)}) \mathbf{G}$$

we see that

$$p_{11}(t) = 1 - p_{12}(t) = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} e^{-t(\alpha+\beta)},$$
$$p_{22}(t) = 1 - p_{21}(t) = \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} e^{-t(\alpha+\beta)},$$

and we find for $t \ge 0$

$$c(t) = \frac{\alpha\beta}{(\alpha+\beta)^2} e^{-t(\alpha+\beta)}, \qquad \rho(t) = e^{-t(\alpha+\beta)}.$$

Thus this process has a spectral density corresponding to a one-sided Cauchy distribution:

$$g(\lambda) = \frac{2(\alpha+\beta)}{\pi((\alpha+\beta)^2 + \lambda^2)}, \quad \lambda \ge 0.$$

Example 6.15 Discrete white noise: a sequence $X_0, X_1, \ldots$ of independent r.v. with zero means and unit variances. This stationary sequence has the uniform spectral density:

$$\rho(n) = 1_{\{n=0\}} = \pi^{-1} \int_0^\pi \cos(n\lambda) d\lambda.$$

Theorem 6.16 *If $\{X(t) : -\infty < t < \infty\}$ is a weakly stationary process with zero mean, unit variance, continuous autocorrelation function and spectral distribution function G, then there exists an orthogonal pair of zero mean random process $(U(\lambda), V(\lambda))$ with uncorrelated increments such that*

$$X(t) = \int_0^\infty \cos(t\lambda) dU(\lambda) + \int_0^\infty \sin(t\lambda) dV(\lambda)$$

and $\mathrm{Var}(U(\lambda)) = \mathrm{Var}(V(\lambda)) = G(\lambda)$.

Theorem 6.17 *If $\{X_n : -\infty < n < \infty\}$ is a discrete-time weakly stationary process with zero mean, unit variance, and spectral distribution function G, then there exists an orthogonal pair of zero mean random process $(U(\lambda), V(\lambda))$ with uncorrelated increments such that*

$$X_n = \int_0^\pi \cos(n\lambda) dU(\lambda) + \int_0^\pi \sin(n\lambda) dV(\lambda)$$

and $\mathrm{Var}(U(\lambda)) = \mathrm{Var}(V(\lambda)) = G(\lambda)$.

6.5 Stochastic integral

Let $\{S(t) : t \in \mathbb{R}\}$ be a complex-valued process on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ such that

- $\mathrm{E}(|S(t)|^2) < \infty$ for all t,
- $\mathrm{E}(|S(t+h) - S(t)|^2) \to 0$ as $h \searrow 0$ for all t,
- orthogonal increments: $\mathrm{E}([S(v) - S(u)][\bar{S}(t) - \bar{S}(s)]) = 0$ whenever $u < v \le s < t$.

Put

$$F(t) := \begin{cases} \mathrm{E}(|S(t) - S(0)|^2), & \text{if } t \ge 0, \\ -\mathrm{E}(|S(t) - S(0)|^2), & \text{if } t < 0. \end{cases}$$

Since the process has orthogonal increments we obtain

$$\mathrm{E}(|S(t) - S(s)|^2) = F(t) - F(s), \quad s < t \qquad (*)$$

implying that F is monotonic and right-continuous.

Let $\psi : \mathbb{R} \to \mathbb{C}$ be a measurable complex-valued function for which

$$\int_{-\infty}^\infty |\psi(t)|^2 dF(t) < \infty.$$

Next comes a two-step definition of a stochastic integral of ψ with respect to S,

$$I(\psi) = \int_{-\infty}^\infty \psi(t) dS(t),$$

possessing the following important property

$$\mathrm{E}(I(\psi_1) I(\psi_2)) = \int_{-\infty}^\infty \psi_1(t) \psi_2(t) dF(t). \qquad (**)$$

1. For an arbitrary step function

$$\phi(t) = \sum_{j=1}^{n-1} c_j 1_{\{a_j \le t < a_{j+1}\}}, \quad -\infty < a_1 < \ldots < a_n < \infty$$

   put

$$I(\phi) := \sum_{j=1}^{n-1} c_j (S(a_{j+1}) - S(a_j)).$$

   Due to orthogonality of increments we obtain $(**)$ and find that "integration is distance preserving"

$$\mathrm{E}(|I(\phi_1) - I(\phi_2)|^2) = \mathrm{E}((I(\phi_1 - \phi_2))^2) = \int_{-\infty}^{\infty} |\phi_1 - \phi_2|^2 dF(t).$$

2. There exists a sequence of step functions such that

$$\|\phi_n - \psi\| := \Big(\int_{-\infty}^{\infty} |\phi_n - \psi|^2 dF(t)\Big)^{1/2} \to 0.$$

   Thus $I(\phi_n)$ is a mean-square Cauchy sequence and there exists a mean-square limit $I(\phi_n) \to I(\psi)$.

Proof sketch of Theorem 6.17 for the complex-valued processes.

Step 1. Let H_X be the set of all r.v of the form $\sum_{j=1}^{n} a_j X_{m_j}$ for $a_1, a_2, \ldots \in \mathbb{C}$, $n \in \mathbb{N}$, $m_1, m_2, \ldots \in \mathbb{Z}$. Similarly, let H_F be the set of linear combinations of sinusoids $f_n(x) := e^{inx}$. Define the linear mapping $\mu : H_F \to H_X$ by $\mu(f_n) := X_n$.

Step 2. The closure $\overline{H}_X$ of H_X is defined to be the space H_X together with all limits of mean-square Cauchy-convergent sequences in H_X. Define the closure $\overline{H}_F$ of H_F as the space H_F together with all limits of Cauchy-convergent sequences $u_n \in H_F$, with the latter meaning by definition that

$$\int_{(-\pi,\pi]} (u_n(\lambda) - u_m(\lambda))\overline{(u_n(\lambda) - u_m(\lambda))} dF(\lambda) \to 0, \qquad n, m \to \infty.$$

For $u = \lim u_n$, where $u_n \in H_F$, define $\mu(u) = \lim \mu(u_n)$ thereby defining a mapping $\mu : \overline{H}_F \to \overline{H}_X$.

Step 3. Define the process $S(\lambda)$ by

$$S(\lambda) = \mu(h_\lambda), \quad -\pi < \lambda \le \pi, \quad h_\lambda(x) := 1_{\{x \in (-\pi,\lambda]\}}$$

and show that it has orthogonal increments and satisfies $(*)$. Prove that

$$\mu(\psi) = \int_{(-\pi,\pi]} \psi(t) dS(t)$$

first for step-functions and then for $\psi(x) = e^{inx}$. It follows that

$$X_n = \int_{(-\pi,\pi]} e^{int} dS(t).$$

The case of real-valued processes.
For the sequence X_n to be real, the following representation

$$\begin{aligned} X_n &= \int_{(-\pi,\pi]} e^{in\lambda} dS(\lambda) = \int_{(0,\pi]} e^{in\lambda} dS(\lambda) + \int_{[0,\pi)} e^{-in\lambda} dS(-\lambda) \\ &= \int_{[0,\pi]} \cos(n\lambda)(dS(\lambda) + dS(-\lambda)) + i \int_{[0,\pi]} \sin(n\lambda)(dS(\lambda) - dS(-\lambda)) \\ &= \int_{[0,\pi]} \cos(n\lambda) dU(\lambda) + \int_{[0,\pi]} \sin(n\lambda) dV(\lambda) \end{aligned}$$

must hold for real processes $U(\lambda)$ and $V(\lambda)$ such that

$$\begin{aligned} dU(\lambda) &= dS(\lambda) + dS(-\lambda), \\ dV(\lambda) &= i(dS(\lambda) - dS(-\lambda)). \end{aligned}$$

This implies that $dS(\lambda) + dS(-\lambda)$ is purely real and $dS(\lambda) - dS(-\lambda)$ is purely imaginary, which implies a symmetry property: $dS(\lambda) = \overline{dS(-\lambda)}$.

Example 6.18 The Wiener process W_t is an example of a process $S(t)$ with orthogonal increments. In this case the key property $(**)$ holds with $F(t) = t$:

$$\mathrm{E}\Big(\int_0^\infty \psi_1(t)dW_t \times \int_0^\infty \psi_2(t)dW_t\Big) = \int_0^\infty \psi_1(t)\psi_2(t)dt.$$

Exercise 6.19 Check the properties of $U(\lambda)$ and $V(\lambda)$ stated in Theorem 6.17 with $dG(\lambda) = 2dF(\lambda)$ for $\lambda > 0$ and $G(0) - G(0-) = F(0) - F(0-)$.

6.6 Ergodic theorem for the weakly stationary processes

Theorem 6.20 *Let $\{X_n, n = 1, 2, \ldots\}$ be a weakly stationary process with mean μ and autocovariance function $c(m)$. There exists a r.v. Y with mean μ and variance*

$$\mathrm{Var}(Y) = c(0)(G(0) - G(0-)) = \lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} c(j),$$

such that

$$\frac{X_1 + \ldots + X_n}{n} \overset{L^2}{\to} Y, \qquad n \to \infty.$$

Proof sketch. Suppose that $\mu = 0$ and $c(0) = 1$, then using a spectral representation

$$X_n = \int_0^\pi \cos(n\lambda)dU(\lambda) + \int_0^\pi \sin(n\lambda)dV(\lambda),$$

we get

$$\bar{X}_n := \frac{X_1 + \ldots + X_n}{n} = \int_0^\pi g_n(\lambda)dU(\lambda) + \int_0^\pi h_n(\lambda)dV(\lambda),$$

where

$$g_n(\lambda) = \frac{\cos(\lambda) + \ldots + \cos(n\lambda)}{n}, \quad h_n(\lambda) = \frac{\sin(\lambda) + \ldots + \sin(n\lambda)}{n}.$$

In terms of complex numbers we have

$$g_n(\lambda) + ih_n(\lambda) = \frac{e^{i\lambda} + \ldots + e^{in\lambda}}{n}.$$

The last expression equals 1 for $\lambda = 0$. For $\lambda \neq 0$, we get

$$\begin{aligned} g_n(\lambda) + ih_n(\lambda) &= \frac{e^{i\lambda} - e^{in\lambda}}{n(1 - e^{i\lambda})} = \frac{(e^{i\lambda} - e^{in\lambda})(1 - e^{-i\lambda})}{2n(1 - \cos(\lambda))} \\ &= \frac{e^{i\lambda} - 1 + e^{i(n-1)\lambda} - e^{-in\lambda}}{4n\sin^2(\lambda/2)}, \end{aligned}$$

so that

$$\begin{aligned} g_n(\lambda) &= \frac{\cos(\lambda) - 1 + \cos((n-1)\lambda) - \cos(n\lambda)}{4n\sin^2(\lambda/2)} = \frac{\sin((n-1)\lambda/2) - \sin(\lambda/2)}{2n\sin(\lambda/2)}, \\ h_n(\lambda) &= \frac{\sin(\lambda) + \sin((n-1)\lambda) - \sin(n\lambda)}{4n\sin^2(\lambda/2)} = \frac{\cos(\lambda/2) - \cos((n-1)\lambda/2)}{2n\sin(\lambda/2)}. \end{aligned}$$

If $\lambda \in [0, \pi]$, then $|g_n(\lambda)| \le 1$, $|h_n(\lambda)| \le 1$, and $g_n(\lambda) \to 1_{\{\lambda=0\}}$, $h_n(\lambda) \to 0$ as $n \to \infty$. It can be shown that

$$\int_0^\pi g_n(\lambda) dU(\lambda) \overset{L^2}{\to} \int_0^\pi 1_{\{\lambda=0\}} dU(\lambda) = U(0) - U(0-),$$
$$\int_0^\pi h_n(\lambda) dV(\lambda) \overset{L^2}{\to} 0.$$

Thus $\bar{X}_n \overset{L^2}{\to} Y := U(0) - U(0-)$ and by Theorem 6.17, we have

$$\operatorname{Var}(Y) = \operatorname{Var}(U(0) - U(0-)) = c(0)(G(0) - G(0-)).$$

It remains to observe that

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^n c(j) = c(0)(G(0) - G(0-)),$$

which follows from the representation

$$n^{-1} \sum_{j=1}^n c(j) = c(0) \int_0^\pi g_n(\lambda) dG(\lambda).$$

6.7 Ergodic theorem for the strongly stationary processes

Theorem 6.21 *If $\{X_n, n = 1, 2, \ldots\}$ is a strongly stationary sequence with a finite mean μ, then there is a random variable Y with the mean μ such that*

$$\bar{X}_n := \frac{X_1 + \ldots + X_n}{n} \to Y \textit{ a.s. and in } L^1.$$

In the ergodic case, when $\operatorname{Var}(Y) = 0$,

$$\bar{X}_n \to \mu \textit{ a.s. and in } L^1.$$

Without a proof.

Example 6.22 Let $Z_1, \ldots, Z_k$ be iid with a finite mean μ. Then the following cyclic process

$$X_1 = Z_1, \ldots, X_k = Z_k,$$
$$X_{k+1} = Z_1, \ldots, X_{2k} = Z_k,$$
$$X_{2k+1} = Z_1, \ldots, X_{3k} = Z_k, \ldots,$$

is a strongly stationary process. The corresponding limit in the ergodic theorem is not the constant μ like in the strong LLN but rather a random variable

$$\frac{X_1 + \ldots + X_n}{n} \to \frac{Z_1 + \ldots + Z_k}{k}.$$

Example 6.23 Let $\{X_n, n = 1, 2, \ldots\}$ be an irreducible positive-recurrent Markov chain with the state space $S = \{0, \pm 1, \pm 2, \ldots\}$. Let $\boldsymbol{\pi} = (\pi_j)_{j \in S}$ be the unique stationary distribution. If X_0 has distribution $\boldsymbol{\pi}$, then X_n is strongly stationary.

For a fixed state $k \in S$ let $I_n = 1_{\{X_n = k\}}$. The stronlgy stationary process I_n has mean $\mu = \pi_k$ and autocovariance function

$$c(m) = \operatorname{Cov}(I_n, I_{n+m}) = \operatorname{E}(I_n I_{n+m}) - \pi_k^2 = \pi_k(p_{kk}^{(m)} - \pi_k).$$

Since $p_{kk}^{(m)} \to \pi_k$ as $m \to \infty$ we have $c(m) \to 0$ and the limit in Theorem 6.20 has zero variance. It follows that $n^{-1}(I_1 + \ldots + I_n)$, the proportion of $(X_1, \ldots, X_n)$ visiting state k, converges to π_k as $n \to \infty$ in L^2. It follows that this process is ergodic and the convergence also holds almost surely.

Example 6.24 Binary expansion. Let X be uniformly distributed on $[0, 1]$ and has a binary expansion $X = \sum_{j=1}^{\infty} X_j 2^{-j}$. Put $Y_n = \sum_{j=n}^{\infty} X_j 2^{n-j-1}$ so that $Y_1 = X$ and $Y_{n+1} = 2Y_n \pmod 1$. We see that (Y_n) is a strictly stationary Markov chain. From $Y_{n+1} = 2^n X \pmod 1$ we derive

$$\begin{aligned}
\mathrm{E}(Y_1 Y_{n+1}) &= \int_0^1 x(2^n x)_{\mathrm{mod} 1} dx = \sum_{j=0}^{2^n-1} \int_{j2^{-n}}^{(j+1)2^{-n}} x(2^n x - j) dx \\
&= 2^{-2n} \sum_{j=0}^{2^n-1} \int_0^1 (y+j) y dy = 2^{-2n} \sum_{j=0}^{2^n-1} \Big(\frac{1}{3} + \frac{j}{2}\Big) = \frac{1}{4} + \frac{2^{-n}}{12}.
\end{aligned}$$

Thus $c(n) = \frac{2^{-n}}{12}$ tends to zero as $n \to \infty$. This implies that $n^{-1} \sum_{j=1}^{n} Y_j \to 1/2$ in L^2 and almost surely.

6.8 Gaussian processes

Definition 6.25 A random process $\{X(t), t \geq 0\}$ is called Gaussian if for any $(t_1, \ldots, t_n)$ the vector $(X(t_1), \ldots, X(t_n))$ has a multivariate normal distribution.

A Gaussian random process is strongly stationary iff it is weakly stationary.

Theorem 6.26 *A Gaussian process $\{X(t), t \geq 0\}$ is Markov iff for any $0 \leq t_1 < \ldots < t_n$*

$$\mathrm{E}(X(t_n)|X(t_1), \ldots, X(t_{n-1})) = \mathrm{E}(X(t_n)|X(t_{n-1})). \qquad (*)$$

Proof. Clearly, the Markov property implies $(*)$. To prove the converse we have to show that in the Gaussian case $(*)$ gives

$$\mathrm{Var}(X(t_n)|X(t_1), \ldots, X(t_{n-1})) = \mathrm{Var}(X(t_n)|X(t_{n-1})),$$

since the conditional distribution is determined by the conditonal mean and variance. The last eaquality is verified using the fact that $X(t_n) - \mathrm{E}\{X(t_n)|X(t_1), \ldots, X(t_{n-1})\}$ is orthogonal to $(X(t_1), \ldots, X(t_{n-1})$, which in the Gaussian case means independence:

$$\begin{aligned}
&\mathrm{E}\Big\{\Big(X(t_n) - \mathrm{E}\{X(t_n)|X(t_1), \ldots, X(t_{n-1})\}\Big)^2 |X(t_1), \ldots, X(t_{n-1})\Big\} \\
&\quad = \mathrm{E}\Big\{\Big(X(t_n) - \mathrm{E}\{X(t_n)|X(t_1), \ldots, X(t_{n-1})\}\Big)^2\Big\} = \mathrm{E}\Big\{\Big(X(t_n) - \mathrm{E}\{X(t_n)|X(t_{n-1})\}\Big)^2\Big\} \\
&\quad = \mathrm{E}\Big\{\Big(X(t_n) - \mathrm{E}\{X(t_n)|X(t_{n-1})\}\Big)^2 |X(t_{n-1})\Big\}.
\end{aligned}$$

Example 6.27 A stationary Gaussian Markov process is called the Ornstein-Uhlenbeck process. It is characterized by three parameters $(\theta, \alpha, \sigma^2)$, where $\theta \in (-\infty, \infty)$ is the stationary mean of the process, $\alpha > 0$ is the rate of attraction to the stationary mean, and σ^2 is the noise variance. In this case the autocorrelation function has the form $\rho(t) = e^{-\alpha t}$, $t \geq 0$. This follows from the equation $\rho(t + s) = \rho(t)\rho(s)$ which is obtained as follows. From the property of the bivariate normal distribution

$$\mathrm{E}(X(t+s)|X(s)) = \theta + \rho(t)(X(s) - \theta)$$

we derive

$$\begin{aligned}
\rho(t+s) = c(0)^{-1}\mathrm{E}((X(t+s) - \theta)(X(0) - \theta)) &= c(0)^{-1}\mathrm{E}\{\mathrm{E}((X(t+s) - \theta)(X(0) - \theta)|X(0), X(s))\} \\
&= \rho(t)c(0)^{-1}\mathrm{E}((X(s) - \theta)(X(0) - \theta)) \\
&= \rho(t)\rho(s).
\end{aligned}$$

See also Example 9.5.

7 Renewal theory and Queues

7.1 Renewal function and excess life

Let $T_0 = 0$ and $T_n = X_1 + \ldots + X_n$, where X_i are iid strictly positive random variables with mean μ. We call T_n the renewal times and X_i are called the inter-arrival times. A renewal process $N(t)$ gives the number of renewal events during the time interval $(0, t]$

$$\{N(t) \geq n\} = \{T_n \leq t\}, \quad T_{N(t)} \leq t < T_{N(t)+1}.$$

The distribution function of $\mathrm{P}(T_k \leq t) = F^{*k}(t)$ is a convolution of $F(t) = \mathrm{P}(X_1 \leq t)$:

$$F^{*0}(t) = 1_{\{t \geq 0\}}, \quad F^{*k}(t) = \int_0^t F^{*(k-1)}(t-u)dF(u).$$

Exercise 7.1 Show that

$$\{N(t) \leq n\} = \{T_{n+1} > t\}.$$

Definition 7.2 Put $m(t) := \mathrm{E}(N(t))$. We will call $U(t) = 1 + m(t)$ the renewal function.

Lemma 7.3 *We have*

$$U(t) = \sum_{k=0}^{\infty} F^{*k}(t),$$

and in terms of the Laplace-Stieltjes transforms $\hat{F}(\theta) := \int_0^\infty e^{-\theta t} dF(t)$

$$\hat{U}(\theta) = \frac{1}{1 - \hat{F}(\theta)}.$$

Proof. Using the recursion

$$N(t) = 1_{\{X_1 \leq t\}}(1 + \tilde{N}(t - X_1)),$$

where $\tilde{N}(t)$ is a renewed copy of the initial process $N(t)$, we find after taking expectations

$$m(t) = F(t) + \int_0^t m(t-u)dF(u).$$

It follows

$$m(t) = F(t) + F^{*2}(t) + \int_0^t m(t-u)dF^{*2}(u) = \ldots = \sum_{k=1}^{\infty} F^{*k}(t),$$

and consequently

$$\hat{m}(\theta) = \int_0^\infty e^{-\theta t} dm(t) = \sum_{k=1}^{\infty} \int_0^\infty e^{-\theta t} dF^{*k}(t) = \sum_{k=1}^{\infty} \hat{F}(\theta)^k = \frac{\hat{F}(\theta)}{1 - \hat{F}(\theta)}.$$

Example 7.4 Poisson process is the Markovian renewal process with exponentially distributed inter-arrival times. Since $F(t) = 1 - e^{-\lambda t}$, we find $\hat{m}(\theta) = \frac{\lambda}{\theta}$, implying $m(t) = \lambda t$. Notice that $\mu = 1/\lambda$ and the rate is $\lambda = 1/\mu$.

7.2 Renewal equation

Definition 7.5 For a measurable function $g(t)$ the relation

$$A(t) = g(t) + \int_0^t A(t-u)dF(u)$$

is called a renewal equation. Clearly, its solution for $t \geq 0$ takes the form $A(t) = \int_0^t g(t-u)dU(u)$.

Definition 7.6 The excess lifetime at t is $E(t) = T_{N(t)+1} - t$. The current lifetime (or age) at t is $C(t) = t - T_{N(t)}$. The total lifetime at t is the sum $C(t) + E(t)$.

Notice that $\{E(t), 0 \leq t < \infty\}$ is a Markov process of deterministic unit speed descent toward zero with upward jumps when the process hits zero.

Lemma 7.7 *The distribution of the excess life $E(t)$ is given by*

$$\mathrm{P}(E(t) > y) = \int_0^t (1 - F(t + y - u))dU(u).$$

The distribution of the age $C(t)$ is given by

$$\mathrm{P}(C(t) \geq y) = 1_{\{y \leq t\}} \int_0^{t-y} (1 - F(t - u))dU(u).$$

In particular, $\mathrm{P}(C(t) = t) = 1 - F(t)$.

Proof. The first claim follows from the following renewal equation for $b(t) := \mathrm{P}(E(t) > y)$

$$\begin{aligned} b(t) &= \mathrm{E}\Big(\mathrm{E}(1_{\{E(t)>y\}}|X_1)\Big) = \mathrm{E}\Big(1_{\{\tilde{E}(t-X_1)>y\}}1_{\{X_1 \leq t\}} + 1_{\{X_1 > t+y\}}\Big) \\ &= 1 - F(t + y) + \int_0^t b(t - x)dF(x). \end{aligned}$$

It is the case that $C(t) \geq y$ if and only if there are no arrivals in $(t - y, t]$. Thus

$$\mathrm{P}(C(t) \geq y) = \mathrm{P}(E(t - y) > y) \qquad (*)$$

and the second claim follows from the first one.

Example 7.8 For the Poisson process with rate λ, the distribution of the excess lifetime

$$\mathrm{P}(E(t) > y) = \int_0^t (1 - F(t + y - u))dU(u) = e^{-\lambda(t+y)} + \lambda \int_0^t e^{-\lambda(t+y-u)}du = e^{-\lambda y}$$

is independent of the observation time t. Therefore in view of $(*)$, we have for $y \leq t$,

$$\mathrm{P}(C(t) \leq y) = 1 - e^{-\lambda y},\ 0 \leq y < t, \qquad \mathrm{P}(C(t) = t) = e^{-\lambda t},$$

implying that the total lifetime has the mean

$$\mathrm{E}(C(t) + E(t)) = 2\mu - \lambda \int_t^\infty (y - t)e^{-\lambda y}dy,$$

which for large t is twice as large as the inter-arrival mean $\mu = 1/\lambda$. This observation is called the "waiting time paradox".

Exercise 7.9 Let the times between the events of a renewal process N be uniformly distributed on $[0, 1]$.
(a) For the mean function $m(t) = \mathbb{E}(N(t))$, show that $m(t) = e^t - 1$ for $0 \leq t \leq 1$.
(b) Show that for $0 \leq t \leq 1$, the variance is

$$\mathrm{Var}(N(t)) = e^t(1 + 2t - e^t).$$

Hint: start by writing down the renewal equations for $m(t)$ and the second moment $m_2(t) = \mathbb{E}(N^2(t))$.

Solution. (a) Since the inter-arrival times are uniformly distributed on $[0, 1]$, we have $F(t) = t \cdot 1_{\{0 \leq t \leq 1\}}$. The renewal property gives

$$N(t) = 1_{\{X_1 \leq t\}}(1 + \tilde{N}(t - X_1)).$$

Taking expectations we see that $m(t) = \mathbb{E}N(t)$ satisfies the renewal equation

$$m(t) = F(t) + \int_0^t m(t-u)dF(u).$$

For $0 \le t \le 1$, we have

$$m(t) = t + \int_0^t m(u)du,$$

and therefore $m'(t) = 1 + m(t)$ with $m(0) = 0$. Thus $m(t) = e^t - 1$ for $0 \le t \le 1$.

(b) From

$$N(t) = 1_{\{X_1 \le t\}}(1 + \tilde{N}(t - X_1))$$

we get

$$N^2(t) = 1_{\{X_1 \le t\}}(1 + 2\tilde{N}(t - X_1) + \tilde{N}^2(t - X_1)).$$

Taking expectations we obtain the renewal equation for the second moment $m_2(t) = \mathbb{E}(N^2(t))$

$$m_2(t) = A(t) + \int_0^t m_2(t-u)dF(u),$$

where

$$A(t) = \int_0^t (1 + 2m(t-u))dF(u).$$

Using the renewal function $U(t) = 1 + m(t)$ we find the solution of this renewal equation as

$$m_2(t) = \int_0^t A(t-u)dU(u).$$

For $0 \le t \le 1$, we have $U(t) = e^t$ and

$$A(t) = \int_0^t (2e^u - 1)du = 2e^t - 2 - t,$$

so that

$$\begin{aligned} m_2(t) &= 2e^t - 2 - t + \int_0^t (2e^{t-u} - 2 - t + u)e^u du \\ &= te^t + \int_0^t ue^u du = 1 - e^t + 2te^t. \end{aligned}$$

Thus for $0 \le t \le 1$, the variance is

$$\mathrm{Var}(N(t)) = 1 - e^t + 2te^t - (e^t - 1)^2 = e^t + 2te^t - e^{2t} = e^t(1 + 2t - e^t).$$

7.3 LLN and CLT for the renewal process

Theorem 7.10 *Law of large numbers: $N(t)/t \overset{\text{a.s.}}{\to} 1/\mu$ as $t \to \infty$. So that $1/\mu$ gives the rate of occurrence of the renewal events.*

Proof. Note that $T_{N(t)} \le t < T_{N(t)+1}$. Thus, if $N(t) > 0$,

$$\frac{T_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{T_{N(t)+1}}{N(t)+1}\Big(1 + \frac{1}{N(t)}\Big).$$

Since $N(t) \to \infty$ as $t \to \infty$, it remains to apply Theorem 4.9, the classical law of large numbers, saying that $T_n/n \to \mu$ almost surely.

Theorem 7.11 *Central limit theorem. If $\sigma^2 = \mathrm{Var}(X_1)$ is positive and finite, then*

$$\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \overset{d}{\to} N(0,1) \text{ as } t \to \infty.$$

Proof. The usual CLT implies that for any x,

$$\mathrm{P}\Big(\frac{T_{a(t)} - \mu a(t)}{\sigma\sqrt{a(t)}} \le x\Big) \to \Phi(x) \text{ as } a(t) \to \infty.$$

For a given x, put $a(t) = \lfloor t/\mu + x\sqrt{t\sigma^2/\mu^3}\rfloor$, and observe that on one hand

$$\mathrm{P}\Big(\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \ge x\Big) = \mathrm{P}(N(t) \ge a(t)) = \mathrm{P}(T_{a(t)} \le t),$$

and on the other hand, $\frac{t-\mu a(t)}{\sigma\sqrt{a(t)}} \to -x$ as $t \to \infty$. We conclude

$$\mathrm{P}\Big(\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \ge x\Big) \to \Phi(-x) = 1 - \Phi(x).$$

7.4 Stopping times and Wald's equation

Definition 7.12 Let M be a r.v. taking values in the set $\{1, 2, \ldots\}$. We call it a stopping time with respect to the sequence (X_n) of random variables, if

$$\{M \le m\} \in \mathcal{F}_m, \quad \text{for all } m = 1, 2, \ldots,$$

where $\mathcal{F}_m = \sigma\{X_1, \ldots, X_m\}$ is the σ-algebra of events generated by the events $\{X_1 \le c_1\}, \ldots, \{X_m \le c_m\}$ for all $c_i \in (-\infty, \infty)$.

Theorem 7.13 *Wald's equation. Let $X_1, X_2, \ldots$ be iid r.v. with finite mean μ, and let M be a stopping time with respect to the sequence (X_n) such that $\mathrm{E}(M) < \infty$. Then*

$$\mathrm{E}(X_1 + \ldots + X_M) = \mu\mathrm{E}(M).$$

Proof. Observe that

$$\sum_{i=1}^\infty X_i 1_{\{M \ge i\}} = \sum_{i=1}^\infty X_i \sum_{j=i}^\infty 1_{\{M=j\}} = \sum_{j=1}^\infty 1_{\{M=j\}} \sum_{i=1}^j X_i = (X_1 + \ldots + X_M) \sum_{j=1}^\infty 1_{\{M=j\}} = X_1 + \ldots + X_M,$$

so that

$$Y_n := \sum_{i=1}^n X_i 1_{\{M \ge i\}} \overset{\text{a.s.}}{\to} X_1 + \ldots + X_M.$$

For all n, we have

$$|Y_n| \le Y, \qquad Y = \sum_{i=1}^\infty |X_i| 1_{\{M \ge i\}},$$

and

$$\mathrm{E}(Y) = \sum_{i=1}^\infty \mathrm{E}(|X_i| 1_{\{M \ge i\}}) = \sum_{i=1}^\infty \mathrm{E}(|X_i|)\mathrm{P}(M \ge i) = \mathrm{E}(|X_1|)\mathrm{E}(M).$$

Here we used independence between $\{M \ge i\}$ and X_i, which follows from the fact that $\{M \ge i\}$ is the complimentary event to $\{M \le i-1\} \in \sigma\{X_1, \ldots, X_{i-1}\}$. By the dominated convergence Theorem 3.21

$$\mathrm{E}(X_1 + \ldots + X_M) = \lim_{n\to\infty} \mathrm{E}\Big(\sum_{i=1}^n X_i 1_{\{M \ge i\}}\Big) = \sum_{i=1}^\infty \mathrm{E}(X_i)\mathrm{P}(M \ge i) = \mu\mathrm{E}(M).$$

Example 7.14 Observe that $M = N(t)$ is not a stopping time and in general $\mathrm{E}(T_{N(t)}) \neq \mu m(t)$. Indeed, for the Poisson process $\mu m(t) = t$ while $T_{N(t)} = t - C(t)$, where $C(t)$ is the current lifetime.

Theorem 7.15 *Elementary renewal theorem: $m(t)/t \to 1/\mu$ as $t \to \infty$, where $\mu = \mathrm{E}(X_1)$ is finite or infinite.*

Proof. Since $M = N(t) + 1$ is a stopping time for (X_n):

$$\{M \le m\} = \{N(t) \le m - 1\} = \{X_1 + \ldots + X_m > t\},$$

the Wald equation implies

$$\mathrm{E}(T_{N(t)+1}) = \mu \mathrm{E}(N(t) + 1) = \mu U(t).$$

From $T_{N(t)+1} = t + E(t)$ we get

$$U(t) = \mu^{-1}(t + \mathrm{E}(E(t))),$$

so that $U(t) \ge \mu^{-1}t$. Moreover, if $\mathrm{P}(X_1 \le a) = 1$ for some finite a, then $U(t) \le \mu^{-1}(t + a)$ and the assertion follows.

If X_1 is unbounded, then consider truncated inter-arrival times $\min(X_i, a)$ with mean μ_a and renewal function $U_a(t)$. It remains to observe that $U_a(t) \sim t\mu_a^{-1}$, $U_a(t) \ge U(t)$, and $\mu_a \to \mu$ as $a \to \infty$.

7.5 Renewal theorems and stationarity

Theorem 7.16 *Renewal theorem. If X_1 is not arithmetic, then for any positive h*

$$U(t + h) - U(t) \to \mu^{-1}h, \quad t \to \infty.$$

Without proof.

Example 7.17 Arithmetic case. A typical arithmetic case is obtained, if we assume that the set of possible values R_X for the inter-arrival times X_i satisfies $R_X \subset \{1, 2, \ldots\}$ and $R_X \not\subset \{k, 2k, \ldots\}$ for any $k = 2, 3, \ldots$, implying $\mu \ge 1$. If again $U(n)$ is the renewal function, then $U(n) - U(n - 1)$ is the probability that a renewal event occurs at time n. A discrete time version of Theorem 7.16 claims that $U(n) - U(n - 1) \to \mu^{-1}$.

Theorem 7.18 *Key renewal theorem. If X_1 is not arithmetic, $\mu < \infty$, and $g : [0, \infty) \to [0, \infty)$ is a monotone function, then*

$$\int_0^t g(t - u)dU(u) \to \mu^{-1} \int_0^\infty g(u)du, \quad t \to \infty.$$

Proof sketch. Using Theorem 7.16, first prove the assertion for indicator functions of intervals, then for step functions, and finally for the limits of increasing sequences of step functions.

Theorem 7.19 *If X_1 is not arithmetic and $\mu < \infty$, then*

$$\lim_{t\to\infty} \mathrm{P}(E(t) \le y) = \mu^{-1} \int_0^y (1 - F(x))dx.$$

Proof. Apply the key renewal theorem and Lemma 7.7.

Definition 7.20 Let $X_1, X_2, \ldots$ be independent positive r.v. such that $X_2, X_3, \ldots$ have the same distribution. If as before, $T_0 = 0$ and $T_n = X_1 + \ldots + X_n$, then $N^{\mathrm{d}}(t) = \max\{n : T_n \le t\}$ is called a *delayed renewal process.* It is described by two distributions $F(t) = \mathrm{P}(X_i \le t)$, $i \ge 2$ and $F^{\mathrm{d}}(t) = \mathrm{P}(X_1 \le t)$.

Lemma 7.21 *The mean $m^{\mathrm{d}}(t) = \mathrm{E}N^{\mathrm{d}}(t)$ satisfies the renewal equation*

$$m^{\mathrm{d}}(t) = F^{\mathrm{d}}(t) + \int_0^t m^{\mathrm{d}}(t - u)dF(u).$$

Proof. First observe that conditioning on X_1 gives

$$m^{\mathrm{d}}(t) = F^{\mathrm{d}}(t) + \int_0^t m(t-u)dF^{\mathrm{d}}(u).$$

Now since

$$m(t) = F(t) + \int_0^t m(t-u)dF(u),$$

we have

$$\begin{aligned} m * F^{\mathrm{d}}(t) := \int_0^t m(t-u)dF^{\mathrm{d}}(u) &= F * F^{\mathrm{d}}(t) + m * F * F^{\mathrm{d}}(t) = F^{\mathrm{d}} * F(t) + m * F^{\mathrm{d}} * F(t) \\ &= (F^{\mathrm{d}} + m * F^{\mathrm{d}}) * F(t) = m^{\mathrm{d}} * F(t) = \int_0^t m^{\mathrm{d}}(t-u)dF(u). \end{aligned}$$

Theorem 7.22 *The process $N^{\mathrm{d}}(t)$ has stationary increments: $N^{\mathrm{d}}(s+t) - N^{\mathrm{d}}(s) \overset{d}{=} N^{\mathrm{d}}(t)$, if and only if*

$$F^{\mathrm{d}}(y) = \mu^{-1} \int_0^y (1 - F(x))dx. \qquad (*)$$

In this case the renewal function is linear $m^{\mathrm{d}}(t) = t/\mu$ and $\mathrm{P}(E^{\mathrm{d}}(t) \le y) = F^{\mathrm{d}}(y)$ independently of t.

Proof. Necessity. If $N^{\mathrm{d}}(t)$ has stationary increments, then $m^{\mathrm{d}}(s+t) = m^{\mathrm{d}}(s) + m^{\mathrm{d}}(t)$, and we get $m^{\mathrm{d}}(t) = tm^{\mathrm{d}}(1)$. Substitute this into the renewal equation of Lemma 7.21 to obtain

$$F^{\mathrm{d}}(t) = m^{\mathrm{d}}(1) \int_0^t (1 - F(x))dx$$

using integration by parts. Let $t \to \infty$ to find $m^{\mathrm{d}}(1) = 1/\mu$ as stated in $(*)$.

Sufficiency. Given $(*)$, we have

$$\hat{F}^{\mathrm{d}}(\theta) = \int_0^\infty e^{-\theta y} dF^{\mathrm{d}}(y) = \mu^{-1} \int_0^\infty e^{-\theta y}(1 - F(y))dy = \frac{1 - \hat{F}(\theta)}{\mu\theta}.$$

Taking the Laplace-Stieltjes transforms in Lemma 7.21 yields

$$\hat{m}^{\mathrm{d}}(\theta) = \frac{1 - \hat{F}(\theta)}{\mu\theta} + \hat{m}^{\mathrm{d}}(\theta)\hat{F}(\theta).$$

It follows that $\hat{m}^{\mathrm{d}}(\theta) = 1/(\mu\theta)$, and therefore $m^{\mathrm{d}}(t) = t/\mu$.

Observe that $N^{\mathrm{d}}(\cdot)$ has stationary increments if and only if the distribution of $E^{\mathrm{d}}(t)$ does not depend on t, and it is enough to check that $\mathrm{P}(E^{\mathrm{d}}(t) > y) = 1 - F^{\mathrm{d}}(y)$. This is obtained from

$$\mathrm{P}(E^{\mathrm{d}}(t) > y) = 1 - F^{\mathrm{d}}(t+y) + \int_0^t \mathrm{P}(E^{\mathrm{d}}(t-u) > y)dF^{\mathrm{d}}(u).$$

and

$$F^{\mathrm{d}} * U(t) = m^{\mathrm{d}}(t) = t/\mu.$$

Indeed, writing $g(t) = 1 - F(t+y)$ we have $\mathrm{P}(E^{\mathrm{d}}(t) > y) = g * U(t)$ and therefore

$$\begin{aligned} \int_0^t \mathrm{P}(E^{\mathrm{d}}(t-u) > y)dF^{\mathrm{d}}(u) = g * U * F^{\mathrm{d}}(t) = g * F^{\mathrm{d}} * U(t) &= \mu^{-1} \int_0^t (1 - F(t+y-u))du \\ &= \mu^{-1} \int_y^{t+y} (1 - F(x))dx = F^{\mathrm{d}}(t+y) - F^{\mathrm{d}}(y). \end{aligned}$$

7.6 Renewal-reward processes

Let $(X_i, R_i), i = 1, 2, \dots$ be iid pairs of possibly dependent random variables: X_i are positive inter-arrival times and R_i the associated rewards. Cumulative reward process

$$W(t) = R_1 + \ldots + R_{N(t)}.$$

Theorem 7.23 *Renewal-reward theorem. Suppose (X_i, R_i) have finite means $\mu = \mathrm{E}(X)$ and $\mathrm{E}(R)$. Then*

$$\frac{W(t)}{t} \overset{\text{a.s.}}{\to} \frac{\mathrm{E}(R)}{\mu}, \qquad \frac{\mathrm{E}W(t)}{t} \to \frac{\mathrm{E}(R)}{\mu}, \qquad t \to \infty.$$

Proof. Applying two laws of large numbers, Theorem 4.9 and Theorem 7.10, we get the first claim

$$\frac{W(t)}{t} = \frac{R_1 + \ldots + R_{N(t)}}{N(t)} \cdot \frac{N(t)}{t} \overset{\text{a.s.}}{\to} \frac{\mathrm{E}(R)}{\mu}.$$

The second claim is an extention of Theorem 7.15. Again, using the Wald equation we find

$$\mathrm{E}W(t) = \mathrm{E}(R_1 + \ldots + R_{N(t)+1}) - \mathrm{E}R_{N(t)+1} = U(t)\mathrm{E}(R) - r(t), \qquad r(t) = \mathrm{E}(R_{N(t)+1}).$$

The result will follow once we have shown that $r(t)/t \to 0$ as $t \to \infty$. By conditioning on X_1 we arrive at the renewal equation

$$r(t) = H(t) + \int_0^t r(t-u)dF(u), \qquad H(t) = \mathrm{E}(R_1 1_{\{X_1 > t\}}).$$

Since $H(t) \to 0$, for any $\epsilon > 0$, there exists a finite $M = M(\epsilon)$ such that $|H(t)| < \epsilon$ for $t \ge M$. Therefore, when $t \ge M$,

$$\begin{aligned} t^{-1}|r(t)| &\le t^{-1} \int_0^{t-M} |H(t-u)|dU(u) + t^{-1} \int_{t-M}^t |H(t-u)|dU(u) \\ &\le t^{-1}\Big(\epsilon U(t) + (U(t) - U(t-M))\mathrm{E}|R|\Big). \end{aligned}$$

Using the renewal theorems we get $\limsup |r(t)/t| \le \epsilon/\mu$ and it remains to let $\epsilon \to 0$.

7.7 Regeneration technique for queues

Consider a general queueing system assuming that customers arrive one by one, and after spending some time in the system they depart from the system. Let $Q(t)$ be the number of customers in the system at time t with $Q(0) = 0$. Let T be the time of first return to zero, so that $Q(T) = 0$. This can be called a regeneration time, because starting from time T the future process $Q(T + t)$ is independent of the past and has the same distribution as the process $Q(t)$.

Assuming the traffic is *light*, that is $\mathrm{P}(T < \infty) = 1$, we get a renewal process of regeneration times $0 = T_0 < T = T_1 < T_2 < T_3 < \ldots$. Write N_i for the number of customers arriving during the cycle $[T_{i-1}, T_i)$ and put $N = N_1$. To be able to apply Theorem 7.23 we shall assume

$$\mathrm{E}(T) < \infty, \qquad \mathrm{E}(N) < \infty, \qquad \mathrm{E}(NT) < \infty.$$

(A) Let the reward associated with the inter-arrival time $X_i = T_i - T_{i-1}$ to be

$$R_i = \int_{T_{i-1}}^{T_i} Q(u)du.$$

Since $R := R_1 \le NT$, we have $\mathrm{E}(R) \le \mathrm{E}(NT) < \infty$, and by Theorem 7.23,

$$t^{-1}\int_0^t Q(u)du \overset{\text{a.s.}}{\to} \mathrm{E}(R)/\mathrm{E}(T) =: L, \quad \text{the long run average queue length.}$$

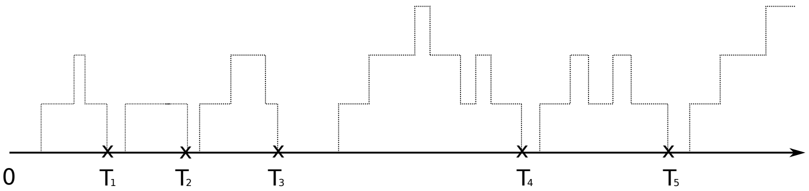


Figure 2: The queue length and the regeneration times. Here $N_1 = 2$, $N_2 = 1$, $N_3 = 2$, $N_4 = 4$, $N_5 = 3$.

(B) Let the reward associated with the inter-arrival time X_i to be N_i. Denote by $W(t)$ the number of customers arrived by time t. Then by Theorem 7.23,

$$W(t)/t \overset{\text{a.s.}}{\to} \mathrm{E}(N)/\mathrm{E}(T) =: \lambda, \quad \text{the long run rate of arrival.}$$

(C) Consider now the reward-renewal process with discrete inter-arrival times N_i, and the associated rewards S_i defined as the total time spent in the system by the customers arrived during $[T_{i-1}, T_i)$. If the n-th customer spends time V_n, then

$$S := S_1 = V_1 + \ldots + V_N.$$

Since $\mathrm{E}(S) \le \mathrm{E}(NT) < \infty$, again by Theorem 7.23, we have

$$n^{-1}\sum_{i=1}^{n} V_i \overset{\text{a.s.}}{\to} \mathrm{E}(S)/\mathrm{E}(N) =: \nu, \quad \text{the long run average time spent by a customer in the system.}$$

Theorem 7.24 *Little's law. If* $\mathrm{E}(T) < \infty$, $\mathrm{E}(N) < \infty$, *and* $\mathrm{E}(NT) < \infty$, *then* $L = \lambda\nu$.

Although it looks intuitively reasonable:

average queue length = average time in the sytem per customer × arrival rate of customers,

it a quite remarkable result, as the relationship is not influenced by the arrival process distribution, the service distribution, the service order, or practically anything else.

PROOF. The total of customer time spent during the first cycle is $\sum_{i=1}^{N} V_i = \int_0^T Q(u)du$. Therefore,

$$\mathrm{E}(S) = \mathrm{E}\Big(\sum_{i=1}^{N} V_i\Big) = \mathrm{E}\Big(\int_0^T Q(u)du\Big) = \mathrm{E}(R).$$

Thus, combining (A), (B), (C) we get

$$\frac{L}{\lambda\nu} = \frac{\mathrm{E}(R)}{\mathrm{E}(T)} \cdot \frac{\mathrm{E}(T)}{\mathrm{E}(N)} \cdot \frac{\mathrm{E}(N)}{\mathrm{E}(S)} = 1.$$

7.8 M/M/1 queues

The most common notation scheme annotates the queueing systems by a triple A/B/s, where A describes the distribution of inter-arrival times of customers, B describes the distribution of service times, and s is the number of servers. It is assumed that the inter-arrival and service times are two independent iid sequences (X_i) and (S_i). Let $Q(t)$ be the queue size at time t.

The simplest queue system has exponential (λ) inter-arrival times and exponential (μ) service times. It is denoted as M/M/1, where the letter M stands for the Markov discipline. It is the only queue with $Q(t)$ forming a Markov chain. In this case $Q(t)$ is a birth-death process with the birth rate λ and the death rate μ for the positive states, and no deaths at state zero. The ratio $\rho = \lambda/\mu$ is called the traffic intensity.

Theorem 7.25 *The probabilities $p_n(t) = \mathrm{P}(Q(t) = n)$ for a M/M/I queue satisfy the Kolmogorov forward equations*

$$\begin{cases} p_n'(t) &= -(\lambda+\mu)p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t) \qquad \text{for } n \geq 1, \\ p_0'(t) &= -\lambda p_0(t) + \mu p_1(t), \end{cases}$$

subject to the boundary condition $p_n(0) = 1_{\{n=0\}}$.

Proof. For $n \geq 1$ and a small positive δ, put

$$R_n(t,\delta) = \mathrm{P}\big(Q(t+\delta) = n, \text{at least two events during } (t, t+\delta]\big).$$

Observe that

$$R_n(t,\delta) \leq \mathrm{P}(\text{at least two events during } (t,t+\delta]) \leq C \cdot \delta^2.$$

For example

$$\begin{aligned} \mathrm{P}(\text{at least two births during } (t,t+\delta]) &= \sum_{k\geq 2} \frac{(\lambda\delta)^k}{k!} e^{-\lambda\delta} \\ &\leq (\lambda\delta)^2 \sum_{j\geq 0} \frac{(\lambda\delta)^j}{j!} e^{-\lambda\delta} = (\lambda\delta)^2. \end{aligned}$$

It follows,

$$\begin{aligned} p_n(t+\delta) &= \mathrm{P}\big(Q(t+\delta) = n, \text{at most one event during } (t,t+\delta]\big) + R_n(t,\delta) \\ &= p_n(t)\mathrm{P}(\text{no events during } (t,t+\delta]) + p_{n-1}(t)\mathrm{P}(\text{single birth during } (t,t+\delta]) \\ &\quad + p_{n+1}(t)\mathrm{P}(\text{single death during } (t,t+\delta]) + o(\delta) \\ &= p_n(t)(1-(\lambda+\mu)\delta) + p_{n-1}(t)\lambda\delta + p_{n+1}(t)\mu\delta + o(\delta). \end{aligned}$$

Similarly,

$$\begin{aligned} p_0(t+\delta) &= p_0(t)\mathrm{P}(\text{no birth during } (t,t+\delta]) + p_1(t)\mathrm{P}(\text{single death during } (t,t+\delta]) + o(\delta) \\ &= p_0(t)(1-\lambda\delta) + p_1(t)\mu\delta + o(\delta). \end{aligned}$$

Exercise 7.26 In terms of the Laplace transforms $\hat{p}_n(\theta) := \int_0^\infty e^{-\theta t} p_n(t)dt$ we obtain

$$\begin{cases} \mu\hat{p}_{n+1}(\theta) - (\lambda+\mu+\theta)\hat{p}_n(\theta) + \lambda\hat{p}_{n-1}(\theta) &= 0, \qquad \text{for } n \geq 1, \\ \mu\hat{p}_1(\theta) - (\lambda+\theta)\hat{p}_0(\theta) &= 1. \end{cases}$$

Solve this system of linear equations.

Solution.

$$\hat{p}_n(\theta) = \theta^{-1}(1-\alpha(\theta))\alpha(\theta)^n, \qquad \alpha(\theta) := \frac{(\lambda+\mu+\theta) - \sqrt{(\lambda+\mu+\theta)^2 - 4\lambda\mu}}{2\mu}.$$

Theorem 7.27 *Stationarity. Condider a M/M/1 queue with the traffic intensity $\rho = \lambda/\mu$. As t $\to \infty$,*

$$\mathrm{P}(Q(t) = n) \to \pi_n := \begin{cases} (1-\rho)\rho^n & \text{if } \rho < 1, \\ 0 & \text{if } \rho \geq 1, \end{cases} \qquad n = 0, 1, 2, \ldots$$

The result asserts that the queue settles down into equilibrium if and only if the service times are shorter than the inter-arrival times on average. Observe that if $\rho \geq 1$, we have $\sum_n \pi_n = 0$ despite $\sum_n p_n(t) = 1$.

Proof. As $\theta \to 0$

$$\alpha(\theta) = \frac{(\lambda+\mu+\theta) - \sqrt{(\lambda-\mu)^2 + 2(\lambda+\mu)\theta + \theta^2}}{2\mu} \to \frac{(\lambda+\mu+\theta) - |\lambda-\mu|}{2\mu} = \begin{cases} \rho & \text{if } \lambda < \mu, \\ 1 & \text{if } \lambda \geq \mu. \end{cases}$$

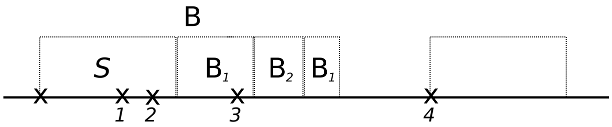


Figure 3: A typical busy period in a M/G/1 queue. Here $Z = 2$.

Thus

$$\theta\hat{p}_n(\theta) = (1-\alpha(\theta))\alpha(\theta)^n \to \begin{cases} (1-\rho)\rho^n & \text{if } \rho < 1, \\ 0 & \text{if } \rho \geq 1. \end{cases}$$

On the other hand, the process $Q(t)$ is an irreducible Markov chain and by the ergodic theorem there are limits $p_n(t) \to \pi_n$ as $t \to \infty$. Therefore, the statement follows from

$$\theta\hat{p}_n(\theta) = \int_0^\infty e^{-u} p_n(u/\theta)du \to \pi_n, \qquad \theta \to 0.$$

Exercise 7.28 Consider the M/M/1 queue with $\rho < 1$. In the stationary regime the average number of customers in the line is $L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$. According to Little's Law the time V spent by a customer in a stationary queue has mean

$$\nu = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}.$$

Verify this by showing that V has an exponential distribution with parameter $\mu - \lambda$.
Hint. Show that $V = S_1 + \ldots + S_{1+Q}$, where S_i are independent Exp(μ) random variables and Q is geometric with parameter $1-\rho$, and then compute the Laplace transform $\mathrm{E}(e^{-uV})$.

7.9 M/G/1 queues

In the M/G/1 queueing system customers arrive according to a Poisson process with intensity λ and the service times S_i have a fixed but unspecified distribution (G for General). In this case the traffic intensity is given by $\rho = \lambda\mathrm{E}(S)$.

Theorem 7.29 *Define a typical busy period of the server as $B = \inf\{t > 0 : Q(t + X_1) = 0\}$, where X_1 is the arrival time of the first customer.*
(i) If $\rho \leq 1$, then $\mathrm{P}(B < \infty) = 1$, and if $\rho > 1$, then $\mathrm{P}(B < \infty) < 1$.
(ii) The Laplace transform $\phi(u) = \mathrm{E}(e^{-uB})$ satisfies the functional equation

$$\phi(u) = \Psi(u + \lambda - \lambda\phi(u)), \qquad \textit{where } \Psi(x) := \mathrm{E}(e^{-xS}).$$

PROOF. We use an embedded branching process construction (recall Section 5.5). Call customer C_2 an offspring of customer C_1, if C_2 joins the queue while C_1 is being served. Given the value of the sirvice time $S = t$, the offspring number Z for a single customer has conditional Pois(λt) distribution. Therefore, the conditional probability generating function of Z is

$$\mathrm{E}(u^Z|S) = \sum_{j=0}^\infty u^j \frac{(\lambda S)^j}{j!} e^{-\lambda S} = e^{S\lambda(u-1)}.$$

This yields the following expression for the probability generating function of the offspring number

$$h(u) := \mathrm{E}(u^Z) = \mathrm{E}(e^{S\lambda(u-1)}) = \Psi(\lambda(1-u)).$$

The mean offspring number is

$$\mathrm{E}(Z) = h'(1) = -\lambda\Psi'(0) = \lambda\mathrm{E}(S) = \rho.$$

Observe that the event $(B < \infty)$ is equivalent to the extinction of the embedded branching process, and the first assertion follows.

The functional equation is derived from the representation $B = S + B_1 + \ldots + B_Z$, where B_i are iid busy times of the offspring customers and Z has the generating function $h(u)$. Indeed, by independence

$$\mathrm{E}(e^{-uB_1} \ldots e^{-uB_Z}|S) = \mathrm{E}(\phi(u)^Z|S) = e^{(\lambda\phi(u)-\lambda)S}.$$

This implies

$$\phi(u) = \mathrm{E}(e^{-uS}e^{-uB_1} \ldots e^{-uB_Z}) = \mathrm{E}(e^{-uS}e^{(\lambda\phi(u)-\lambda)S}) = \Psi(u + \lambda - \lambda\phi(u)).$$

Exercise 7.30 Show that for the M/M/1 queue, Z is geometric with parameter $\frac{1}{1+\rho}$. Compute the generating function $\phi(u)$ for the busy period B.

Theorem 7.31 *Stationarity. As $t \to \infty$, for every $n \geq 0$,*

$$\mathrm{P}(Q(t) = n) \to \begin{cases} \pi_n & \text{if } \rho < 1, \text{ where } \sum_{j=0}^{\infty} \pi_j u^j = (1-\rho)(1-u)\frac{h(u)}{h(u)-u}, \\ 0 & \text{if } \rho \geq 1. \end{cases}$$

Partial proof. Let D_n be the number of customers in the system right after the n-th customer left the system. Denoting Z_n the offspring number of the n-th customer, we get

$$D_{n+1} = D_n + Z_{n+1} - 1_{\{D_n>0\}}.$$

Clearly, D_n forms a Markov chain with the state space $\{0, 1, 2, \ldots\}$ and transition probabilities

$$\begin{pmatrix} \delta_0 & \delta_1 & \delta_2 & \ldots \\ \delta_0 & \delta_1 & \delta_2 & \ldots \\ 0 & \delta_0 & \delta_1 & \ldots \\ 0 & 0 & \delta_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \qquad \delta_j = \mathrm{P}(Z = j) = \mathrm{E}\Big(\frac{(\lambda S)^j}{j!}e^{-\lambda S}\Big).$$

The stationary distribution of this chain can be found from the equation

$$D \stackrel{d}{=} D - 1_{\{D>0\}} + Z.$$

For the probability generating function $\psi(u) = \mathrm{E}(u^D)$, we have

$$\psi(u) = \mathrm{E}(u^{D-1_{\{D>0\}}})h(u) = \Big(\frac{\psi(u) - \psi(0)}{u} + \psi(0)\Big)h(u).$$

Thus

$$\psi(u) = \frac{\psi(0)(u-1)h(u)}{u - h(u)}.$$

If $\rho < 1$, we get

$$1 = \psi(1) = \frac{\psi(0)}{1 - h'(1)} = \frac{\psi(0)}{1-\rho}$$

and therefore

$$\mathrm{E}(u^D) = (1-\rho)(1-u)\frac{h(u)}{h(u)-u}. \qquad (*)$$

gives the same stationary distribution as in the theorem statement (only for the embedded chain).

From this analysis of the chain (D_n) one can be derive (not shown here) the stated convergence for the queue size distribution.

Theorem 7.32 *Suppose a customer joins the queue after some large time has elapsed. She will wait a period W of time before her service begins. If $\rho < 1$, then $\mathrm{P}(W = 0) = 1 - \rho$ and*

$$\mathrm{E}(e^{-uW}) = \frac{(1-\rho)u}{u - \lambda + \lambda\mathrm{E}(e^{-uS})}.$$

PROOF. Suppose that a customer waits for a period of length W and then is served for a period of length S. In the stationary regime the length D of the queue on the departure of this customer is distributed according to $(*)$. On the other hand, D conditionally on (W, S) has a Poisson distribution with parameter $\lambda(W + S)$:

$$(1-\rho)(1-s)\frac{h(s)}{h(s)-s} = \mathrm{E}(s^D) = \mathrm{E}(e^{\lambda(W+S)(s-1)}) = \mathrm{E}(e^{\lambda W(s-1)})h(s).$$

Thus, with $u = \lambda(1-s)$,

$$\mathrm{E}(e^{-uW}) = \mathrm{E}(e^{\lambda(s-1)W}) = \frac{(1-\rho)(s-1)}{s-\mathrm{E}(e^{\lambda(s-1)S})} = \frac{(1-\rho)u}{u-\lambda+\lambda\mathrm{E}(e^{-uS})}.$$

Exercise 7.33 Using the previous theorem show that for the M/M/1 queue we have

$$\mathrm{E}(e^{-uW}) = 1 - \frac{\rho u}{\mu - \lambda + u}.$$

Find $\mathrm{P}(W = 0)$. Show that conditionally on $W > 0$, this distribution is exponential.

Theorem 7.34 *Heavy traffic. Let $\rho = \lambda d$ be the traffic intensity of an M/G/1 queue with $M = M(\lambda)$ and $G = D(d)$, meaning that the service time S is detreministic in that* $\mathrm{P}(S = d) = 1$ *for some $d > 0$. For $\rho < 1$, let Q_ρ be a r.v. with the equilibrium queue length distribution. Then, as $\rho \nearrow 1$, the distribution of $(1-\rho)Q_\rho$ converges to the exponential distribution with parameter 2.*

PROOF. Applying Theorem 7.31 with $u = e^{-s(1-\rho)}$ and $h(u) = e^{\rho(u-1)}$, we find the Laplace transform of the scaled queue length

$$\mathrm{E}(e^{-s(1-\rho)Q_\rho}) = \sum_{j=0}^{\infty}\pi_j u^j = (1-\rho)(1-u)\frac{e^{\rho(u-1)}}{e^{\rho(u-1)}-u} = \frac{(1-\rho)(1-e^{-s(1-\rho)})}{1-e^{\rho(1-u)}e^{-s(1-\rho)}},$$

which converges to $\frac{2}{2+s}$ as $\rho \nearrow 1$.

Exercise 7.35 For $\rho = 0.99$ find $\mathrm{P}(Q_\rho > 60)$ using the previous theorem. Answer 30%.

7.10 G/M/1 queues

Customers' arrival times form a renewal process with inter-arrival times (X_n) having a general ditribution G, and the service times are exponentially distributed with parameter μ. The traffic intensity is $\rho = (\mu\mathrm{E}(X))^{-1}$.

An embedded Markov chain. Consider the moment T_n at which the n-th customer joins the queue, and let A_n be the number of customers who are ahead of this customer in the system. Define V_n as the number of departures from the system during the interval $[T_n, T_{n+1})$. Conditionally on A_n and $X_{n+1} = T_{n+1} - T_n$, the r.v. V_n has a truncated Poisson distribution

$$\mathrm{P}(V_n = v | A_n = a, X_{n+1} = x) = \begin{cases} \frac{(\mu x)^v}{v!}e^{-\mu x} & \text{if } v \le a, \\ \sum_{i \ge a+1}\frac{(\mu x)^i}{i!}e^{-\mu x} & \text{if } v = a+1. \end{cases}$$

The sequence (A_n) satisfies

$$A_{n+1} = A_n + 1 - V_n$$

and forms a Markov chain with transition probabilities

$$\Pi_A = \begin{pmatrix} 1-\alpha_0 & \alpha_0 & 0 & 0 & \ldots \\ 1-\alpha_0-\alpha_1 & \alpha_1 & \alpha_0 & 0 & \ldots \\ 1-\alpha_0-\alpha_1-\alpha_2 & \alpha_2 & \alpha_1 & \alpha_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \qquad \alpha_j = \mathrm{E}\Big(\frac{(\mu X)^j}{j!}e^{-\mu X}\Big).$$

Theorem 7.36 *If $\rho < 1$, then the A-chain is ergodic with a unique stationary distribution*

$$\pi_j = (1-\eta)\eta^j, \quad j = 0, 1, \ldots,$$

where η is the smallest positive root of $\eta = \mathrm{E}(e^{X\mu(\eta-1)})$. If $\rho = 1$, then the chain (A_n) is null recurrent. If $\rho > 1$, then the chain (A_n) is transient.

Partial proof. Let $\rho < 1$ and put $\bar{\pi} = (\pi_0, \pi_1, \ldots)$. To see that $\eta \in (0,1)$ observe that $a(x) = \mathrm{E}(e^{-X\mu(1-x)})$ has a convex graph. Since $a(0) > 0$, $a(1) = 1$, and $a'(1) = \rho^{-1} > 1$, we conclude that there is an intersection $\eta \in (0,1)$ of the lines $y = x$ and $y = a(x)$. The vector $\bar{\pi}$ gives the stationary distribution of the A-chain, since we have $\bar{\pi} = \bar{\pi}\Pi_A$ due to the following equalities

$$\sum_{j=0}^{\infty} \alpha_j \eta^j = \eta, \quad \pi_j + \pi_{j+1} + \ldots = \eta^j.$$

Example 7.37 Consider a deterministic arrival process with $\mathrm{P}(X=1) = 1$ so that $A_n = Q(n) - 1$. Then $\rho = 1/\mu$ and for $\mu > 1$ we have the stationary distribution

$$\mathrm{P}(A = j) = (1-\eta)\eta^j,$$

where $\eta = e^{\mu(\eta-1)}$. For $t \in (0,1)$, we have $Q(n+t) = A_n + 1 - V_n(t)$, where

$$p_t(a) := \mathrm{P}(V_n(t) = v | A_n = a) = \begin{cases} \frac{(\mu t)^v}{v!} e^{-\mu t} & \text{if } v \le a, \\ \sum_{i \ge a+1} \frac{(\mu t)^i}{i!} e^{-\mu t} & \text{if } v = a+1. \end{cases}$$

Using the stationary distribution for A_n we get

$$\mathrm{P}(Q(n+t) = k) = \sum_{j=k-1}^{\infty} (1-\eta)\eta^j p_t(j).$$

This example demonstrates that unlike (D_n) in the case of M/G/1, the stationary distribution of (A_n) need not to be the limiting distribution of the queue size $Q(t)$.

Theorem 7.38 *Let $\rho < 1$, and assume that the chain (A_n) is in equilibrium. Then the waiting time W of an arriving customer has an atom of size $1-\eta$ at zero and for $x \ge 0$*

$$\mathrm{P}(W > x) = \eta e^{-\mu(1-\eta)x}.$$

Proof. If $A_n > 0$, then the waiting time of the n-th customer is

$$W_n = S_1^* + S_2 + S_3 + \ldots + S_{A_n},$$

where S_1^* is the residual service time of the customer under service, and $S_2, S_3, \ldots, S_{A_n}$ are the service times of the others in the queue. By the lack-of-memory property, this is a sum of A_n iid exponentials. Use the equilibrium distribution of A_n to find that

$$\mathrm{E}(e^{-uW}) = \mathrm{E}((\mathrm{E}(e^{-uS}))^A) = \mathrm{E}\Big(\Big(\frac{\mu}{\mu+u}\Big)^A\Big) = \frac{(\mu+u)(1-\eta)}{\mu+u-\mu\eta} = (1-\eta) + \eta\frac{\mu(1-\eta)}{\mu(1-\eta)+u}.$$

Exercise 7.39 Using the previous theorem, show that for the M/M/1 queue we have

$$\mathrm{E}(e^{-uW}) = 1 - \rho + \rho\frac{\mu-\lambda}{\mu-\lambda+u}.$$

7.11 G/G/1 queues

Now the arrivals of customers form a renewal process with inter-arrival times X_n having an arbitrary common distribution. The service times S_n have another fixed distribution. The traffic intensity is given by $\rho = \mathrm{E}(S)/\mathrm{E}(X)$. We exclude the trivial case when $\mathrm{P}(S = X) = 1$.

Lemma 7.40 *Lindley equation. Let W_n be the waiting time of the n-th customer. Then*

$$W_{n+1} = \max\{0, W_n + S_n - X_{n+1}\}.$$

PROOF. The n-th customer is in the system for a length $W_n + S_n$ of time. If $X_{n+1} > W_n + S_n$, then the queue is empty at the $(n+1)$-th arrival, and so $W_{n+1} = 0$. If $X_{n+1} \leq W_n + S_n$, then the $(n+1)$-th customer arrives while the n-th is still present. In the second case the new customer waits for a period of $W_{n+1} = W_n + S_n - X_{n+1}$.

Theorem 7.41 *Note that $U_n = S_n - X_{n+1}$ is a collection of iid r.v. Denote by $G(x) = \mathrm{P}(U \leq x)$ their common distribution function. Let $F_n(x) = \mathrm{P}(W_n \leq x)$. Then for $x \geq 0$*

$$F_{n+1}(x) = \int_{-\infty}^{x} F_n(x-y)dG(y).$$

There exists a limit $F(x) = \lim_{n\to\infty} F_n(x)$ which satisfies the Wiener-Hopf equation

$$F(x) = \int_{-\infty}^{x} F(x-y)dG(y) = \mathrm{E}(F(x-U); U \leq x).$$

In terms of an embedded random walk $\Sigma_n = U_1 + \ldots + U_n$,

$$F(x) = \mathrm{P}(\Sigma_n \leq x \text{ for all } n), \qquad x \geq 0.$$

PROOF. If $x \geq 0$ then due to the Lindley equation and independence between W_n and $U_n = S_n - X_{n+1}$

$$\mathrm{P}(W_{n+1} \leq x) = \mathrm{P}(W_n + U_n \leq x) = \int_{-\infty}^{x} \mathrm{P}(W_n \leq x - y)dG(y)$$

and the first part is proved. We claim that

$$F_{n+1}(x) \leq F_n(x) \text{ for all } x \geq 0 \text{ and } n \geq 1. \qquad (*)$$

If $(*)$ holds, then the second result follows immediately. We prove $(*)$ by induction. Observe that $F_2(x) \leq 1 = F_1(x)$, and suppose that $(*)$ holds for $n = k-1$. Then

$$F_{k+1}(x) - F_k(x) = \int_{-\infty}^{x} (F_k(x-y) - F_{k-1}(x-y))dG(y) \leq 0.$$

Turning to the embedded random walk observe that Lemma 7.40 implies

$$W_{n+1} = \max\{0, U_n, U_n + U_{n-1}, \ldots, U_n + U_{n-1} + \ldots + U_1\} \overset{d}{=} \max\{\Sigma_0, \ldots, \Sigma_n\}.$$

Therefore, for $x \geq 0$,

$$F_{n+1}(x) = \mathrm{P}(\Sigma_1 \leq x, \ldots, \Sigma_n \leq x),$$

and we conclude that F is the distribution of $\max\{\Sigma_0, \Sigma_1, \ldots\}$.

Theorem 7.42 *If $\rho < 1$, then F from the previous theorem is a non-defective distribution function. On the other hand, if $\rho \geq 1$, then $F(x) = 0$ for all x.*

PROOF. Note that $\mathrm{E}(U) = \mathrm{E}(S) - \mathrm{E}(X)$, and $\mathrm{E}(U) < 0$ is equivalent to $\rho < 1$. If $\mathrm{E}(U) < 0$, then

$$\mathrm{P}(\Sigma_n > 0 \text{ for infinitely many } n) = \mathrm{P}(n^{-1}\Sigma_n > 0 \text{ i.o.}) = \mathrm{P}(n^{-1}\Sigma_n - \mathrm{E}(U) > |\mathrm{E}(U)| \text{ i.o.}) = 0$$

due to the LLN. Thus $\max\{\Sigma_0, \Sigma_1, \ldots\}$ is either zero or the maximum of only finitely many positive terms, and F is a non-defective distribution.

Next suppose that $\mathrm{E}(U) > 0$ and pick any $x > 0$. For $n \geq 2x/\mathrm{E}(U)$

$$\mathrm{P}(\Sigma_n \geq x) = \mathrm{P}(n^{-1}\Sigma_n - \mathrm{E}(U) \geq n^{-1}x - \mathrm{E}(U)) \geq \mathrm{P}(n^{-1}\Sigma_n - \mathrm{E}(U) \geq -\mathrm{E}(U)/2) = \mathrm{P}(n^{-1}\Sigma_n \geq \mathrm{E}(U)/2).$$

Since $1 - F(x) \geq \mathrm{P}(\Sigma_n \geq x)$, the weak LLN implies $F(x) = 0$. In the case when $\mathrm{E}(U) = 0$ we need a more precise measure of the fluctuations of Σ_n. According to the law of the iterated logarithm the fluctuations of Σ_n are of order $O(\sqrt{n \log\log n})$ in both positive and negative directions with probability 1, and so for any given $x \geq 0$,

$$1 - F(x) = \mathrm{P}(\Sigma_n > x \text{ for some } n) = 1.$$

Definition 7.43 Define an increasing sequence of r.v. by

$$L(0) = 0, \quad L(n+1) = \min\{k > L(n) : \Sigma_k > \Sigma_{L(n)}\}.$$

The $L(n)$ are called ladder points of the random walk Σ, these are the times when the random walk Σ reaches its new maximal values.

Lemma 7.44 *Let*

$$\eta = \mathrm{P}(\Sigma_n > 0 \textit{ for some } n) = 1 - F(0).$$

The total number Λ of ladder points has a geometric distribution

$$\mathrm{P}(\Lambda = k) = (1-\eta)\eta^k, \quad k = 0, 1, 2, \ldots.$$

Proof. The underlying random walk has a special Markov property: going from one record height to another is a sequence of Benoulli trials with the probability of success η.

Theorem 7.45 *If $\rho < 1$, the equilibrium waiting time $W \overset{d}{=} \max\{\Sigma_0, \Sigma_1, \ldots\}$ has the Laplace transform*

$$\mathrm{E}(e^{-uW}) = \frac{1-\eta}{1 - \eta\mathrm{E}(e^{-uY})},$$

where $Y = \Sigma_{L(1)}$ is the first ladder hight of the embedded random walk.

Proof. In terms of the consecutive increments of the record heights $Y_j = \Sigma_{L(j)} - \Sigma_{L(j-1)}$, which are iid copies of $Y = Y_1$, we have

$$\max\{\Sigma_0, \Sigma_1, \ldots\} = \Sigma_{L(\Lambda)} = Y_1 + \ldots + Y_\Lambda.$$

Therefore,

$$\mathrm{E}(e^{-uW}) = \phi(\mathrm{E}e^{-uY}), \qquad \phi(s) = \mathrm{E}(s^\Lambda) = \frac{1-\eta}{1-\eta s}.$$

8 Martingales

8.1 Definitions and examples

Example 8.1 Martingale: a betting strategy. After tossing a fair coin a gambler wins for heads and looses for tails. Let X_n be the gain of a gambler doubling the bet after each loss. The game stops after the first win. The distribution of X_n evolves as follows

$X_0 = 0$,

$X_1 = 1$ with probability $1/2$ and $X_1 = -1$ with probability $1/2$,

$X_2 = 1$ with probability $3/4$ and $X_2 = -3$ with probability $1/4$,

$X_3 = 1$ with probability $7/8$ and $X_3 = -7$ with probability $1/8$,

and so on. After the n-th tossing the gambler is winning little, $X_n = 1$, with a high probability $1 - 2^{-n}$, or loosing big, $X_n = -2^n + 1$, with a small probability 2^{-n}. The game is fair in that $\mathrm{E}X_n = 0$.

In general, we have $X_{n+1} = (2X_n - 1)$ or $X_{n+1} = 1$ depending on whether it is tails or heads on the $(n+1)$-th tossing. This yields in terms of the conditional expectations

$$\mathrm{E}(X_{n+1}|X_n) = (2X_n - 1) \cdot \frac{1}{2} + \frac{1}{2} = X_n.$$

If N is the number of coin tosses until one gets heads, then $\mathrm{P}(N=n)=2^{-n}$, $n=1,2,\ldots$ with $\mathrm{E}(N)=2$. Interestingly, the state X_{N-1} prior to the winning toss has the mean

$$\mathrm{E}(X_{N-1})=\mathrm{E}(1-2^{N-1})=1-\sum_{n=1}^{\infty}2^{n-1}2^{-n}=-\infty.$$

Definition 8.2 A sequence of sigma-fields $(\mathcal{F}_n)$ such that $\mathcal{F}_0\subset\mathcal{F}_1\subset\ldots\subset\mathcal{F}_n\subset\ldots\subset\mathcal{F}$ is called a filtration. A sequence of r.v. (Y_n) is called adapted to $(\mathcal{F}_n)$ if Y_n is $\mathcal{F}_n$-measurable for all n. Consider an adapted sequence with $\mathrm{E}(|Y_n|)<\infty$ for all $n\ge 0$. If for all $n\ge 0$,

- martingale property $\mathrm{E}(Y_{n+1}|\mathcal{F}_n)=Y_n$,
- submartingale property $\mathrm{E}(Y_{n+1}|\mathcal{F}_n)\ge Y_n$,
- supermartingale property $\mathrm{E}(Y_{n+1}|\mathcal{F}_n)\le Y_n$,

then $(Y_n,\mathcal{F}_n)$ is called a martingale, submartingale, or supermartingale.

Definition 8.3 The sequence $(Y_n)_{n\ge1}$ is a martingale with respect to the sequence $(X_n)_{n\ge1}$, if $(Y_n,\mathcal{F}_n)$ is a martingale for $\mathcal{F}_n=\sigma\{X_1,\ldots,X_n\}$, the σ-algebras generated by $(X_1,\ldots,X_n)$. It follows that $Y_n=\psi_n(X_1,\ldots,X_n)$. We sometimes just say that (Y_n) is a martingale.

The origin of the term "martingale".

Exercise 8.4 Consider the sequence of means $m_n=\mathrm{E}(Y_n)$. We have $m_{n+1}\ge m_n$ for submartingales, $m_{n+1}\le m_n$ for supermartingales, and $m_{n+1}=m_n$ for martingales. A martingale is both a sub- and supermartingale. If (Y_n) is a submartingale, then $(-Y_n)$ is a supermartingale.

Example 8.5 Consider a simple random walk $S_n=S_0+X_1+\ldots+X_n$ with $\mathrm{P}(X_i=1)=p$, $\mathrm{P}(X_i=-1)=q$, and $S_0=k$. The centered sequence $S_n-n(p-q)$ is a martingale:

$$\mathrm{E}(S_{n+1}-(n+1)(p-q)|X_1,\ldots,X_n)=S_n+\mathrm{E}(X_{n+1})-(n+1)(p-q)=S_n-n(p-q).$$

Another martingale is $Y_n=(q/p)^{S_n}$:

$$\mathrm{E}(Y_{n+1}|X_1,\ldots,X_n)=p(q/p)^{S_n+1}+q(q/p)^{S_n-1}=(q/p)^{S_n}=Y_n$$

with $\mathrm{E}(Y_n)=\mathrm{E}(Y_0)=(q/p)^k$. It is called De Moivre martingale.

Example 8.6 Let $S_n=X_1+\ldots+X_n$, where X_i are iid r.v. with zero means and finite variances σ^2. Then $S_n^2-n\sigma^2$ is a martingale

$$\mathrm{E}(S_{n+1}^2-(n+1)\sigma^2|X_1,\ldots,X_n)=S_n^2+2S_n\mathrm{E}(X_{n+1})+\mathrm{E}(X_{n+1}^2)-(n+1)\sigma^2=S_n^2-n\sigma^2.$$

Example 8.7 Application of the optional stopping theorem. Consider a simple random walk $S_n=S_0+X_1+\ldots+X_n$ with $\mathrm{P}(X_i=1)=p$, $\mathrm{P}(X_i=-1)=q$, and $S_0=k$. Let T be the first time when it hits 0 or N, where is $0\le k\le N$. Next we show that for $p\ne q$,

$$\mathrm{P}(S_T=0)=\frac{(p/q)^{N-k}-1}{(p/q)^N-1}.$$

Put $p_k:=\mathrm{P}(S_T=0)$ so that $\mathrm{P}(S_T=N)=1-p_k$. According to Definition 7.12, T is a stopping time. Applying Theorem 8.40 to De Moivre martingale we get $\mathrm{E}(Y_T)=\mathrm{E}(Y_0)$. Since $\mathrm{P}(Y_T=1)=p_k$ and $\mathrm{P}(Y_T=(q/p)^N)=1-p_k$, it follows

$$(q/p)^0p_k+(q/p)^N(1-p_k)=(q/p)^k,$$

and we immediately derive the stated equality.

In terms of the simple random walk starting at zero and stopped either at $-k$ or at N we find

$$\mathrm{P}(S_T=-k)=\frac{(p/q)^N-1}{(p/q)^{N+k}-1}.$$

If $p>q$ and $N\to\infty$, we obtain $\mathrm{P}(S_T=-1)\to\frac{q}{p}$. Thus, if $(-X)$ is the lowest level visited, then X has a shifted geometric distribution distribution with parameter $1-\frac{q}{p}$.

Example 8.8 Stopped de Moivre martingale. Consider the same simple random walk and let T be a stopping time of the random walk. Denote $Z_n = Y_{T\wedge n}$, where (Y_n) is the de Moivre martingale. It is easy to see that Z_n is also a martingale:

$$\begin{aligned}\mathrm{E}(Z_{n+1}|X_1,\ldots,X_n) &= \mathrm{E}(Y_{n+1}1_{\{T>n\}} + \sum_{j\le n} Y_j 1_{\{T=j\}}|X_1,\ldots,X_n)\\ &= Y_n 1_{\{T>n\}} + \sum_{j\le n} Y_j 1_{\{T=j\}} = Z_n.\end{aligned}$$

Example 8.9 Branching processes (recall Section 5.5). Let Z_n be a branching process with $Z_0 = 1$ and the mean offspring number μ. In the supercritical case, $\mu > 1$, the extinction probability $\eta \in [0,1)$ of Z_n is identified as a solution of the equation $\eta = h(\eta)$, where $h(s) = \mathrm{E}(s^X)$ is the generating function of the offspring number. The process $V_n = \eta^{Z_n}$ is a martingale

$$\mathrm{E}(V_{n+1}|Z_1,\ldots,Z_n) = \mathrm{E}(\eta^{X_1+\ldots+X_{Z_n}}|Z_1,\ldots,Z_n) = h(\eta)^{Z_n} = V_n.$$

8.2 Convergence in L^2

Lemma 8.10 *If (Y_n) is a martingale with $\mathrm{E}(Y_n^2) < \infty$, then $Y_{n+1} - Y_n$ and Y_n are uncorrelated. It follows that (Y_n^2) is a submartingale.*

Proof. The first assertion comes from

$$\mathrm{E}(Y_n(Y_{n+1} - Y_n)|\mathcal{F}_n) = Y_n(\mathrm{E}(Y_{n+1}|\mathcal{F}_n) - Y_n) = 0.$$

The second claim is derived as follows

$$\begin{aligned}\mathrm{E}(Y_{n+1}^2|\mathcal{F}_n) &= \mathrm{E}((Y_{n+1} - Y_n)^2 + 2Y_n(Y_{n+1} - Y_n) + Y_n^2|\mathcal{F}_n)\\ &= \mathrm{E}((Y_{n+1} - Y_n)^2|\mathcal{F}_n) + Y_n^2 \ge Y_n^2.\end{aligned}$$

Notice that

$$\mathrm{E}(Y_{n+1}^2) = \mathrm{E}(Y_n^2) + \mathrm{E}((Y_{n+1} - Y_n)^2)$$

so that $\mathrm{E}(Y_n^2)$ is non-decreasing and there always exists a finite or infinite limit

$$M = \lim_{n\to\infty} \mathrm{E}(Y_n^2).$$

Exercise 8.11 Let $J(x)$ be a convex function. If (Y_n) is a martingale with $\mathrm{E}J(Y_n) < \infty$, then $J(Y_n)$ is a submartingale. (Hint: use Jensen inequality.)

Lemma 8.12 *Doob-Kolmogorov inequality. If (Y_n) is a martingale with $\mathrm{E}(Y_n^2) < \infty$, then for any $\epsilon > 0$*

$$\mathrm{P}(\max_{1\le i\le n} |Y_i| \ge \epsilon) \le \frac{\mathrm{E}(Y_n^2)}{\epsilon^2}.$$

Proof. Let $B_1 = \{|Y_1| \ge \epsilon\}$ and $B_k = \{|Y_1| < \epsilon, \ldots, |Y_{k-1}| < \epsilon, |Y_k| \ge \epsilon\}$. Then using a submartingale property (in the second inequality) we get

$$\mathrm{E}(Y_n^2) \ge \sum_{i=1}^n \mathrm{E}(Y_n^2 1_{B_i}) \ge \sum_{i=1}^n \mathrm{E}(Y_i^2 1_{B_i}) \ge \epsilon^2 \sum_{i=1}^n \mathrm{P}(B_i) = \epsilon^2 \mathrm{P}(\max_{1\le i\le n} |Y_i| \ge \epsilon).$$

Theorem 8.13 *If (Y_n) is a martingale with $\mathrm{E}(Y_n^2) \le M$ for some finite M, then there exists a random variable Y such that $Y_n \to Y$ a.s. and in mean square as $n \to \infty$.*

Proof. Step 1. For

$$A_m(\epsilon) = \bigcup_{i\ge 1}\{|Y_{m+i} - Y_m| \ge \epsilon\}$$

we will show that

$$\mathrm{P}(A_m(\epsilon)) \to 0, \quad m \to \infty \text{ for any } \epsilon > 0.$$

For a given m, put $S_n = Y_{m+n} - Y_m$. It is also a martingale, since

$$\mathrm{E}(S_{n+1}|S_1,\ldots,S_n) = \mathrm{E}(\mathrm{E}(S_{n+1}|\mathcal{F}_{m+n})|S_1,\ldots,S_n) = \mathrm{E}(S_n|S_1,\ldots,S_n) = S_n.$$

Apply the Doob-Kolmogorov inequality to this martingale to find that

$$\mathrm{P}(|Y_{m+i} - Y_m| \geq \epsilon \text{ for some } i \in [1,n]) \leq \epsilon^{-2}\mathrm{E}((Y_{m+n} - Y_m)^2) = \epsilon^{-2}(\mathrm{E}(Y_{m+n}^2) - \mathrm{E}(Y_m^2)).$$

Letting $n \to \infty$ we obtain

$$\mathrm{P}(A_m(\epsilon)) \leq \epsilon^{-2}(M - \mathrm{E}(Y_m^2))$$

and hence the step 1 statement follows.

Step 2. Since $A_m(\epsilon_1) \subset A_m(\epsilon_2)$ for $\epsilon_1 > \epsilon_2$, we have, by step 1,

$$\mathrm{P}\Big(\bigcup_{\epsilon>0}\bigcap_{m\geq 1} A_m(\epsilon)\Big) = \lim_{\epsilon\to 0}\mathrm{P}\Big(\bigcap_{m\geq 1} A_m(\epsilon)\Big) \leq \lim_{\epsilon\to 0}\lim_{m\to\infty}\mathrm{P}(A_m(\epsilon)) = 0.$$

Therefore, the sequence (Y_n) is a.s. Cauchy convergent:

$$\mathrm{P}\Big(\bigcap_{\epsilon>0}\bigcup_{m\geq 1} A_m^c(\epsilon)\Big) = 1,$$

which implies the existence of Y such that $Y_n \to Y$ a.s.

Step 3. Finally, we prove the convergence in mean square using the Fatou lemma

$$\begin{aligned}\mathrm{E}((Y_n - Y)^2) &= \mathrm{E}(\liminf_{m\to\infty}(Y_n - Y_m)^2) \leq \liminf_{m\to\infty}\mathrm{E}((Y_n - Y_m)^2)\\ &= \liminf_{m\to\infty}\mathrm{E}(Y_m^2) - \mathrm{E}(Y_n^2) = M - \mathrm{E}(Y_n^2) \to 0, \quad n \to \infty.\end{aligned}$$

Example 8.14 Branching processes (recall Section 5.5). Let Z_n be a branching process with $Z_0 = 1$ and the offspring numbers having mean μ and variance σ^2. Since $\mathrm{E}(Z_{n+1}|Z_n) = \mu Z_n$, the ratio $W_n = \mu^{-n}Z_n$ is a martingale with

$$\mathrm{E}(W_n^2) = 1 + (\sigma/\mu)^2(1 + \mu^{-1} + \ldots + \mu^{-n+1}).$$

In the supercritical case, $\mu > 1$, we have $\mathrm{E}(W_n^2) \to 1 + \frac{\sigma^2}{\mu(\mu-1)}$, and there is a r.v. W such that $W_n \to W$ a.s. and in L^2.

The Laplace transform of the limit $\phi(\theta) = \mathrm{E}(e^{-\theta W})$ satisfies the functional equation $\phi(\mu\theta) = h(\phi(\theta))$.

8.3 Doob decomposition

Definition 8.15 The sequence $(S_n, \mathcal{F}_n)$ is called predictable if $S_0 = 0$, and S_n is $\mathcal{F}_{n-1}$-measurable for all $n \geq 1$. It is also called increasing if $\mathrm{P}(S_n \leq S_{n+1}) = 1$ for all $n \geq 0$.

Theorem 8.16 *Doob decomposition. A submartingale $(Y_n, \mathcal{F}_n)$ can be expressed in the form $Y_n = M_n + S_n$, where $(M_n, \mathcal{F}_n)$ is a martingale and $(S_n, \mathcal{F}_n)$ is an increasing predictable process (called the compensator of the submartingale). This decomposition is unique.*

Proof. We define M and S explicitly: $M_0 = Y_0$, $S_0 = 0$, and for $n \geq 0$

$$M_{n+1} - M_n = Y_{n+1} - \mathrm{E}(Y_{n+1}|\mathcal{F}_n), \qquad S_{n+1} - S_n = \mathrm{E}(Y_{n+1}|\mathcal{F}_n) - Y_n.$$

To see uniqueness suppose another such decomposition $Y_n = M_n' + S_n'$. Then

$$M_{n+1}' - M_n' + S_{n+1}' - S_n' = M_{n+1} - M_n + S_{n+1} - S_n.$$

Taking conditional expectations given $\mathcal{F}_n$ we get $S_{n+1}' - S_n' = S_{n+1} - S_n$. This in view of $S_0' = S_0 = 0$ implies $S_n' = S_n$.

Definition 8.17 Let (Y_n) be adapted to $(\mathcal{F}_n)$, and (S_n) be predictable. The sequence

$$Z_n = Y_0 + \sum_{i=1}^{n} S_i(Y_i - Y_{i-1})$$

is called the transform of (Y_n) by (S_n).

Example 8.18 Such transforms with $S_n \geq 0$ are usually interpreted as gambling systems, with (Y_n) being a supermartingale defined as the capital of a gambler waging a unit stake at each round. Optional skipping is one such strategy: the gambler either wagers a unit stake, $S_n = 1$, or skips the round, $S_n = 0$. The following result says that there are no sure ways to beat the casino.

Theorem 8.19 *Let (Z_n) be the transform of (Y_n) by (S_n) such that $\mathrm{E}|Z_n| < \infty$ for all n. Then*

(i) If (Y_n) is a martingale, then (Z_n) is a martingale.

(ii) If (Y_n) is a submartingale (supermartingale) and in addition $S_n \geq 0$ for all n, then (Z_n) is a submartingale (supermartingale).

Proof. All the assertions follow immediately from

$$\mathrm{E}(Z_{n+1}|\mathcal{F}_n) - Z_n = \mathrm{E}(Z_{n+1} - Z_n|\mathcal{F}_n) = \mathrm{E}(S_{n+1}(Y_{n+1} - Y_n)|\mathcal{F}_n) = S_{n+1}(\mathrm{E}(Y_{n+1}|\mathcal{F}_n) - Y_n).$$

Example 8.20 Optional stopping. Consider a gambler waging a unit stake on each play until a random time T. In this case $S_n = 1_{\{n \leq T\}}$ and $Z_n = Y_{T \wedge n}$. If S_n is predictable, then $\{T = n\} = \{S_n = 1, S_{n+1} = 0\} \in \mathcal{F}_n$, so that T is a stopping time.

Example 8.21 Optional starting. If a gambler does not play until the $(T + 1)$-th round, where T is a stopping time, then

$$Z_n = Y_0 + (Y_n - Y_T)1_{\{n \geq T+1\}} = Y_0 + \sum_{i=T+1}^{n} (Y_i - Y_{i-1}).$$

The corresponding predictable sequence is $S_n = 1_{\{T \leq n-1\}}$.

8.4 Hoeffding inequality

Definition 8.22 Let $(Y_n, \mathcal{F}_n)$ be a martingale. The sequence of martingale differences is defined by $D_n = Y_n - Y_{n-1}$, so that D_n is $\mathcal{F}_n$-measurable and

$$\mathrm{E}|D_n| < \infty, \qquad \mathrm{E}(D_{n+1}|\mathcal{F}_n) = 0, \qquad Y_n = Y_0 + D_1 + \ldots + D_n.$$

Sums of martingale differences extend the classical setting of summing independent variables.

Theorem 8.23 *Let $(Y_n, \mathcal{F}_n)$ be a martingale with bounded martingale differences:* $\mathrm{P}(|D_n| \leq K_n) = 1$ *for a sequence of real numbers K_n. Then for any $x > 0$*

$$\mathrm{P}(|Y_n - Y_0| \geq x) \leq 2\exp\Big(-\frac{x^2}{2(K_1^2 + \ldots + K_n^2)}\Big).$$

Proof. Let $\theta > 0$. Later in the proof we put $\theta = x/\sum_{i=1}^{n} K_i^2$.

Step 1. The function $e^{\theta x}$ is convex over $x \in [-1, 1]$, therefore

$$e^{\theta d} \leq \frac{1}{2}(1 - d)e^{-\theta} + \frac{1}{2}(1 + d)e^{\theta} \text{ for all } |d| \leq 1.$$

Hence if D is a zero-mean random variable, such that $\mathrm{P}(|D| \leq 1) = 1$, then

$$\mathrm{E}(e^{\theta D}) \leq \frac{e^{-\theta} + e^{\theta}}{2} + \frac{e^{\theta} - e^{-\theta}}{2}\mathrm{E}(D) = \frac{e^{-\theta} + e^{\theta}}{2} = \sum_{k=0}^{\infty} \frac{\theta^{2k}}{(2k)!} < \sum_{k=0}^{\infty} \frac{\theta^{2k}}{2^k(k)!} = e^{\theta^2/2}.$$

Step 2. Applying the previous step to the scaled martingale differences $D'_n = D_n/K_n$, we obtain

$$\mathrm{E}(e^{\theta(Y_n-Y_0)}|\mathcal{F}_{n-1}) = e^{\theta(Y_{n-1}-Y_0)}\mathrm{E}(e^{\theta D_n}|\mathcal{F}_{n-1}) = e^{\theta(Y_{n-1}-Y_0)}\mathrm{E}(e^{\theta K_n D'_n}|\mathcal{F}_{n-1}) \le e^{\theta(Y_{n-1}-Y_0)}e^{\theta^2 K_n^2/2}.$$

Take expectations and iterate to find

$$\mathrm{E}(e^{\theta(Y_n-Y_0)}) \le \mathrm{E}(e^{\theta(Y_{n-1}-Y_0)})e^{\theta^2 K_n^2/2} \le \exp\Big(\frac{\theta^2}{2}\sum_{i=1}^n K_i^2\Big).$$

Step 3. Due to the Markov inequality we have for any $x > 0$

$$\mathrm{P}(Y_n - Y_0 \ge x) = \mathrm{P}(e^{\theta(Y_n-Y_0)} \ge e^{\theta x}) \le e^{-\theta x}\mathrm{E}(e^{\theta(Y_n-Y_0)}) \le \exp\Big(-\theta x + \frac{\theta^2}{2}\sum_{i=1}^n K_i^2\Big).$$

Set $\theta = x/\sum_{i=1}^n K_i^2$ to minimize the exponent. Then

$$\mathrm{P}(Y_n - Y_0 \ge x) \le \exp\Big(-\frac{x^2}{2(K_1^2 + \ldots + K_n^2)}\Big).$$

Since $(-Y_n)$ is also a martingale, we get

$$\mathrm{P}(Y_n - Y_0 \le -x) = \mathrm{P}(-Y_n + Y_0 \ge x) \le \exp\Big(-\frac{x^2}{2(K_1^2 + \ldots + K_n^2)}\Big).$$

Example 8.24 An application to large deviations. Let X_n be iid Ber(p) random variables. If $S_n = X_1 + \ldots + X_n$, then $Y_n = S_n - np$ is a martingale with

$$|Y_n - Y_{n-1}| = |X_n - p| \le \max(p, 1-p).$$

Due to the Hoeffding inequality for any $x > 0$

$$\mathrm{P}\Big(\Big|\frac{S_n - np}{\sqrt{n}}\Big| \ge x\Big) = \mathrm{P}(|S_n - np| \ge x\sqrt{n}) \le 2\exp\Big(-\frac{x^2}{2(\max(p,1-p))^2}\Big).$$

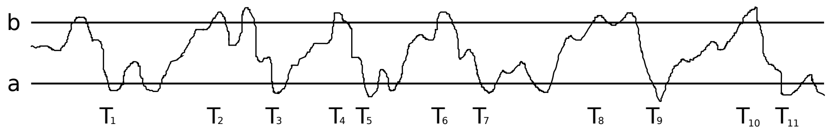
In particular, if $p = 1/2$,

$$\mathrm{P}(|S_n - n/2| \ge x\sqrt{n}) \le 2e^{-2x^2}.$$

Putting here $x = 3$ we get $\mathrm{P}(|S_n - n/2| \ge 3\sqrt{n}) \le 3 \cdot 10^{-8}$.

8.5 Convergence in L^1

On the figure below five time intervals are shown: $(T_1, T_2]$, $(T_3, T_4]$, ..., $(T_9, T_{10}]$ for the uppcrossings of the interval (a, b). If for all rational intervals (a, b) the number of uppcrossings $U(a, b)$ is finite, then the corresponding trajectory has a (possibly infinite) limit.



All T_k are stopping times. Definition 7.12 for the stopping times can be extended as follows.

Definition 8.25 A random variable T taking values in the set $\{0, 1, 2, \ldots\} \cup \{\infty\}$ is called a stopping time with respect to the filtration $(\mathcal{F}_n)$, if

$$\{T = n\} \in \mathcal{F}_n, \quad \text{for all } n = 0, 1, 2, \ldots.$$

Exercise 8.26 Define more exactly the crossing times T_k. Let $(Y_n, \mathcal{F}_n)_{n\geq 0}$ be an adapted process, and take $a < b$. Put $T_1 = n$ for the smallest n such that $Y_n \leq a$. In particular, if $Y_0 \leq a$, then $T_1 = 0$.

For an even natural number $k \geq 2$ put $T_k = n$ if $Y_i < b$ for $T_{k-1} \leq i \leq n-1$, and $Y_n \geq b$. For an odd natural number $k \geq 3$ put $T_k = n$ if $Y_i > a$ for $T_{k-1} \leq i \leq n-1$, and $Y_n \leq a$. Show that all crossing times T_k are stopping times. (Hint: use induction over k.)

Lemma 8.27 *Snell uppcrossing inequality. Let $a < b$ and $U_n(a,b)$ is the number of uppcrossings of the interval (a,b) for a submartingale $(Y_0,\ldots,Y_n)$. Then*

$$\mathrm{E}[U_n(a,b)] \leq \frac{\mathrm{E}[(Y_n - a) \vee 0]}{b-a}.$$

Proof. Let I_i be the indicator of the event that $i \in (T_{2k-1}, T_{2k}]$ for some k. Note that I_i is $\mathcal{F}_{i-1}$-measurable, since

$$\{I_i = 1\} = \bigcup_k \{T_{2k-1} \leq i-1 < T_{2k}\} = \bigcup_k \{T_{2k-1} \leq i-1\} \cap \{T_{2k} \leq i-1\}^c$$

is an event that depends on $(Y_0,\ldots,Y_{i-1})$ only. Now, since $Z_n = (Y_n - a) \vee 0$ forms a submartingale, we get

$$\begin{aligned}\mathrm{E}((Z_i - Z_{i-1}) \cdot I_i) &= \mathrm{E}[\mathrm{E}(I_i \cdot (Z_i - Z_{i-1})|\mathcal{F}_{i-1})] = \mathrm{E}[I_i \cdot (\mathrm{E}(Z_i|\mathcal{F}_{i-1}) - Z_{i-1})] \\ &\leq \mathrm{E}[\mathrm{E}(Z_i|\mathcal{F}_{i-1}) - Z_{i-1}] = \mathrm{E}(Z_i) - \mathrm{E}(Z_{i-1}).\end{aligned}$$

Now, observe that (inspect the figure above)

$$(b-a) \cdot U_n(a,b) \leq \sum_{i=1}^n (Z_i - Z_{i-1}) I_i.$$

Thus

$$(b-a)\mathrm{E}\,U_n(a,b) \leq \mathrm{E}\sum_{i=1}^n (\mathrm{E}(Z_i|\mathcal{F}_{i-1}) - Z_{i-1}) I_i \leq \mathrm{E}\sum_{i=1}^n (\mathrm{E}(Z_i|\mathcal{F}_{i-1}) - Z_{i-1}) = \mathrm{E}(Z_n) - \mathrm{E}(Z_0) \leq \mathrm{E}(Z_n).$$

Theorem 8.28 *Suppose $(Y_n, \mathcal{F}_n)$ is a submartingale such that for some finite constant M,*

$$\sup_{n\geq 1} \mathrm{E}(Y_n \vee 0) \leq M.$$

Then

(i) there exists a r.v. Y such that $Y_n \to Y$ almost surely,

(ii) the limit Y has a finite mean,

(iii) if (Y_n) is uniformly integrable, see Definition 3.22, then $Y_n \to Y$ in L^1.

Proof. (i) Using Snell inequality we obtain that $U(a,b) = \lim U_n(a,b)$ satisfies

$$\mathrm{E}U(a,b) \leq \frac{M + |a|}{b-a}.$$

Therefore, $\mathrm{P}(U(a,b) < \infty) = 1$ for any given interval (a,b). Since there are only countably many rationals, it follows that

$$\mathrm{P}\{U(a,b) < \infty \text{ for all rational } (a,b)\} = 1,$$

which implies that $Y_n \to Y$ almost surely.

(ii) Denote $Y_n^+ = Y_n \vee 0$. Since $|Y_n| = 2Y_n^+ - Y_n$ and $\mathrm{E}(Y_n|\mathcal{F}_0) \geq Y_0$, we get

$$\mathrm{E}(|Y_n||\mathcal{F}_0) \leq 2\mathrm{E}(Y_n^+|\mathcal{F}_0) - Y_0.$$

By Fatou lemma

$$\mathrm{E}(|Y||\mathcal{F}_0) = \mathrm{E}(\varliminf_{n\to\infty} |Y_n||\mathcal{F}_0) \leq \varliminf_{n\to\infty} \mathrm{E}(|Y_n||\mathcal{F}_0) \leq 2 \varliminf_{n\to\infty} \mathrm{E}(Y_n^+|\mathcal{F}_0) - Y_0,$$

and it remains to observe that $\mathrm{E}(\varliminf_{n\to\infty} \mathrm{E}(Y_n^+|\mathcal{F}_0)) \leq M$, again due to Fatou lemma.

(iii) Finally, recall that according to Theorem 3.25, given $Y_n \xrightarrow{\mathrm{P}} Y$, the uniform integrability of (Y_n) is equivalent to $\mathrm{E}|Y_n| < \infty$ for all n, $\mathrm{E}|Y| < \infty$, and $Y_n \xrightarrow{L^1} Y$.

Corollary 8.29 *Any martingale, submartingale or supermartingale $(Y_n, \mathcal{F}_n)$ satisfying $\sup_n \mathrm{E}|Y_n| \le M$ converges almost surely to a r.v. with a finite mean.*

Corollary 8.30 *A non-negative supermartingale converges almost surely with a finite mean. A non-positive submartingale converges almost surely with a finite mean.*

Example 8.31 De Moivre martingale $Y_n = (q/p)^{S_n}$ is non-negative and hence converges a.s. to some limit Y. Let $p \ne q$. Since $S_n \to \infty$ for $p > q$ and $S_n \to -\infty$ for $p < q$, in both cases we get $\mathrm{P}(Y = 0) = 1$. Note that Y_n does not converge in mean, since $\mathrm{E}(Y_n) = \mathrm{E}(Y_0) \ne 0$.

Exercise 8.32 Knapsack problem. It is required to pack a knapsack of volume c to maximum benefit. Suppose you have n objects, the ith object having volume V_i and worth W_i, where $V_1, \ldots, V_n, W_1, \ldots, W_n$ are independent non-negative random variables with finite means. You wish to find a vector $(z_1, \ldots, z_n)$ of 0 and 1 such that

$$z_1 V_1 + \ldots + z_n V_n \le c$$

and which maximizes $W = z_1 W_1 + \ldots + z_n W_n$. Let Z be the maximal possible W, and show that given $W_i \le 1$ for all i,

$$\mathrm{P}(|Z - \mathrm{E}Z| \ge x) \le 2e^{-\frac{x^2}{2n}}.$$

In particular,

$$\mathrm{P}(|Z - \mathrm{E}Z| \ge 3\sqrt{n}) \le 0.022.$$

Solution. Consider a filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_i = \sigma(V_1, W_1, \ldots, V_i, W_i)$ for $i = 1, \ldots, n$. And let $Y_i = \mathrm{E}(Z|\mathcal{F}_i)$, $i = 0, 1, \ldots, n$ be a Doob martingale. The assertion follows from the Hoeffding inequality if we verify that $|Y_i - Y_{i-1}| \le 1$. To this end define $Z(j)$ as the maximal worth attainable without using the jth object. Clearly,

$$\mathrm{E}(Z(j)|\mathcal{F}_j) = \mathrm{E}(Z(j)|\mathcal{F}_{j-1}).$$

Since obviously $Z(j) \le Z \le Z(j) + 1$, we have

$$Y_i - Y_{i-1} = \mathrm{E}(Z|\mathcal{F}_i) - \mathrm{E}(Z|\mathcal{F}_{i-1}) \le 1 + \mathrm{E}(Z(i)|\mathcal{F}_i) - \mathrm{E}(Z(i)|\mathcal{F}_{i-1}) = 1,$$

and

$$Y_i - Y_{i-1} = \mathrm{E}(Z|\mathcal{F}_i) - \mathrm{E}(Z|\mathcal{F}_{i-1}) \ge \mathrm{E}(Z(i)|\mathcal{F}_i) - \mathrm{E}(Z(i)|\mathcal{F}_{i-1}) - 1 = -1.$$

8.6 Doob's martingale

Theorem 8.33 *Let Z be a r.v. on $(\Omega, \mathcal{F}, \mathrm{P})$ such that $\mathrm{E}(|Z|) < \infty$. For a filtration $(\mathcal{F}_n)$ define $Y_n = \mathrm{E}(Z|\mathcal{F}_n)$. Then $(Y_n, \mathcal{F}_n)$ is a uniformly integrable martingale, which is called Doob's martingale.*

Proof. Put $Z_n = \mathrm{E}(|Z|\,|\mathcal{F}_n)$ and observe that $|Y_n| \le Z_n$ by the Jensen inequality. The finiteness of the means follows from

$$\mathrm{E}(|Y_n|) \le \mathrm{E}(Z_n) = \mathrm{E}(|Z|),$$

and the martingale property is straightforward

$$\mathrm{E}(Y_{n+1}|\mathcal{F}_n) = \mathrm{E}(\mathrm{E}(Z|\mathcal{F}_{n+1})|\mathcal{F}_n)) = \mathrm{E}(Z|\mathcal{F}_n) = Y_n.$$

To show uniform integrability observe that the inequality $|Y_n| \le Z_n$ implies that for any a,

$$|Y_n| 1_{\{|Y_n| \ge a\}} \le Z_n 1_{\{Z_n \ge a\}}.$$

Since on the other hand, by Definition 2.7 of conditional expectation,

$$\mathrm{E}\big((|Z| - Z_n) 1_{\{Z_n \ge a\}}\big) = 0,$$

we find that

$$\mathrm{E}(|Y_n| 1_{\{|Y_n| \ge a\}}) \le \mathrm{E}(|Z| 1_{\{Z_n \ge a\}}).$$

To derive from here the uniform integrability

$$\sup_n \mathrm{E}(|Y_n| 1_{\{|Y_n|\geq a\}}) \to 0, \quad a \to \infty,$$

it is enough to show that

$$\sup_n \mathrm{E}(|Z| 1_{\{Z_n \geq a\}}) \to 0, \quad a \to \infty.$$

The latter is obtained using the Markov inequality. Indeed, for arbitrary $C > 0$,

$$\mathrm{E}(|Z| 1_{\{Z_n \geq a\}}) \leq C \cdot \mathrm{P}(Z_n \geq a) + \mathrm{E}(|Z| 1_{\{|Z|\geq C\}}) \leq C \cdot \frac{\mathrm{E}(Z_n)}{a} + \mathrm{E}(|Z| 1_{\{|Z|\geq C\}}),$$

implying that

$$\sup_n \mathrm{E}(|Z| 1_{\{Z_n \geq a\}}) \leq C \cdot \frac{\mathrm{E}|Z|}{a} + \mathrm{E}(|Z| 1_{\{|Z|\geq C\}}).$$

It remains to let first $a \to \infty$ and then $C \to \infty$.

Remarks

- By Theorem 8.28, Doob's martingale converges $\mathrm{E}(Z|\mathcal{F}_n) \to Y$ a.s. and in mean as $n \to \infty$.
- It is actually the case (without proof) that the limit $Y = \mathrm{E}(Z|\mathcal{F}_\infty)$, where $\mathcal{F}_\infty$ is the smallest σ-algebra containing all $\mathcal{F}_n$.
- There is an important converse result (without proof): if a martingale $(Y_n, \mathcal{F}_n)$ converges in mean, then there exists a r.v. Z with finite mean such that $Y_n = \mathrm{E}(Z|\mathcal{F}_n)$.

8.7 Bounded stopping times. Optional sampling theorem

The stopped de Moivre martingale from Example 8.8 is also a martingale. A general statement of this type follows next.

Theorem 8.34 *Let $(Y_n, \mathcal{F}_n)$ be a submartingale and let T be a stopping time. Then both $(Y_{T\wedge n}, \mathcal{F}_n)$ and $(Y_n - Y_{T\wedge n}, \mathcal{F}_n)$ are also submartingales.*

Proof. The random variable $Z_n = Y_{T\wedge n}$ is $\mathcal{F}_n$-measurable:

$$Z_n = \sum_{i=0}^{n-1} Y_i 1_{\{T=i\}} + Y_n 1_{\{T\geq n\}},$$

with

$$\mathrm{E}|Z_n| \leq \sum_{i=0}^{n} \mathrm{E}|Y_i| < \infty.$$

It remains to see that the equality

$$Z_{n+1} - Z_n = (Y_{n+1} - Y_n) 1_{\{T>n\}}$$

implies on one hand,

$$\mathrm{E}(Z_{n+1} - Z_n|\mathcal{F}_n) = \mathrm{E}(Y_{n+1} - Y_n|\mathcal{F}_n) 1_{\{T>n\}} \geq 0,$$

and on the other hand,

$$\mathrm{E}(Y_{n+1} - Z_{n+1} - Y_n + Z_n|\mathcal{F}_n) = \mathrm{E}(Y_{n+1} - Y_n|\mathcal{F}_n) 1_{\{T\leq n\}} \geq 0.$$

Corollary 8.35 *If $(Y_n, \mathcal{F}_n)$ is a martingale, then it is both a submartingale and a supermartingale, and therefore, for a given stopping time T, both $(Y_{T\wedge n}, \mathcal{F}_n)$ and $(Y_n - Y_{T\wedge n}, \mathcal{F}_n)$ are martingales.*

Definition 8.36 For a stopping time T with respect to filtration $(\mathcal{F}_n)$, denote by $\mathcal{F}_T$ the σ-algebra of all events A such that

$$A \cap \{T = n\} \in \mathcal{F}_n \quad \text{for all } n.$$

The stopping time T is called bounded if $\mathrm{P}(T \le N) = 1$ for some finite constant N.

Exercise 8.37 If $T \le S$ are two stopping times, then $\mathcal{F}_T \subset \mathcal{F}_S$.

Theorem 8.38 *Optional sampling. Let $(Y_n, \mathcal{F}_n)$ be a submartingale.*

(i) If T is a bounded stopping time, then $\mathrm{E}|Y_T| < \infty$ and $\mathrm{E}(Y_T|\mathcal{F}_0) \ge Y_0$.

(ii) If $0 = T_0 \le T_1 \le T_2 \le \ldots$ is a sequence of bounded stopping times, then $(Y_{T_j}, \mathcal{F}_{T_j})$ is a submartingale.

Proof. (i) Let $\mathrm{P}(T \le N) = 1$ where N is a positive constant. By the previous theorem, $Z_n = Y_{T\wedge n}$ is a submartingale. Therefore $Z_N = Y_T$, we have $\mathrm{E}|Y_T| = \mathrm{E}|Z_N| < \infty$ and

$$\mathrm{E}(Y_T|\mathcal{F}_0) = \mathrm{E}(Z_N|\mathcal{F}_0) \ge Z_0 = Y_0.$$

(ii) It is easy to see that for an arbitrary stopping time S, the random variable Y_S is $\mathcal{F}_S$-measurable:

$$\{Y_S \le x\} \cap \{S = n\} = \{Y_n \le x\} \cap \{S = n\} \in \mathcal{F}_n.$$

Consider two bounded stopping times $S \le T \le N$ and put $W = \mathrm{E}(Y_T|\mathcal{F}_S)$. To show that $W \ge Y_S$, observe that for $A \in \mathcal{F}_S$ we have

$$\mathrm{E}(W 1_A) = \mathrm{E}(Y_T 1_A) = \sum_{k\le N} \mathrm{E}(Y_T 1_{A\cap\{S=k\}}) = \sum_{k \le N} \mathrm{E}\Big(1_{A\cap\{S=k\}} \mathrm{E}(Y_T|\mathcal{F}_k)\Big),$$

and since in view of Theorem 8.34,

$$\mathrm{E}(Y_T|\mathcal{F}_k) = \mathrm{E}(Y_{T\wedge N}|\mathcal{F}_k) \ge Y_{T\wedge k} \quad \text{for all } k \le N,$$

we conclude

$$\mathrm{E}(W 1_A) \ge \mathrm{E}\Big(\sum_{k\le N} 1_{A\cap\{S=k\}} Y_{T\wedge k}\Big) = \mathrm{E}\Big(\sum_{k \le N} 1_{A\cap\{S=k\}} Y_k\Big) = \mathrm{E}(Y_S 1_A),$$

so that $\mathrm{E}((W - Y_S)1_A) \ge 0$ for all $A \in \mathcal{F}_S$. In particular, for any given $\epsilon > 0$, we can put $A_\epsilon = \{W \le Y_S - \epsilon\}$ and obtain

$$0 \le \mathrm{E}((W - Y_S)1_{A_\epsilon}) \le -\epsilon \mathrm{P}(A_\epsilon).$$

Thus $\mathrm{P}(A_\epsilon) = 0$ and $W \ge Y_S$ with probability 1.

Exercise 8.39 Let the process (Y_n) be a martingale and T be a bounded stopping time. Show that according to Theorem 8.38 we have $\mathrm{E}(Y_T|\mathcal{F}_0) = Y_0$ and $\mathrm{E}(Y_T) = \mathrm{E}(Y_0)$.

8.8 Unbounded stopping times

The martingale property in Exercise 8.39 is not enough for Example 8.7 because the corresponding stopping time T is not bounded. However, according to Section 5.1,

$$\mathrm{E}(T) = \frac{1}{q-p}\left[k - N \cdot \frac{1-(q/p)^k}{1-(q/p)^N}\right]$$

is finite and the following two optional stopping theorems work.

Theorem 8.40 *Let $(Y_n, \mathcal{F}_n)$ be a martingale and T be a stopping time. Then $\mathrm{E}(Y_T) = \mathrm{E}(Y_0)$ if*

(a) $\mathrm{P}(T < \infty) = 1$,

(b) $\mathrm{E}|Y_T| < \infty$,

(c) $\mathrm{E}(Y_n 1_{\{T>n\}}) \to 0$ as $n \to \infty$.

Proof. From $Y_T = Y_{T\wedge n} + (Y_T - Y_n)1_{\{T>n\}}$ using that $\mathrm{E}(Y_{T\wedge n}) = \mathrm{E}(Y_0)$ we obtain

$$\mathrm{E}(Y_T) = \mathrm{E}(Y_0) + \mathrm{E}(Y_T 1_{\{T>n\}}) - \mathrm{E}(Y_n 1_{\{T>n\}}).$$

It remains to apply (c) and observe that using (a), (b) and the dominated convergence theorem we get

$$\mathrm{E}(Y_T 1_{\{T>n\}}) \to 0.$$

Theorem 8.41 *Let $(Y_n, \mathcal{F}_n)$ be a martingale and T be a stopping time. Then $\mathrm{E}(Y_T) = \mathrm{E}(Y_0)$, if $\mathrm{E}(T) < \infty$ and there exists a constant c such that for any n*

$$\mathrm{E}(|Y_{n+1} - Y_n| \big| \mathcal{F}_n) 1_{\{T>n\}} \le c 1_{\{T>n\}}.$$

Proof. Since $T \wedge n \to T$, we have $Y_{T\wedge n} \to Y_T$ a.s. It follows that

$$\mathrm{E}(Y_0) = \mathrm{E}(Y_{T\wedge n}) \to \mathrm{E}(Y_T)$$

as long as $(Y_{T\wedge n})$ is uniformly integrable. By Exercise 3.23, to prove the uniform integrability it is enough to observe that

$$\sup_n |Y_{T\wedge n}| \le |Y_0| + W, \quad W := |Y_1 - Y_0| + \ldots + |Y_T - Y_{T-1}|,$$

and to verify that $\mathrm{E}(W) < \infty$. Indeed, since $\mathrm{E}(|Y_i - Y_{i-1}| 1_{\{T\ge i\}} \big| \mathcal{F}_{i-1}) \le c 1_{\{T\ge i\}}$, we have

$$\mathrm{E}(|Y_i - Y_{i-1}| 1_{\{T\ge i\}}) \le c\mathrm{P}(T \ge i),$$

and therefore

$$\begin{aligned} \mathrm{E}(W) &= \mathrm{E}\Big(\sum_{i=1}^{T} |Y_i - Y_{i-1}|\Big) = \sum_{i=1}^{\infty} \mathrm{E}(|Y_i - Y_{i-1}| 1_{\{T\ge i\}}) = \sum_{i=1}^{\infty} \mathrm{E}\Big(\mathrm{E}(|Y_i - Y_{i-1}| 1_{\{T\ge i\}} \big| \mathcal{F}_{i-1})\Big) \\ &\le c \sum_{i=1}^{\infty} \mathrm{P}(T \ge i) \le c\mathrm{E}(T) < \infty. \end{aligned}$$

Example 8.42 Wald's equality. Let (X_n) be iid r.v. with finite mean μ and $S_n = X_1 + \ldots + X_n$, then $Y_n = S_n - n\mu$ is a martingale with respect to $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. Now

$$\mathrm{E}(|Y_{n+1} - Y_n| \big| \mathcal{F}_n) = \mathrm{E}|X_{n+1} - \mu| = \mathrm{E}|X_1 - \mu| < \infty.$$

We deduce from Theorem 8.41 that $\mathrm{E}(Y_T) = \mathrm{E}(Y_0)$ for *any* stopping time T with finite mean, implying that $\mathrm{E}(S_T) = \mu\mathrm{E}(T)$.

Lemma 8.43 *Wald's identity. Let (X_n) be iid r.v. with a finite $M(t) = \mathrm{E}(e^{tX})$. Put $S_n = X_1 + \ldots + X_n$. If T is a stopping time with finite mean such that $|S_n| 1_{\{T>n\}} \le c 1_{\{T>n\}}$, then*

$$\mathrm{E}(e^{tS_T} M(t)^{-T}) = 1 \textit{ whenever } M(t) \ge 1.$$

Proof. Define $Y_0 = 1$, $Y_n = e^{tS_n} M(t)^{-n}$, and let $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. It is clear that (Y_n) is a martingale and thus the claim follows from Theorem 8.41 after we verify the key condition. By the definition of Y_n,

$$\mathrm{E}(|Y_{n+1} - Y_n| \big| \mathcal{F}_n) = Y_n \mathrm{E}|e^{tX} M(t)^{-1} - 1| \le Y_n \mathrm{E}(e^{tX} M(t)^{-1} + 1) = 2Y_n.$$

Furthermore, given $M(t) \ge 1$,

$$Y_n 1_{\{T>n\}} = e^{tS_n} M(t)^{-n} 1_{\{T>n\}} \le e^{tS_n} 1_{\{T>n\}} \le e^{|t||S_n|} 1_{\{T>n\}} \le e^{c|t|} 1_{\{T>n\}}.$$

Example 8.44 Simple random walk S_n with $\mathrm{P}(X_i = 1) = p$ and $\mathrm{P}(X_i = -1) = q$. Let T be the first exit time of $(-a, b)$ for some $a > 0$ and $b > 0$. Clearly, $|S_n|1_{\{T>n\}} \leq c1_{\{T>n\}}$ with $c = a \vee b$.

By Lemma 8.43 with $M(t) = pe^t + qe^{-t}$,

$$e^{-at}\mathrm{E}(M(t)^{-T}1_{\{S_T=-a\}}) + e^{bt}\mathrm{E}(M(t)^{-T}1_{\{S_T=b\}}) = 1 \text{ whenever } M(t) \geq 1.$$

Setting $pe^t + qe^{-t} = s^{-1}$, we obtain a quadratic equation for e^t having two solutions $\lambda_i = \lambda_i(s)$:

$$\lambda_1 = \frac{1 + \sqrt{1 - 4pqs^2}}{2ps}, \qquad \lambda_2 = \frac{1 - \sqrt{1 - 4pqs^2}}{2ps}, \quad s \in [0, 1].$$

This observation yields two linear equations

$$\lambda_i^{-a}\mathrm{E}(s^T 1_{\{S_T=-a\}}) + \lambda_i^b \mathrm{E}(s^T 1_{\{S_T=b\}}) = 1, \quad i = 1, 2,$$

resulting in

$$\mathrm{E}(s^T 1_{\{S_T=-a\}}) = \frac{\lambda_1^a \lambda_2^a (\lambda_1^b - \lambda_2^b)}{\lambda_1^{a+b} - \lambda_2^{a+b}}, \qquad \mathrm{E}(s^T 1_{\{S_T=b\}}) = \frac{\lambda_1^a - \lambda_2^a}{\lambda_1^{a+b} - \lambda_2^{a+b}}.$$

Summing up these two relations we get a formula for the probability generating function of the stopping time T

$$\mathrm{E}(s^T) = \frac{\lambda_1^a(1 - \lambda_2^{a+b}) + \lambda_2^a(\lambda_1^{a+b} - 1)}{\lambda_1^{a+b} - \lambda_2^{a+b}}.$$

8.9 Maximal inequality

For a random variable X, we write $X^+ = X \wedge 0$ and $X^- = (-X) \wedge 0$, so that

$$X = X^+ - X^-, \qquad |X| = X^+ + X^-.$$

Theorem 8.45 *Maximal inequality.*

(i) If (Y_n) is a submartingale, then for any $\epsilon > 0$,

$$\mathrm{P}(\max_{0 \leq i \leq n} Y_i \geq \epsilon) \leq \frac{\mathrm{E}(Y_n^+)}{\epsilon}.$$

(ii) If (Y_n) is a supermartingale, then for any $\epsilon > 0$,

$$\mathrm{P}(\max_{0 \leq i \leq n} Y_i \geq \epsilon) \leq \frac{\mathrm{E}(Y_0) + \mathrm{E}(Y_n^-)}{\epsilon}.$$

Proof. (i) If (Y_n) is a submartingale, then (Y_n^+) is a non-negative submartingale. Introduce a stopping time

$$T = \min\{n : Y_n \geq \epsilon\} = \min\{n : Y_n^+ \geq \epsilon\}$$

and notice that

$$\{T \leq n\} = \{\max_{0 \leq i \leq n} Y_i \geq \epsilon\}.$$

By the second part of Theorem 8.34, $\mathrm{E}(Y_n^+ - Y_{T\wedge n}^+) \geq 0$. Therefore,

$$\mathrm{E}(Y_n^+) \geq \mathrm{E}(Y_{T\wedge n}^+) = \mathrm{E}(Y_T^+ 1_{\{T \leq n\}}) + \mathrm{E}(Y_n^+ 1_{\{T>n\}}) \geq \mathrm{E}(Y_T^+ 1_{\{T \leq n\}}) \geq \epsilon\mathrm{P}(T \leq n),$$

implying the first stated inequality.

Furthermore, since $\mathrm{E}(Y_n^+ 1_{\{T>n\}}) = \mathrm{E}(Y_{T\wedge n}^+ 1_{\{T>n\}})$, we have

$$\mathrm{E}(Y_n^+ 1_{\{T \leq n\}}) = \mathrm{E}(Y_n^+) - \mathrm{E}(Y_{T\wedge n}^+ 1_{\{T>n\}}) \geq \mathrm{E}(Y_{T\wedge n}^+ 1_{\{T \leq n\}}) = \mathrm{E}(Y_T^+ 1_{\{T \leq n\}}) \geq \epsilon\mathrm{P}(T \leq n).$$

Using this we get even a stronger inequality

$$\mathrm{P}(A) \leq \frac{\mathrm{E}(Y_n^+ 1_A)}{\epsilon}, \text{ where } A = \{\max_{0 \leq i \leq n} Y_i \geq \epsilon\}. \qquad (*)$$

(ii) If (Y_n) is a supermartingale, then by the first part of Theorem 8.34, $\mathrm{E}(-Y_{T\wedge n}) \geq \mathrm{E}(-Y_0)$. Thus

$$\mathrm{E}(Y_0) \geq \mathrm{E}(Y_{T\wedge n}) = \mathrm{E}(Y_T 1_{\{T \leq n\}}) + \mathrm{E}(Y_n 1_{\{T>n\}}) \geq \epsilon\mathrm{P}(T \leq n) - \mathrm{E}(Y_n^- 1_{\{T>n\}})$$

giving the second assertion.

Corollary 8.46 *If (Y_n) is a submartingale, then for any $\epsilon > 0$,*

$$P(\max_{0\le i\le n} |Y_i| \ge \epsilon) \le \frac{2E(Y_n^+) - E(Y_0)}{\epsilon}.$$

Proof. Let $\epsilon > 0$. If (Y_n) is a submartingale, then $(-Y_n)$ is a supermartingale so that according to Theorem 8.45 (ii),

$$P(\min_{0\le i\le n} Y_i \le -\epsilon) = P(\max_{0\le i\le n}(-Y_i) \ge \epsilon) \le \frac{E(-Y_0) + E[(-Y_n)^-]}{\epsilon} = \frac{E(Y_n^+) - E(Y_0)}{\epsilon}.$$

Adding this to Theorem 8.45 (i) we arrive at the asserted inequality.

Exercise 8.47 Show that if (Y_n) is a martingale, then

$$P(\max_{0\le i\le n} |Y_i| \ge \epsilon) \le \frac{E|Y_n|}{\epsilon}.$$

Corollary 8.48 *Doob-Kolmogorov inequality. If (Y_n) is a martingale with finite second moments, then (Y_n^2) is a submartingale, and by Theorem 8.45 (i), for any $\epsilon > 0$,*

$$P(\max_{1\le i\le n} |Y_i| \ge \epsilon) = P(\max_{1\le i\le n} Y_i^2 \ge \epsilon^2) \le \frac{E(Y_n^2)}{\epsilon^2}.$$

Corollary 8.49 *Kolmogorov inequality. Let (X_n) are independent r.v. with zero means and finite variances (σ_n^2), then for any $\epsilon > 0$,*

$$P(\max_{1\le i\le n} |X_1 + \ldots + X_i| \ge \epsilon) \le \frac{\sigma_1^2 + \ldots + \sigma_n^2}{\epsilon^2}.$$

Theorem 8.50 *Convergence in L^r. Let $r > 1$. Suppose $(Y_n, \mathcal{F}_n)$ is a martingale such that $E(|Y_n|^r) \le M$ for some constant M and all n. Then there exists Y such that $Y_n \to Y$ a.s. as $n \to \infty$, and moreover, $Y_n \to Y$ in L^r.*

Proof. Combining Corollary 8.29 and Lyapunov inequality we get the a.s. convergence $Y_n \to Y$. We prove $Y_n \overset{L^r}{\to} Y$ using Theorem 3.26. For this we need to verify that $(|Y_n^r|)_{n\ge 0}$ is uniformly integrable. Observe first that

$$\bar{Y}_n := \max_{0\le i\le n} |Y_i|$$

have finite r-th moment

$$E(\bar{Y}_n^r) \le E(|Y_0|^r + \ldots + |Y_n|^r) < \infty.$$

By Exercise 3.20,

$$E(\bar{Y}_n^r) = \int_0^\infty r x^{r-1} P(\bar{Y}_n > x) dx.$$

Applying $(*)$ to $A(x) = \{\bar{Y}_n \ge x\}$, we obtain

$$E(\bar{Y}_n^r) \le \int_0^\infty r x^{r-2} E(|Y_n| 1_{A(x)}) dx = r E\Big(|Y_n| \int_{0\le x\le \bar{Y}_n} x^{r-2} dx\Big) = \frac{r}{r-1} E\Big(|Y_n| \bar{Y}_n^{r-1}\Big).$$

Now, applying the Hölder inequality we find

$$E(\bar{Y}_n^r) \le \frac{r}{r-1} E\Big(|Y_n| \bar{Y}_n^{r-1}\Big) \le \frac{r}{r-1} \Big(E(|Y_n|^r)\Big)^{1/r} \Big(E(\bar{Y}_n^r)\Big)^{(r-1)/r},$$

and we conclude

$$E(\max_{0\le i\le n} |Y_i|^r) = E(\bar{Y}_n^r) \le \Big(\frac{r}{r-1}\Big)^r E(|Y_n|^r) \le \Big(\frac{r}{r-1}\Big)^r M.$$

Thus, by monotone convergence, $E\big(\sup_n |Y_n|^r\big) < \infty$ implying that $(|Y_n^r|)_{n\ge 0}$ is uniformly integrable, see Exercise 3.23.

8.10 Backward martingales. Martingale proof of the strong LLN

Definition 8.51 Let $(\mathcal{G}_n)$ be a decreasing sequence of σ-algebras:

$$\mathcal{G}_0 \supset \mathcal{G}_1 \supset \mathcal{G}_2 \supset \ldots,$$

and (Y_n) be a sequence of adapted r.v. The sequence $(Y_n, \mathcal{G}_n)$ is called a backward (or reversed) martingale if, for all $n \geq 0$,

$$\begin{aligned}&\mathrm{E}(|Y_n|) < \infty,\\ &\mathrm{E}(Y_n|\mathcal{G}_{n+1}) = Y_{n+1}.\end{aligned}$$

Theorem 8.52 *Let $(Y_n, \mathcal{G}_n)$ be a backward martingale. Then $Y_n = \mathrm{E}(Y_0|\mathcal{G}_n)$ and there is a r.v. Y such that $Y_n \to Y$ almost surely and in mean.*

Proof. Using the tower property for conditional expectations we prove the first statement

$$Y_n = \mathrm{E}(Y_{n-1}|\mathcal{G}_n) = \mathrm{E}(\mathrm{E}(Y_{n-2}|\mathcal{G}_{n-1})|\mathcal{G}_n) = \mathrm{E}(Y_{n-2}|\mathcal{G}_n) = \ldots = \mathrm{E}(Y_0|\mathcal{G}_n).$$

To prove a.s. convergence, we apply Lemma 8.27 to the martingale $(Y_n, \mathcal{G}_n), \ldots, (Y_0, \mathcal{G}_0)$. The number $U_n(a,b)$ of $[a,b]$ uppcrossings by $(Y_n, \ldots, Y_0)$ satisfies

$$\mathrm{E}\, U_n(a,b) \leq \frac{\mathrm{E}(Y_0 - a)^+}{b-a},$$

so that letting $n \to \infty$ and repeating the proof of Theorem 8.28 we arrive at the stated a.s. convergence.

The convergence in mean follows from the observation that the sequence $Y_n = \mathrm{E}(Y_0|\mathcal{G}_n)$ is uniformly integrable, which is obtained by repeating the proof of Theorem 8.33 dealing with the Doob martingale.

Theorem 8.53 *Strong LLN. Let $X_1, X_2, \ldots$ be iid random variables defined on the same probability space. If $\mathrm{E}|X_1| < \infty$, then*

$$\frac{X_1 + \ldots + X_n}{n} \to \mathrm{E}X_1$$

almost surely and in L^1. On the other hand, if

$$\frac{X_1 + \ldots + X_n}{n} \overset{\text{a.s.}}{\to} \mu$$

for some constant μ, then $\mathrm{E}|X_1| < \infty$ and $\mu = \mathrm{E}X_1$.

Proof. Let $\mathrm{E}|X_1| < \infty$. Set $S_n = X_1 + \ldots + X_n$ and let $\mathcal{G}_n = \sigma(S_n, S_{n+1}, \ldots)$, then by symmetry,

$$\mathrm{E}(S_n|\mathcal{G}_{n+1}) = \mathrm{E}(S_n|S_{n+1}) = \sum_{i=1}^{n} \mathrm{E}(X_i|S_{n+1}) = n\mathrm{E}(X_1|S_{n+1}).$$

On the other hand,

$$S_{n+1} = \mathrm{E}(S_{n+1}|S_{n+1}) = (n+1)\mathrm{E}(X_1|S_{n+1}),$$

implying

$$n^{-1}\mathrm{E}(S_n|\mathcal{G}_{n+1}) = \mathrm{E}(X_1|S_{n+1}) = (n+1)^{-1}S_{n+1}.$$

We conclude that S_n/n is a backward martingale, and according to Theorem 8.52 there exists Y such that $S_n/n \to Y$ a.s. and in mean. By Kolmogorov zero-one law, $\mathrm{P}(Y > c)$ is either 0 or 1 for any given constant c. Therefore, $\mathrm{P}(Y = \mu) = 1$ for a finite constant μ. It remains to see that convergence in mean yields

$$\mathrm{E}(X_1) = \mathrm{E}(S_n/n) = \mu.$$

To prove the reverse part, assume that $S_n/n \overset{\text{a.s.}}{\to} \mu$ for some finite constant μ. Then $X_n/n \overset{\text{a.s.}}{\to} 0$ by the theory of convergent real series. Indeed, from $(a_1 + \ldots + a_n)/n \to \mu$ it follows that

$$\frac{a_n}{n} = \frac{a_1 + \ldots + a_{n-1}}{n(n-1)} + \frac{a_1 + \ldots + a_n}{n} - \frac{a_1 + \ldots + a_{n-1}}{n-1} \to 0$$

Now, in view of $X_n/n \overset{\text{a.s.}}{\to} 0$, the second Borell-Cantelli lemma gives

$$\sum_n \mathrm{P}(|X_n| \geq n) < \infty,$$

since otherwise $\mathrm{P}(n^{-1}|X_n| \geq 1 \text{ i.o.}) = 1$. Thus

$$\mathrm{E}|X_1| = \int_0^\infty \mathrm{P}(|X_1| > x)dx \leq \sum_n \mathrm{P}(|X_1| \geq n) = \sum_n \mathrm{P}(|X_n| \geq n) < \infty.$$

Apllying the first part of this theorem we see that $\mu = \mathrm{E}X_1$.

9 Diffusion processes

9.1 The Wiener process

Definition 9.1 The standard Wiener process $W(t) = W_t$ is a continuous time analogue of a simple symmetric random walk. It is characterized by three properties:

- $W_0 = 0$,
- W_t has independent increments with $W_t - W_s \sim \mathrm{N}(0, t-s)$ for $0 \leq s < t$,
- the path $t \to W_t$ is continuous with probability 1.

Next we sketch a construction of W_t for $t \in [0,1]$. First observe that the following theorem is not enough.

Theorem 9.2 *Kolmogorov extension theorem. Assume that for any vector $(t_1, \ldots, t_n)$ with $t_i \in [0,1]$ there given a joint distribution function $F_{(t_1,\ldots,t_n)}(x_1, \ldots, x_n)$. Suppose that these distribution functions satisfy two consistency conditions*

(i) $F_{(t_1,\ldots,t_n,t_{n+1})}(x_1, \ldots, x_n, \infty) = F_{(t_1,\ldots,t_n)}(x_1, \ldots, x_n)$,

(ii) if π is a permutation of $(1, \ldots, n)$, then $F_{(t_{\pi(1)},\ldots,t_{\pi(n)})}(x_{\pi(1)}, \ldots, x_{\pi(n)}) = F_{(t_1,\ldots,t_n)}(x_1, \ldots, x_n)$. Put $\Omega = \{functions\ \omega : [0,1] \to \mathbb{R}\}$ and $\mathcal{F}$ is the σ-algebra generated by the finite-dimensional sets $\{\omega : \omega(t_i) \in B_i, i = 1, \ldots, n\}$, where B_i are Borel subsets of $\mathbb{R}$. Then there is a unique probability measure P *on $(\Omega, \mathcal{F})$ such that a stochastic process defined by $X(t, \omega) = \omega(t)$ has the finite-dimensional distributions $F_{(t_1,\ldots,t_n)}(x_1, \ldots, x_n)$.*

The problem is that the set $\{\omega : t \to \omega(t) \text{ is continuous}\}$ does not belong to $\mathcal{F}$, since all events in $\mathcal{F}$ may depend on only countably many coordinates.

The above problem can be fixed if we focus of the subset $\mathcal{Q}$ be the set of dyadic rationals $\{t = m2^{-n}$ for some $0 \leq m \leq 2^n, n \geq 1\}$.

Step 1. Let $(X_{m,n})$ be a collection of Gaussian r.v. such that if we put $X(t) = X_{m,n}$ for $t = m2^{-n}$, then

- $X(0) = 0$,
- $X(t)$ has independent increments with $X(t) - X(s) \sim \mathrm{N}(0, t-s)$ for $0 \leq s < t$, $s, t \in \mathcal{Q}$.

According to Theorem 9.2 the process $X(t)$, $t \in \mathcal{Q}$ can be defined on $(\Omega_q, \mathcal{F}_q, \mathrm{P}_q)$ where index q means the restriction $t \in \mathcal{Q}$.

Step 2. For $m2^{-n} \leq t < (m+1)2^{-n}$ define

$$X_n(t) = X_{m,n} + 2^n(t - m2^{-n})(X_{m+1,n} - X_{m,n}).$$

For each n the process $X_n(t)$ has continuous paths for $t \in [0,1]$. Think of $X_{n+1}(t)$ as being obtained from $X_n(t)$ by repositioning the centers of the line segments by iid normal amounts. If $t \in \mathcal{Q}$, then $X_n(t) = X(t)$ for all large n. Thus $X_n(t) \to X(t)$ for all $t \in \mathcal{Q}$.

Step 3. Show that $X(t)$ is a.s. uniformly continuous over $t \in \mathcal{Q}$. It follows from $X_n(t) \to X(t)$ a.s. uniformly over $t \in \mathcal{Q}$. To prove the latter we use the Weierstrass M-test by observing that $X_n(t) = Z_1(t) + \ldots + Z_n(t)$, where $Z_i(t) = X_i(t) - X_{i-1}(t)$, and showing that

$$\sum_{i=1}^{\infty} \sup_{t \in \mathcal{Q}} |Z_i(t)| < \infty. \qquad (*)$$

Using independence and normality of the increments one can show for $x_i = c\sqrt{i2^{-i}\log 2}$ that

$$\mathrm{P}(\sup_{t \in \mathcal{Q}} |Z_i(t)| > x_i) \le 2^{i-1} \frac{2^{-ic^2}}{c\sqrt{i \log 2}}.$$

The first Borel-Cantelli lemma implies that for $c > 1$ the events $\sup_{t \in \mathcal{Q}} |Z_i(t)| > x_i$ occur finitely many times and $(*)$ follows.

Step 4. Define $W(t)$ for any $t \in [0,1]$ by moving our probability measure to $(C, \mathcal{C})$, where $C =$ continuous $\omega : [0,1) \to \mathbb{R}$ and $\mathcal{C}$ is the σ-algebra generated by the coordinate maps $t \to \omega(t)$. To do this, we observe that the map ψ that takes a uniformly continuous point in Ω_q to its unique continuous extension in C is measurable, and we set $\mathrm{P}(A) = \mathrm{P}_q(\psi^{-1}(A))$.

9.2 Properties of the Wiener process

We will prove some of the following properties of the standard Wiener process.

(i) The vector $(W(t_1), \ldots, W(t_n))$ has the multivariate normal distribution with zero means and covariances $\mathrm{Cov}(W(t_i), W(t_j)) = \min(t_i, t_j)$.

(ii) For any positive s the shifted process $\tilde{W}_t = W_{t+s} - W_s$ is a standard Wiener process. This implies the (weak) Markov property.

(iii) For any non-negative stopping time T the shifted process $W_{t+T} - W_T$ is a standard Wiener process. This implies the strong Markov property.

(iv) Let $T(x) = \inf\{t : W(t) = x\}$ be the first passage time. It is a stopping time for the Wiener process.

(v) The r.v. $M(t) = \max\{W(s) : 0 \le s \le t\}$ has the same distribution as $|W(t)|$ and has density function $f(x) = \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ for $x \ge 0$.

(vi) The r.v. $T(x) \overset{d}{=} (x/Z)^2$, where $Z \sim$ N(0,1), has density function $f(t) = \frac{|x|}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}}$ for $t \ge 0$.

(vii) If $\mathcal{F}_t$ is the filtration generated by $(W(u), u \le t)$, then $(e^{\theta W(t) - \theta^2 t/2}, \mathcal{F}_t)$ is a martingale.

(viii) Consider the Wiener process on $t \in [0, \infty)$ with a negative drift $W(t) - mt$ with $m > 0$. Its maximum is exponentially distributed with parameter $2m$.

Proof. (i) If $0 \le s < t$, then

$$\mathrm{E}(W(s)W(t)) = \mathrm{E}[W(s)^2 + W(s)(W(t) - W(s))] = \mathrm{E}[W(s)^2] = s.$$

(v) For $x > 0$ we have $\{T(x) \le t\} = \{M(t) \ge x\}$. This and

$$\mathrm{P}(M(t) \ge x) = \mathrm{P}(M(t) \ge x, W(t) - x \ge 0) + \mathrm{P}(M(t) \ge x, W(t) - x < 0)$$

imply

$$\begin{aligned}\mathrm{P}(M(t) \geq x, W(t) < x) &= \mathrm{P}(M(t) \geq x, W(t) - W(T(x)) < 0, T(x) \leq t)\\ &= \mathrm{P}(M(t) \geq x, W(t) - W(T(x)) \geq 0, T(x) \leq t) = \mathrm{P}(M(t) \geq x, W(t) \geq x).\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{P}(M(t) \geq x) &= 2\mathrm{P}(M(t) \geq x, W(t) \geq x) = 2\mathrm{P}(W(t) \geq x)\\ &= \mathrm{P}(W(t) \geq x) + \mathrm{P}(W(t) \leq -x) = \mathrm{P}(|W(t)| \geq x).\end{aligned}$$

(vi) We have

$$\mathrm{P}(T(x) \leq t) = \mathrm{P}(M(t) \geq x) = \mathrm{P}(|W(t)| \geq x) = \frac{2}{\sqrt{2\pi t}} \int_x^\infty e^{-\frac{y^2}{2t}} dy = \int_0^t \frac{|x|}{\sqrt{2\pi u^3}} e^{-\frac{x^2}{2u}} du.$$

(vii) Bringing $e^{\theta W(s)}$ outside

$$\mathrm{E}(e^{\theta W(t)}|\mathcal{F}_s) = e^{\theta W(s)}\mathrm{E}(e^{\theta(W(t)-W(s))}|\mathcal{F}_s) = e^{\theta W(s)}e^{\theta^2(t-s)/2}.$$

(viii) It suffices to prove that for any $x > 0$

$$\mathrm{P}(W(t) - mt = x \text{ for some } t) = e^{-2mx}.$$

Let $T(a, b)$ be the first exit time from the interval (a, b). Applying a continuous version of the optional stopping theorem to the martingale $U(t) = e^{2mW(t)-2m^2t}$ we obtain $\mathrm{E}(U(T(a, x))) = \mathrm{E}(U(0)) = 1$. Thus

$$1 = e^{2mx}\mathrm{P}(U(T(a, x)) = x) + e^{2ma}\mathrm{P}(U(T(a, x)) = a).$$

Letting $a \to -\infty$ we obtain the desired relation.

9.3 Examples of diffusion processes

Definition 9.3 An Ito diffusion process $X(t) = X_t$ is a Markov process with continuous sample paths characterized by the standard Wiener process W_t in terms of a stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \qquad (*)$$

Here $\mu(t, x)$ and $\sigma^2(t, x)$ are the instantaneous mean and variance for the increments of the diffusion process.

The second term of the stochastic differential equation is defined in terms of the Ito integral leading to the integrated form of the equation

$$X_t - X_0 = \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

The Ito integral $J_t = \int_0^t Y_s dW_s$ is defined for a certain class of adapted processes Y_t. The process J_t is a martingale (cf Theorem 8.19).

Example 9.4 The Wiener process with a drift $W_t + mt$ corresponds to $\mu(t, x) = m$ and $\sigma(t, x) = \sigma^2$.

Example 9.5 The Ornstein-Uhlenbeck process: $\mu(t, x) = -\alpha(x - \theta)$ and $\sigma(t, x) = \sigma^2$. Given the initial value X_0 the process is described by the stochastic differential equation

$$dX_t = -\alpha(X_t - \theta)dt + \sigma dW_t,$$

which is a continuous version of an AR(1) process $X_n = aX_{n-1} + Z_n$. This process can be interpreted as the evolution of a phenotypic trait value (like logarithm of the body size) along a lineage of species in terms of the adaptation rate $\alpha > 0$, the optimal trait value θ, and the noise size $\sigma > 0$.

Let $f(t,y|s,x)$ be the density of the distribution of X_t given the position at an earlier time $X_s = x$. Then

$$\text{forward equation} \qquad \frac{\partial f}{\partial t} = -\frac{\partial}{\partial y}[\mu(t,y)f] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(t,y)f],$$

$$\text{backward equation} \qquad \frac{\partial f}{\partial s} = -\mu(s,x)\frac{\partial f}{\partial x} - \frac{1}{2}\sigma^2(s,x)\frac{\partial^2 f}{\partial x^2}.$$

Example 9.6 The Wiener process W_t corresponds to $\mu(t,x) = 0$ and $\sigma(t,x) = 1$. The forward and backward equations for the density $f(t,y|s,x) = \frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(y-x)^2}{2(t-s)}}$ are

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}, \quad \frac{\partial f}{\partial s} = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2}.$$

Example 9.7 For $dX_t = \sigma(t)dW_t$ the equations

$$\frac{\partial f}{\partial t} = \frac{\sigma^2(t)}{2}\frac{\partial^2}{\partial y^2}, \quad \frac{\partial f}{\partial s} = -\frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial x^2}$$

imply that X_t has a normal distribution with zero mean and variance $\int_0^t \sigma^2(u)du$.

9.4 The Ito formula

Main rule: $(dW_t)^2$ should be replaced by dt.

Theorem 9.8 *Let $f(t,x)$ be twice continuously differentiable on $[0,\infty)\times\mathbb{R}$ and X_t is given by $(*)$. Then $Y_t = f(t,B_t)$ is also an Ito process given by*

$$dY_t = \{f_t(t,X_t) + f_x(t,X_t)\mu(t,X_t) + \frac{1}{2}f_{xx}(t,X_t)\sigma^2(t,X_t)\}dt + f_x(t,X_t)\sigma(t,X_t)dW_t,$$

where

$$f_x(t,X_t) = \frac{\partial}{\partial x}f(t,x)|_{x=X_t}, \quad f_t(t,X_t) = \frac{\partial}{\partial t}f(t,x)|_{x=X_t}, \quad f_{xx}(t,X_t) = \frac{\partial^2}{\partial x^2}f(t,x)|_{x=X_t}.$$

Example 9.9 The distribution of the Ornstein-Uhlenbeck process X_t is normal with

$$\mathrm{E}(X_t) = \theta + e^{-\alpha t}(X_0 - \theta), \quad \mathrm{Var}(X_t) = \sigma^2(1 - e^{-2\alpha t})/2\alpha, \qquad (**)$$

implying that X_t looses the effect of the ancestral state X_0 at an exponential rate. In the long run X_0 is forgotten, and the OU–process acquires a stationary normal distribution with mean θ and variance $\sigma^2/2\alpha$.

To verify these formula for the mean and variance we apply the following simple version of Ito lemma: if $dX_t = \mu_t dt + \sigma_t dW_t$, then for any nice function $f(t,x)$

$$df(t,X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu_t dt + \sigma_t dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2 dt.$$

Let $f(t,x) = xe^{\alpha t}$. Then using the equation from Example 9.5 we obtain

$$d(X_t e^{\alpha t}) = \alpha X_t e^{\alpha t}dt + e^{\alpha t}\Big(\alpha(\theta - X_t)dt + \sigma dW_t\Big) = \theta e^{\alpha t}\alpha dt + \sigma e^{\alpha t}dW_t.$$

Integration gives

$$X_t e^{\alpha t} - X_0 = \theta(e^{\alpha t} - 1) + \sigma\int_0^t e^{\alpha u}dW_u,$$

implying $(**)$, since in view of Example 9.7 and the formula in Example 6.18 , we have

$$\mathrm{E}\Big(\int_0^t e^{\alpha u}dW_u\Big)^2 = \int_0^t e^{2\alpha u}du = \frac{e^{2\alpha t} - 1}{2\alpha}.$$

Observe that the correlation coefficient between $X(s)$ and $X(s+t)$ equals

$$\rho(s,s+t) = e^{-\alpha t}\sqrt{\frac{1 - e^{-2\alpha s}}{1 - e^{-2\alpha(s+t)}}} \to e^{-\alpha t}, \qquad s \to \infty.$$

Example 9.10 Geometric Brownian motion $Y_t = e^{\mu t+\sigma W_t}$. Due to the Ito formula

$$dY_t = (\mu + \frac{1}{2}\sigma^2)Y_t dt + \sigma Y_t dW_t,$$

so that $\mu(t,x) = (\mu + \frac{1}{2}\sigma^2)x$ and $\sigma^2(t,x) = \sigma^2 x^2$. The process Y_t is a martingale iff $\mu = -\frac{1}{2}\sigma^2$.

Example 9.11 Take $f(t,x) = x^2$. Then the Ito formula gives

$$dX_t^2 = 2X_t dX_t + \sigma(t,X_t)^2 dt.$$

In particular, $dW_t^2 = 2W_t dW_t + dt$ so that $\int_0^t W_t dW_t = (W_t^2 - t)/2$.

Example 9.12 Product rule. If

$$dX_t = \mu_1(t,X_t)dt + \sigma_1(t,X_t)dW_t, \quad dY_t = \mu_2(t,Y_t)dt + \sigma_2(t,Y_t)dW_t,$$

then

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \sigma_1(t,X_t)\sigma_2(t,Y_t)dt.$$

This follows from $2XY = (X+Y)^2 - X^2 - Y^2$ and

$$dX_t^2 = 2X_t dX_t + \sigma_1(t,X_t)^2 dt, \quad dY_t^2 = 2Y_t dY_t + \sigma_2(t,X_t)^2 dt,$$
$$d(X_t + Y_t)^2 = 2(X_t + Y_t)(dX_t + dY_t) + (\sigma_1(t,X_t) + \sigma_2(t,X_t))^2 dt.$$

9.5 The Black-Scholes formula

Lemma 9.13 *Let $(W_t, 0 \le t \le T)$ be the standard Wiener process on $(\Omega, \mathcal{F}, \mathrm{P})$ and let $\nu \in \mathbb{R}$. Define another measure by $\mathbb{Q}(A) = \mathrm{E}(e^{\nu W_T - \nu^2 T/2} 1_{\{A\}})$. Then $\mathbb{Q}$ is a probability measure and $\tilde{W}_t = W_t - \nu t$, regarded as a process on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, is the standard Wiener process.*

Proof. By definition, $\mathbb{Q}(\Omega) = e^{-\nu^2 T/2}\mathrm{E}(e^{\nu W_T}) = 1$. For the finite-dimensional distributions let $0 = t_0 < t_1 < \ldots < t_n = T$ and $x_0, x_1, \ldots, x_n \in \mathbb{R}$. Writing $\{W(t_i) \in dx_i\}$ for the event $\{x_i < W(t_i) \le x_i + dx_i\}$ we have that

$$\begin{aligned}\mathbb{Q}(W(t_1) \in dx_1, \ldots, W(t_n) \in dx_n) &= \mathrm{E}\big(e^{\nu W_T - \nu^2 T/2} 1_{\{W(t_1)\in dx_1,\ldots,W(t_n)\in dx_n\}}\big) \\ &= e^{\nu x_n - \nu^2 T/2} \prod_{i=1}^n \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\Big(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\Big)dx_i \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\Big(-\frac{(x_i - x_{i-1} - \nu(t_i - t_{i-1}))^2}{2(t_i - t_{i-1})}\Big)dx_i.\end{aligned}$$

Black-Scholes model. Writing B_t for the cost of one unit of a risk-free bond (so that $B_0 = 1$) at time t we have that

$$dB_t = rB_t dt \text{ or } B_t = e^{rt}.$$

The price (per unit) S_t of a stock at time t satisfies the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t) \text{ with solution } S_t = \exp\{(\mu - \sigma^2/2)t + \sigma W_t\}.$$

This is a geometric Brownian motion, and parameter σ is called volatility of the price process.

European call option. The buyer of the option may purchase one unit of stock at the exercise date T (fixed time) and the strike price K:

- if $S_T > K$, an immediate profit will be $S_T - K$,
- if $S_T \le K$, the call option will not be exercised.

The value of the option at time $t < T$ is

$$V_t = e^{-r(T-t)}(S_T - K)^+,$$

where S_T is not known.

Theorem 9.14 *Black-Scholes formula. Let $t < T$. The value of the European call option at time t is*

$$V_t = S_t\Phi(d_1(t,S_t)) - Ke^{-r(T-t)}\Phi(d_2(t,S_t)),$$

where $\Phi(x)$ is the standard normal distribution function and

$$d_1(t,x) = \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2(t,x) = \frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Definition 9.15 Let $\mathcal{F}_t$ be the σ-algebra generated by $(S_u, 0 \le u \le t)$. A portfolio is a pair (α_t, β_t) of $\mathcal{F}_t$-adapted processes. The value of the portfolio is $V_t(\alpha,\beta) = \alpha_t S_t + \beta_t B_t$. The portfolio is called self-financing if

$$dV_t(\alpha,\beta) = \alpha_t dS_t + \beta_t dB_t.$$

We say that a self-financing portfolio (α_t,β_t) replicates the given European call if $V_T(\alpha,\beta) = (S_T - K)^+$ almost surely.

Proof. We are going to apply Lemma 9.13 with $\nu = \frac{r-\mu}{\sigma}$. Note also that under $\mathbb{Q}$ the process

$$e^{-rt}S_t = \exp\{(\nu\sigma - \sigma^2/2)t + \sigma W_t\} = e^{\sigma\tilde{W}_t - \sigma^2 t/2}$$

is a martingale. Take without a proof that there exists a self-financing portfolio (α_t,β_t) replicating the European call option in question. If the market contains no arbitrage opportunities, we have $V_t = V_t(\alpha,\beta)$ and therefore

$$\begin{aligned} d(e^{-rt}V_t) &= e^{-rt}dV_t - re^{-rt}V_t dt = e^{-rt}\alpha_t(dS_t - rS_t dt) + e^{-rt}\beta_t(dB_t - rB_t dt) \\ &= \alpha_t e^{-rt}S_t((\mu - r)dt + \sigma dW_t) = \alpha_t e^{-rt}S_t\sigma d\tilde{W}_t. \end{aligned}$$

This defines a martingale under $\mathbb{Q}$:

$$e^{-rt}V_t = V_0 + \int_0^t \alpha_u e^{-ru}S_u\sigma d\tilde{W}_u.$$

Thus

$$V_t = e^{rt}\mathrm{E}_{\mathbb{Q}}(e^{-rT}V_T|\mathcal{F}_t) = e^{-r(T-t)}\mathrm{E}_{\mathbb{Q}}((S_T-K)^+|\mathcal{F}_t) = e^{-r(T-t)}\mathrm{E}_{\mathbb{Q}}((ae^Z - K)\vee 0|\mathcal{F}_t),$$

where $a = S_t$ with

$$Z = \exp\{(\mu - \sigma^2/2)(T-t) + \sigma(W_T - W_t)\} = \exp\{(r - \sigma^2/2)(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)\}.$$

Since $(Z|\mathcal{F}_t)_{\mathbb{Q}} \sim \mathrm{N}(\gamma,\tau^2)$ with $\gamma = (r - \sigma^2/2)(T-t)$ and $\tau^2 = (T-t)\sigma^2$. It remains to observe that for any constant a and $Z \sim \mathrm{N}(\gamma,\tau^2)$

$$\mathrm{E}(ae^Z - K)^+ = ae^{\gamma+\tau^2/2}\Phi\Big(\frac{\log(a/K)+\gamma}{\tau} + \tau\Big) - K\Phi\Big(\frac{\log(a/K)+\gamma}{\tau}\Big).$$