

TMA225 Differential Equations and Scientific
Computing, part A

Solutions to Problems Week 5

October 2, 2002

Week 5:

Problem 1. Consider the triangulation of the unit square $\Omega = [0, 1] \times [0, 1]$ into 8 triangles drawn in Figure 1.

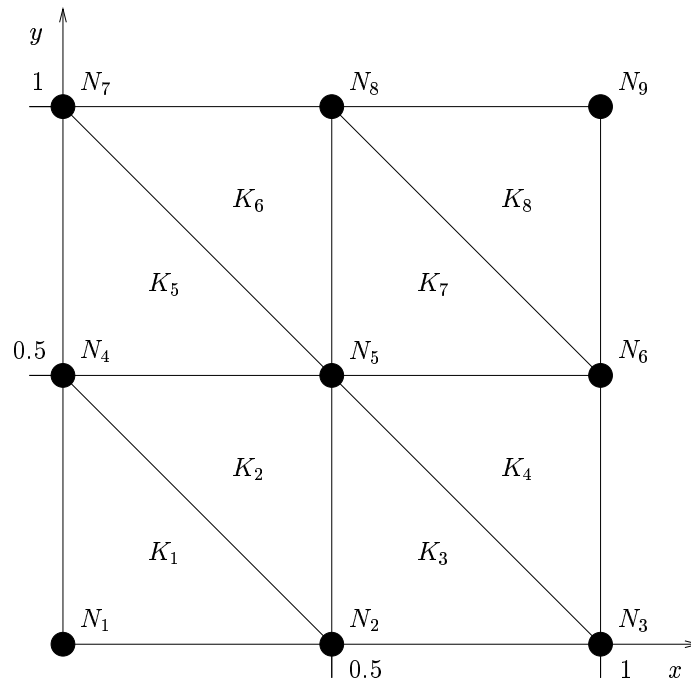


Figure 1: Problem 1 (Week 5). The triangulation of Ω .

- Compute the length of the largest side h_{K_j} , and the smallest angle τ_{K_j} of the triangles.
- Determine the *point matrix* \mathbf{p} that describes this triangulation in Matlab. *Hint:* Since node 1 is located at the origin, the first column in \mathbf{p} is $[0; 0]$.
- Determine the *triangle matrix* \mathbf{t} that describes this triangulation in Matlab. *Hint:* Since triangle 1 has corners in node number 1, 2 and 4, the first column in \mathbf{t} can e.g. be $[1; 2; 4]$. It is not important which node comes first, but they must be listed in a *counter-clockwise* order.
- Verify your results by creating \mathbf{p} and \mathbf{t} in Matlab:

```
>> p(:, 1) = [0; 0]
>> p(:, 2) = ...
...
>> p(:, 9) = ...
>> t(:, 1) = [1; 2; 4]
>> t(:, 2) = ...
...
>> t(:, 8) = ...
```

and plot the triangulation by the Matlab-command:

```
>> pdemesh(p, [], t)
```

Solution:

(a) Pythagoras' theorem and simple trigonometry gives that $h_{K_j} = 1/\sqrt{2}$ and $\tau_{K_j} = \pi/4$ for all triangles K_j .

(b)

$$p = \begin{bmatrix} 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \end{bmatrix}$$

(c) For example:

$$t = \begin{bmatrix} 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 \\ 2 & 5 & 3 & 6 & 5 & 8 & 6 & 9 \\ 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 \end{bmatrix}$$

(d) -

□

Problem 2. Consider the same triangulation as in *Problem 1 (Week 5)*.

(a) The continuous piecewise linear function $\varphi_2(x, y)$ is defined by:

$$\varphi_2(N_2) = 1; \quad \varphi_2(N_j) = 0 \text{ for } j \neq 2.$$

Compute the analytical expression for φ_2 . *Hint:* The analytical expressions on K_1 , K_2 and K_3 may be determined by solving linear systems of equations as you have seen in the lecture. On the other triangles, $\varphi_2 \equiv 0$. Why?

(b) Plot φ_2 in Matlab by giving the command:

```
>> pdesurf(p, t, [0; 1; 0; 0; 0; 0; 0; 0; 0])
```

or

```
>> pdemesh(p, [], t, [0; 1; 0; 0; 0; 0; 0; 0; 0])
```

Try both! The argument `[0; 1; 0; 0; 0; 0; 0; 0; 0]` is a *column vector* containing the *nodal values* of φ_2 . Try also to plot some other “tent functions” φ_j !

(c) Since an arbitrary continuous piecewise linear function v can be written as a linear combination of “tent functions”:

$$v(x, y) = v(N_1) \varphi_1(x, y) + \dots + v(N_9) \varphi_9(x, y)$$

the “tent functions” $\{\varphi_i\}_{i=1}^9$ form a *basis* for the vector space V_h of continuous piecewise linear functions on the triangulation in Figure 1. What is the *dimension* of V_h ?

(d) Try plotting some different functions in V_h using the Matlab commands `pdesurf` and `pdemesh`. *Hint:* Cf. how you plotted φ_2 .

Solution:

(a) The analytical expression for φ_2 is different on each triangle. Since φ_2 is equal to one at node N_2 and zero at all other nodes, it is only non-zero on triangles K_1 , K_2 and K_3 , and therefore $\varphi_2(x, y) = 0$ on $K_4 \cup K_5 \cup K_6 \cup K_7 \cup K_8$.

On each triangle φ_2 is a linear function $\varphi_2(x, y) = c_0 + c_1x + c_2y$, where c_0 , c_1 , and c_2 are constants to be determined for the three triangles K_1 , K_2 and K_3 . These constants can be computed by solving the linear system of equations (see *Applied Mathematics: B&S*, Part D, page 1032):

$$\begin{pmatrix} 1 & a_1^1 & a_2^1 \\ 1 & a_1^2 & a_2^2 \\ 1 & a_1^3 & a_2^3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \varphi_2(a_1^1, a_2^1) \\ \varphi_2(a_1^2, a_2^2) \\ \varphi_2(a_1^3, a_2^3) \end{pmatrix}$$

where (a_1^1, a_2^1) , (a_1^2, a_2^2) and (a_1^3, a_2^3) are the node coordinates of the triangle. On triangle K_1 , with nodes N_1 , N_2 and N_4 (in that order), we get:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0 \\ 1 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which has the solution $c_0 = 0$, $c_1 = 2$ and $c_2 = 0$. That is: on K_1 , $\varphi_2(x, y) = c_0 + c_1x + c_2y = 2x$.

Similarly, on triangle K_2 , with nodes N_2 , N_5 and N_4 (in that order), we get:

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 1 & 0.5 & 0.5 \\ 1 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which has the solution $c_0 = 1$, $c_1 = 0$ and $c_2 = -2$. That is: on K_2 , $\varphi_2(x, y) = 1 - 2y$.

Finally, on triangle K_3 , with nodes N_2 , N_3 and N_5 (in that order), we get:

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 1 & 1 & 0 \\ 1 & 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which has the solution $c_0 = 2$, $c_1 = -2$ and $c_2 = -2$. That is: on K_3 , $\varphi_2(x, y) = c_0 + c_1x + c_2y = 2 - 2(x + y)$.

(b) -

(c) The dimension of V_h is 9, since there are 9 nodes and therefore 9 basis functions.

(d) To plot $v(x, y) = 2\varphi_1(x, y) + 3\varphi_8(x, y)$:

```
>> pdesurf(p, t, [2; 0; 0; 0; 0; 0; 0; 0; 3; 0])
```

or

```
>> pdemesh(p, [], t, [2; 0; 0; 0; 0; 0; 0; 0; 3; 0])
```

Comment: `[]` is an empty matrix. We don't need to use this argument but we still have to pass something to the function `pdemesh`, which expects an argument (actually the "edge matrix" `e`) between `p` and `t`. See `>> help pdemesh` □