94 Taylor's formula

Suppose that f has Lipschitz continuous derivative of order $n+1$ on the interval $[a,b]$ and let $\bar{x} \in (a,b)$. Then f satisfies *Taylor's formula of order n at $\bar{x}$*:

$$\begin{aligned} f(x) &= f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + f''(\bar{x})\frac{(x-\bar{x})^2}{2} + f'''(\bar{x})\frac{(x-\bar{x})^3}{3!} \\ &\quad + \cdots + f^{(n)}(\bar{x})\frac{(x-\bar{x})^n}{n!} + R_n(x,\bar{x}) \\ &= \sum_{k=0}^{n} f^{(k)}(\bar{x})\frac{(x-\bar{x})^k}{k!} + R_n(x,\bar{x}), \quad \text{for all } x \in [a,b], \end{aligned}$$

where the *remainder* R_n is given by

$$\begin{aligned} R_n(x,\bar{x}) &= \int_{\bar{x}}^{x} \frac{(x-y)^n}{n!} f^{(n+1)}(y)\,dy \\ &= f^{(n+1)}(\hat{x})\frac{(x-\bar{x})^{n+1}}{(n+1)!}, \end{aligned}$$

and $\hat{x}$ is an unknown number between x and $\bar{x}$. See AMBS Ch 28.15 and Problem 28.11.

The polynomial

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(\bar{x})\frac{(x-\bar{x})^k}{k!}$$

is called *the Taylor polynomial of f of degree n at $\bar{x}$*. Remember that *n factorial* ("n fakultet") means

$$n! = 1 \cdot 2 \cdot 3 \cdots n, \quad 0! = 1.$$

The main importance of the formula is that the remainder is smaller than the terms in the polynomial, when x is close to $\bar{x}$. For example, if we know that

$$|f^{(n+1)}(x)| \le M, \quad \text{for all } x \in [a,b],$$

then

$$|R_n(x,\bar{x})| = |f^{(n+1)}(\hat{x})|\,\frac{|x-\bar{x}|^{n+1}}{(n+1)!} \le M\frac{|x-\bar{x}|^{n+1}}{(n+1)!}.$$

Problems

94.1. Write down Taylor's formula of order n at $\bar{x}=0$ for the following functions:

(a) $\log(1+x)$

(b) $\exp(x)$

(c) $\sin(x)$

(d) $\cos(x)$

94.2. Use Taylor's formula of order 2 (or 3 or 4) to compute approximations of the following. Estimate the error.

(a) $\log(1.1)$

(b) $\exp(-0.1)$

(c) $\sin(0.1)$

(d) $\cos(0.1)$

Answers and solutions

94.1.

(a)

$$\begin{aligned}
&f(x) = \log(1+x) && f(0) = 0\\
&f'(x) = \frac{1}{1+x} && f'(0) = 1\\
&f''(x) = \frac{-1}{(1+x)^2} && f''(0) = -1\\
&f'''(x) = \frac{(-1)(-2)}{(1+x)^3} = \frac{(-1)^2 2!}{(1+x)^3} && f'''(0) = 2 = (-1)^2 2!\\
&f''''(x) = \frac{(-1)(-2)(-3)}{(1+x)^4} = \frac{(-1)^3 3!}{(1+x)^4} && f''''(0) = -6 = (-1)^3 3!\\
&\vdots && \vdots\\
&f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k} && f^{(k)}(0) = (-1)^{k-1}(k-1)!
\end{aligned}$$

$$\begin{aligned}
\log(1+x) &= 0 + x + (-1)\frac{x^2}{2!} + (-1)^2 2!\frac{x^3}{3!} + (-1)^3 3!\frac{x^4}{4!} + \cdots + (-1)^{n-1}(n-1)!\frac{x^n}{n!} + R_n(x,0)\\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + R_n(x,0)\\
&= \sum_{k=0}^{n}(-1)^{k-1}\frac{x^k}{k} + R_n(x,0)\\
R_n(x,0) &= \frac{(-1)^n n!}{(1+\hat{x})^{n+1}}\frac{x^{n+1}}{(n+1)!} = \frac{(-1)^n}{(1+\hat{x})^{n+1}}\frac{x^{n+1}}{n+1}, \quad \text{where } \hat{x} \text{ is between } x \text{ and } 0.
\end{aligned}$$

(b)

$$\begin{aligned}
\exp(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + R_n(x,0)\\
R_n(x,0) &= e^{\hat{x}}\frac{x^{n+1}}{(n+1)!}, \quad \text{where } \hat{x} \text{ is between } x \text{ and } 0
\end{aligned}$$

(c)

$$\begin{aligned}
&f(x) = \sin(x) && (k=0,\ m=1) && f(0) = 0\\
&f'(x) = \cos(x) && (k=1,\ m=1) && f'(0) = 1\\
&f''(x) = -\sin(x) && (k=2,\ m=2) && f''(0) = 0\\
&f'''(x) = -\cos(x) && (k=3,\ m=2) && f'''(0) = -1\\
&f''''(x) = \sin(x) && (k=4,\ m=3) && f''''(0) = 0\\
&f^{(5)}(x) = \cos(x) && (k=5,\ m=3) && f^{(5)}(0) = 1\\
&\vdots && \vdots && \vdots\\
&f^{(2m-2)}(x) = (-1)^{m-1}\sin(x) && (k=2m-2 \text{ even}) && f^{(2m-2)}(0) = 0\\
&f^{(2m-1)}(x) = (-1)^{m-1}\cos(x) && (k=2m-1 \text{ odd}) && f^{(2m-1)}(0) = (-1)^{m-1}
\end{aligned}$$

$$
\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + R_{2n}(x,0)\\
R_{2n}(x,0) &= (-1)^n \cos(\hat{x})\frac{x^{2n+1}}{(2n+1)!}, \quad \text{where } \hat{x} \text{ is between } x \text{ and } 0
\end{aligned}
$$

(d)

$$
\begin{aligned}
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + R_{2n+1}(x,0)\\
R_{2n+1}(x,0) &= (-1)^{n+1} \cos(\hat{x})\frac{x^{2n+2}}{(2n+2)!}, \quad \text{where } \hat{x} \text{ is between } x \text{ and } 0
\end{aligned}
$$

94.2.

(a) Taylor of order 2:

$$
\begin{aligned}
\log(1+x) &= x - \frac{x^2}{2} + R_2(x,0)\\
R_2(x,0) &= \frac{1}{(1+\hat{x})^3}\frac{x^3}{3}\\
\log(1.1) = \log(1+0.1) &\approx 0.1 - \frac{(0.1)^2}{2} = 0.1 - 0.005 = 0.095\\
|R_2(0.1,0)| &= \left|\frac{1}{(1+\hat{x})^3}\frac{(0.1)^3}{3}\right| = \frac{1}{(1+\hat{x})^3}\frac{(0.1)^3}{3} \le \frac{1}{3}\cdot 10^{-3}
\end{aligned}
$$

because $\hat{x} \in [0, 0.1]$ implies $1+\hat{x} \ge 1$, so that $\frac{1}{(1+\hat{x})^3} \le 1$. Thus, $\log(1.1) \approx 0.095$ with 3 correct decimals.

(b) Taylor of order 2:

$$
\begin{aligned}
\exp(x) &= 1 + x + \frac{x^2}{2} + R_2(x,0)\\
R_2(x,0) &= e^{\hat{x}}\frac{x^3}{3!}\\
\exp(-0.1) &\approx 1 + (-0.1) + \frac{(-0.1)^2}{2} = 0.905\\
|R_2(-0.1,0)| &= \left|e^{\hat{x}}\frac{(-0.1)^3}{3!}\right| = \frac{1}{6}e^{\hat{x}}10^{-3} \le \frac{1}{6}\cdot 10^{-3}
\end{aligned}
$$

because $\hat{x} \in [-0.1, 0]$ implies $e^{\hat{x}} \le 1$. Thus, $\exp(-0.1) \approx 0.905$ with 3 correct decimals.

(c) Taylor of order 4:

$$
\begin{aligned}
\sin(x) &= x - \frac{x^3}{6} + R_4(x,0)\\
R_4(x,0) &= (-1)^2\cos(\hat{x})\frac{x^5}{5!}\\
\sin(0.1) &\approx 0.1 - \frac{(0.1)^3}{6} \approx 0.099833333\\
|R_4(0.1,0)| &= \left|(-1)^2\cos(\hat{x})\frac{(0.1)^5}{5!}\right| = |\cos(\hat{x})|\frac{1}{120}10^{-5} \le \frac{1}{120}\cdot 10^{-5} < 10^{-7}
\end{aligned}
$$

because $|\cos(\hat{x})| \le 1$. Thus, $\sin(0.1) \approx 0.099833$ with 6 correct decimals.

(d) ...