Chapter 63 The curve integral

The length of a curve

Let

$$g : \mathbb{R} \to \mathbb{R}^3 \quad (\text{or } g : \mathbb{R} \to \mathbb{R}^n,\ n \geq 2)$$

be used in the parametrization of a curve Γ:

$$x = g(t), \quad t \in I = [a, b].$$

What is the length of Γ?

Figure!!

We introduce a mesh in the parameter interval: $a = t_0 < t_1 < \cdots < t_{i-1} < t_i < \cdots < t_N = b$ with sub-intervals $I_i = (t_{i-1}, t_i)$ and steps $h_i = t_i - t_{i-1}$.

The linearization of g at t_{i-1} is: $\tilde{g}(t) = g(t_{i-1}) + g'(t_{i-1})(t - t_{i-1})$. At $t = t_i$ we get $\tilde{g}(t_i) = g(t_{i-1}) + g'(t_{i-1})h_i$.

The curve segment from $g(t_{i-1})$ to $g(t_i)$ is approximated by the straight line segment Γ_i which goes from $\tilde{g}(t_{i-1}) = g(t_{i-1})$ to $\tilde{g}(t_i) \approx g(t_i)$. The length of Γ_i is

$$|\Gamma_i| = \|\tilde{g}(t_i) - g(t_{i-1})\| = \|g'(t_{i-1})h_i\| = \|g'(t_{i-1})\|h_i.$$

The length of Γ is approximated by

$$\sum_{i=1}^{N} |\Gamma_i| = \sum_{i=1}^{N} \|g'(t_{i-1})\|h_i.$$

When $h_i \to 0$ this approaches to the integral

$$\int_a^b \|g'(t)\|\,dt$$

because the sum is the rectangle rule for this integral. We define this to be the length of Γ:

$$|\Gamma| = L(\Gamma) = \int_a^b \|g'(t)\|\,dt.$$

We also write it as

$$\int_\Gamma ds = \int_a^b \|g'(t)\|\,dt.$$

The symbol ds is called the *element of arclength* (båglängdselementet)

$$ds = \|g'(t)\|\,dt.$$

Example: The length of the half circle

$$\begin{cases} x_1 = \cos(t), \\ x_2 = \sin(t), \end{cases} \quad t \in [0, \pi].$$

The tangent: $g'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$.

The element of arclength: $ds = \|g'(t)\|\,dt = \sqrt{(-\sin(t))^2 + (\cos(t))^2}\,dt = dt$.

The length of Γ: $\int_\Gamma ds = \int_0^\pi dt = \pi$.

You may think of t as the time variable and $x = g(t)$ as the motion of a particle that moves with constant speed $\|g'(t)\| = 1$ along the curve.

Another parametrization:

$$\begin{cases} x_1 = \tau, \\ x_2 = \sqrt{1-\tau^2}, \end{cases} \qquad \tau \in [-1, 1].$$

The tangent: $g'(\tau) = \begin{bmatrix} 1 \\ \frac{-\tau}{\sqrt{1-\tau^2}} \end{bmatrix}$.

The element of arclength: $ds = \|g'(\tau)\|\,d\tau = \sqrt{1^2 + \Big(\frac{-\tau}{\sqrt{1-\tau^2}}\Big)^2}\,d\tau = \frac{1}{\sqrt{1-\tau^2}}\,d\tau$.

The length of Γ: $\displaystyle\int_\Gamma ds = \int_{-1}^{1} \frac{1}{\sqrt{1-\tau^2}}\,d\tau = \Big[\arcsin(\tau)\Big]_{-1}^{1} = \pi$.

Note that the curve goes in the opposite direction with this parametrization. And if τ is the time, then the speed of the particle is not constant $\|g'(\tau)\| = \frac{1}{\sqrt{1-\tau^2}}$. If we want the same direction then we can use

$$\begin{cases} x_1 = -\tau, \\ x_2 = \sqrt{1-\tau^2}, \end{cases} \qquad \tau \in [-1, 1].$$

Compute the length of Γ with this parametrization!

The curve integral

Now let

$$u : \mathbb{R}^3 \to \mathbb{R} \quad (\text{or } u : \mathbb{R}^n \to \mathbb{R},\ n \geq 2)$$

be a scalar-valued function (scalar field). We define the integral of u along Γ:

$$\int_\Gamma u\,ds = \int_a^b u(g(t))\|g'(t)\|\,dt.$$

Reparametrization

If we have two parametrizations

$$\begin{aligned} x &= g(t), \quad t \in I = [a, b], \\ x &= h(\tau), \quad \tau \in J = [c, d], \end{aligned}$$

of the same curve Γ, then we also have two definitions of the curve integral:

$$\begin{aligned} \int_\Gamma u\,ds &= \int_a^b u(g(t))\|g'(t)\|\,dt \\ \int_\Gamma u\,ds &= \int_c^d u(h(\tau))\|h'(\tau)\|\,d\tau. \end{aligned}$$

These must be the same, otherwise we have a bad definition. We check this now.

Since g and h parametrize the same curve there is for each $\tau \in J$ a unique $t \in I$ such that $g(t) = h(\tau)$, and vice versa. This means that there is an invertible function $t = f(\tau)$, $\tau = f^{-1}(t)$ such that $g(t) = h(f^{-1}(t)) = h(\tau)$. We use this as a change of variables in the first integral:

$$\int_a^b u(g(t))\|g'(t)\|\,dt = \left\{\begin{array}{l} t = f(\tau),\ \ \tau = f^{-1}(t) \\ dt = f'(\tau)\,d\tau \\ g(t) = h(f^{-1}(t)) \\ g'(t) = \frac{d}{dt}h(f^{-1}(t)) = h'(f^{-1}(t))\frac{d}{dt}f^{-1}(t) = h'(\tau)\frac{1}{f'(\tau)} \end{array}\right\}$$

$$= \int_{f^{-1}(a)}^{f^{-1}(b)} u(h(\tau))\left\|\frac{h'(\tau)}{f'(\tau)}\right\| f'(\tau)\,d\tau$$

$$= \int_{f^{-1}(a)}^{f^{-1}(b)} u(h(\tau))\|h'(\tau)\|\frac{f'(\tau)}{|f'(\tau)|}\,d\tau = \int_c^d u(h(\tau))\|h'(\tau)\|\,d\tau.$$

Here we assumed that both parametrizations go in the same direction so that $a = f(c)$, $b = f(d)$ and $f'(\tau) > 0$. What happens in this calculation if the curve is parametrized in the opposite directions by the two parametrizations?

In our example we have $\tau = \cos(t)$, $t = \arccos(\tau)$, so that $f(\tau) = \arccos(\tau)$ and $f(-1) = \pi$, $f(1) = 0$, $f'(\tau) = -1/\sqrt{1-\tau^2} < 0$. Opposite directions.

We conclude that the curve integral is independent of the choice of parametrization of the curve.

Example: We integrate $u(x) = u(x_1, x_2) = x_2$ along the half circle. With the first parametrization:

$$\int_\Gamma u\,ds = \int_0^\pi g_2(t)\|g'(t)\|\,dt = \int_0^\pi \sin(t)\,dt = 2.$$

With the second parametrization:

$$\int_\Gamma u\,ds = \int_{-1}^1 g_2(\tau)\|g'(\tau)\|\,d\tau = \int_{-1}^1 \frac{\sqrt{1-\tau^2}}{\sqrt{1-\tau^2}}\,d\tau = 2.$$

Tangent curve integral

Let

$$F : \mathbb{R}^3 \to \mathbb{R}^3 \quad (\text{or } F : \mathbb{R}^n \to \mathbb{R}^n,\ n \geq 2)$$

be a vector-valued function (vector field).

You may think of F as a force field and we want to compute the work done by the force on a particle that moves along the curve Γ:

$$x = g(t), \quad t \in I = [a, b].$$

Figure!!

The work done by the *constant force* $F(g(t_{i-1}))$ when the particle moves along the *straight line segment* Γ_i is equal to the component of the force along the tangent times the distance traveled:

$$W_i = \underbrace{F(g(t_{i-1})) \cdot \underbrace{\frac{g'(t_{i-1})}{\|g'(t_{i-1})\|}}_{\text{normalized tangent}}}_{\text{force component}} \underbrace{\|g'(t_{i-1})\| h_i}_{\text{distance}} = F(g(t_{i-1})) \cdot g'(t_{i-1}) h_i.$$

The sum of these terms

$$\sum_{i=1}^{N} W_i = \sum_{i=1}^{N} F(g(t_{i-1})) \cdot g'(t_{i-1}) h_i$$

approaches the integral

$$\int_a^b F(g(t)) \cdot g(t)\, dt$$

when $h_i \to 0$ (the rectangle rule). This is the work done by F when the particle moves along Γ:

$$W = \int_\Gamma F \cdot\, ds = \int_a^b F(g(t)) \cdot g'(t)\, dt.$$

The symbol $ds = g'(t)\, dt$ is called the tangent-arclength-element. Note that it is a vector and it is not the same as the arclength element that we introduced before.

In general (when F is not a necessarily a force field) the integral

$$\int_\Gamma F \cdot\, ds = \int_a^b F(g(t)) \cdot g'(t)\, dt$$

is called the tangent curve integral of the vector field F along the curve Γ. It is also independent of the choice of parametrization, except that the direction of parametrization is now important, the integral changes sign if the direction is changed. This is proved in the same way as for the curve integral.

Example: We integrate $F(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ along the half circle. With the first parametrization:

$$\int_\Gamma F \cdot ds = \int_0^\pi \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} dt = -\int_0^\pi \sin(t)\, dt = -2.$$

With the second parametrization:

$$\int_\Gamma F \cdot ds = \int_{-1}^1 \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{-\tau}{\sqrt{1-\tau^2}} \end{bmatrix} d\tau = \int_{-1}^1 d\tau = 2.$$

Note the opposite direction and the opposite sign in the result.

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