# Adaptive Variational Multiscale Methods Based on A Posteriori Error Estimation

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# The Model Problem

**Poisson Equation.**

$$-\nabla \cdot a\nabla u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

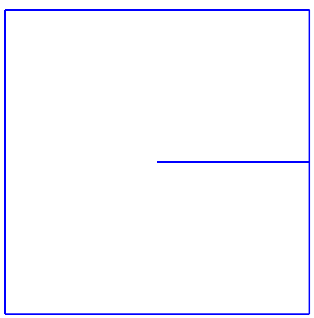
where $a > 0$ bounded, and $\Omega$ is a domain in $\mathbf{R}^d$, $d = 1, 2, 3$.

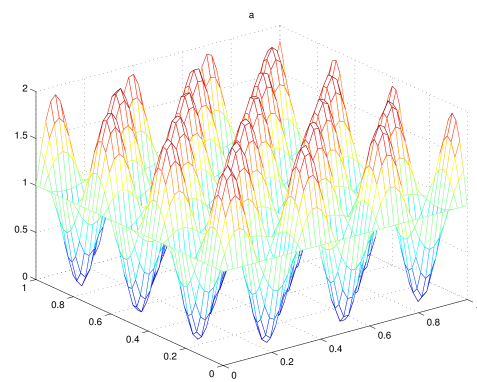
**Weak Form.** Find $u \in H_0^1(\Omega)$ such that

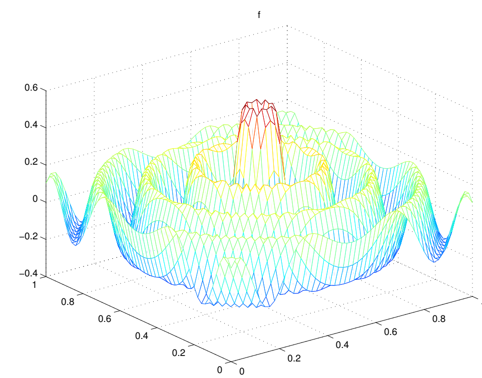
$$a(u, v) = (a\nabla u, \nabla v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega).$$

# Multiscale Problems

Below are three examples of multiscale problems.






The first one represents difficulties in the domain (cracks, holes, ...) the second one oscillations in $a$ and the third one oscillations in $f$.

# Motivation

- Very important applications including materials, flow in porous media, ...
- The problems are very computationally challenging so error estimation and efficient algorithms are crucial.
- Attempts on using adaptive algorithms are not common in literature.

# Variational Multiscale Method

We introduce two spaces $\mathcal{V}_c$ and $\mathcal{V}_f$ such that $\mathcal{V}_c \oplus \mathcal{V}_f = H_0^1(\Omega)$.

- $\mathcal{V}_c$ is a finite dimensional approximation of $H_0^1(\Omega)$. (finite element space)
- $\mathcal{V}_f$ is can be chosen in different ways e.g.
  (i) Hierarchical basis.
  (ii) $L^2(\Omega)$-orthogonal to $\mathcal{V}_c$.
  (iii) Wavelet modified hierarchical basis.

# Variational Multiscale Method

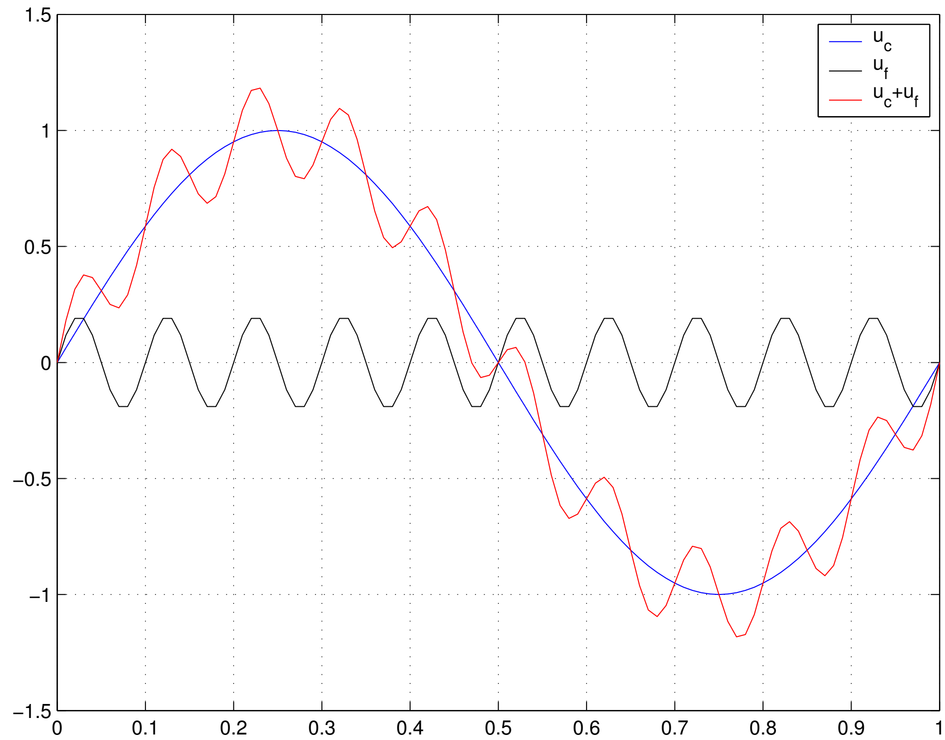



Figure 1: $u_c$, $u_f$, and $u_c + u_f$.

# Variational Multiscale Method

Find $u_c \in \mathcal{V}_c$ and $u_f \in \mathcal{V}_f$ such that

$$\begin{aligned} a(u_c, v_c) + a(u_f, v_c) &= (f, v_c) \quad \text{for all } v_c \in \mathcal{V}_c, \\ a(u_f, v_f) = (f, v_f) - a(u_c, v_f) & \\ := (R(u_c), v_f) &\quad \text{for all } v_f \in \mathcal{V}_f. \end{aligned}$$

Fine scale information is used to modify the coarse scale equation: Find $u_c \in \mathcal{V}_c$ such that

$$a(u_c, v_c) + a(\hat{A}_f^{-1} R(u_c), v_c) = (f, v_c) \;\; \forall v_c \in \mathcal{V}_c.$$

# Our Basic Idea

- Discretization of $\mathcal{V}_f$ (analytical estimates are more common).
- Solve localized fine scale problems for each coarse node (or some coarse nodes) in parallel.
- Error estimation framework.
- Adaptive strategy for this setting.

# Decouple Fine Scale Equations

Remember the fine scale equations:

$$a(u_f, v_f) = (R(u_c), v_f), \quad \text{for all } v_f \in \mathcal{V}_f.$$

Include a partition of unity,

$$a(u_f, v_f) = (R(u_c), v_f) \quad = \sum_{i=1}^{n} (R(u_c), \varphi_i v_f),$$

let $u_f = \sum_i^n u_{f,i}$ where $a(u_{f,i}, v_f) = (R(u_c), \varphi_i v_f)$.

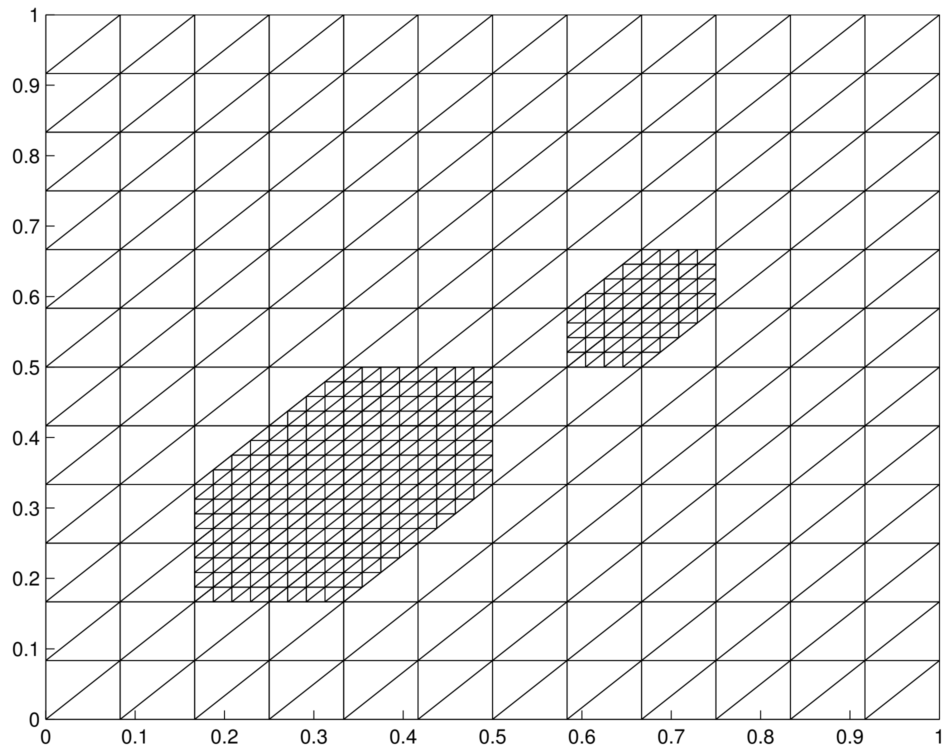
# Approximate Solution

Since $\varphi_i$ has support on a star $S_i^1$ in node $i$ we solve the fine scale equations approximately on $\omega_i$ with $U_{f,i} = 0$ on $\partial\omega_i$.

Find $U_c \in \mathcal{V}_c$ and $U_f = \sum_i^n U_{f,i}$ where $U_{f,i} \in \mathcal{V}_f^h(\omega_i)$ such that

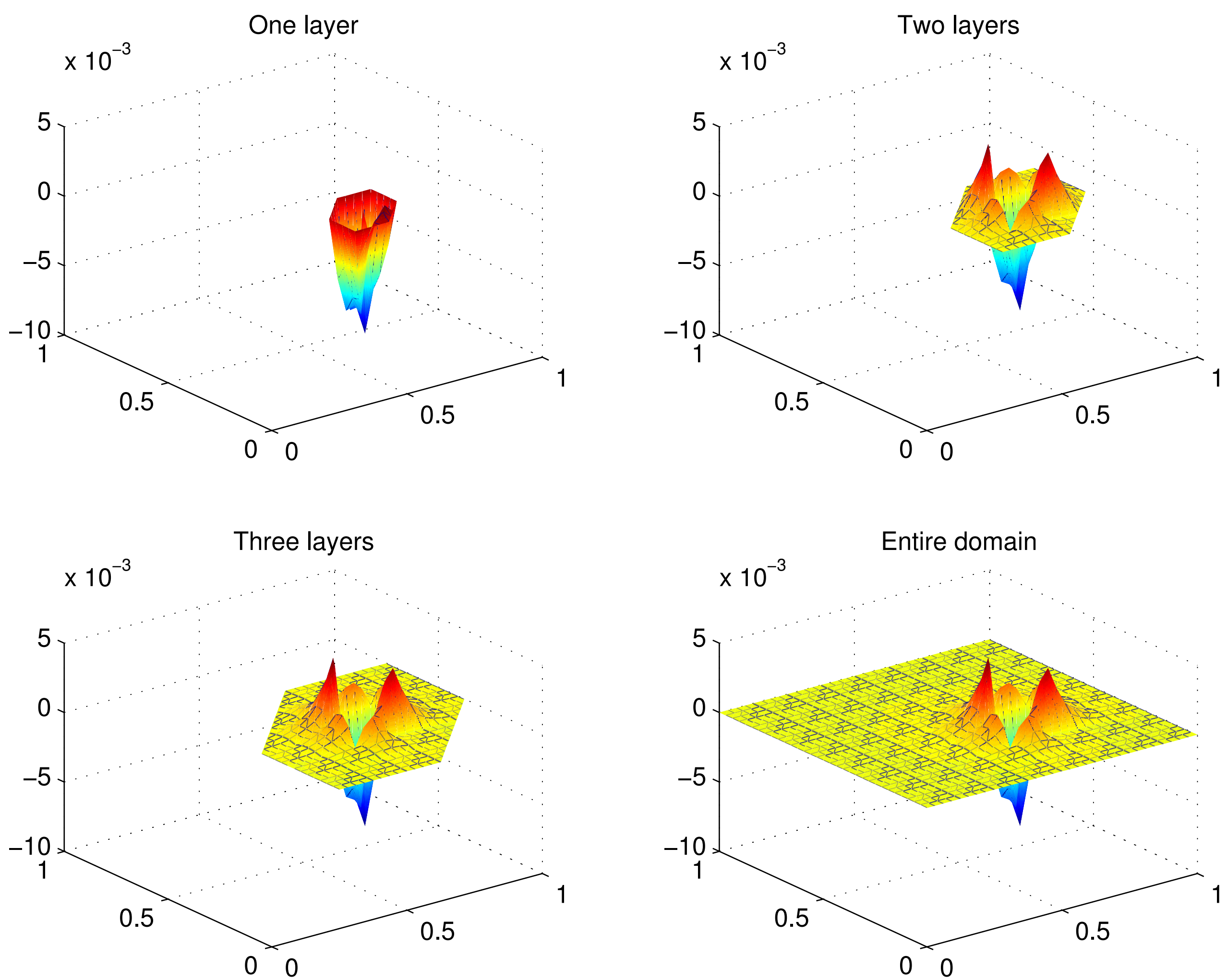
$$a(U_c, v_c) + a(U_f, v_c) = (f, v_c) \quad \text{for all } v_c \in \mathcal{V}_c,$$
$$a(U_{f,i}, v_f) = (R(U_c), \varphi_i v_f) \quad \text{for all } v_f \in \mathcal{V}_f^h(\omega_i).$$

# Refinement and Layers




One and two layer stars.

# Localized Fine Scale Solution



# Energy Norm Estimate

$$\|\sqrt{a}\nabla e\| \leq \sum_{i\in\mathcal{C}} C_i \|H\mathcal{R}(U_c)\|_{\omega_i} + \sum_{i\in\mathcal{F}} C_i \left( \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i} \right)$$

- The first term is coarse mesh error.
- The second term is the normal derivative of the fine scale solutions on $\partial\omega_i$.
- The third term is fine scale error.

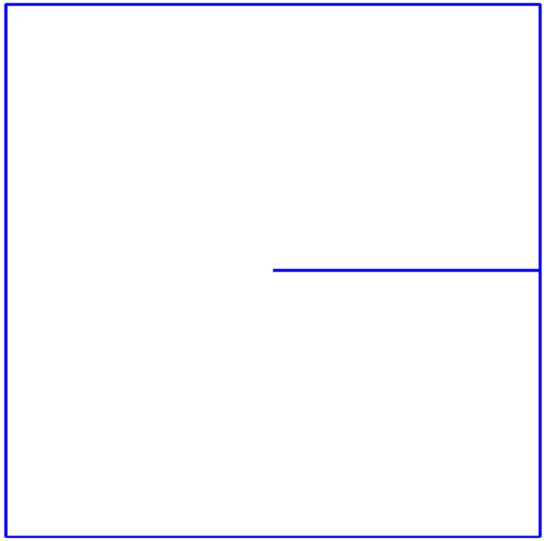
# Adaptive Strategy

$$\|\sqrt{a}\nabla e\| \leq \sum_{i\in\mathcal{C}} C_i \|H\mathcal{R}(U_c)\|_{\omega_i} + \sum_{i\in\mathcal{F}} C_i \left( \|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i} \right)$$

- We calculate these for each $i \in \{$coarse fine$\}$.
- Large values $i \in$ coarse $\rightarrow$ more local problems.
- Large values $i \in$ fine $\rightarrow$ more layers or smaller $h$.

# Numerical Examples

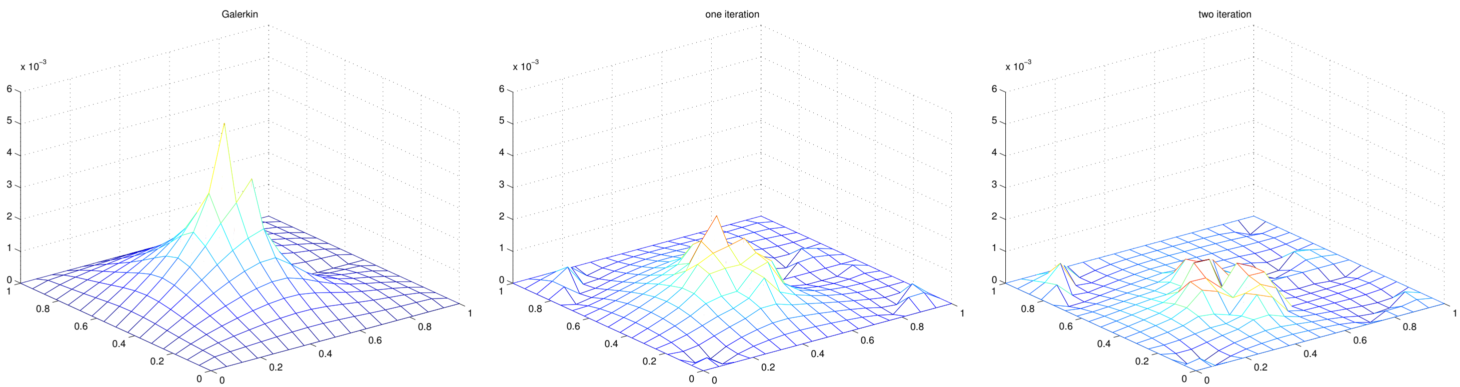
We start with a unit square containing a crack.



We let the coefficient $a = 1$ and solve, $-\triangle u = f$ with $u = 0$ on the boundary including the crack.

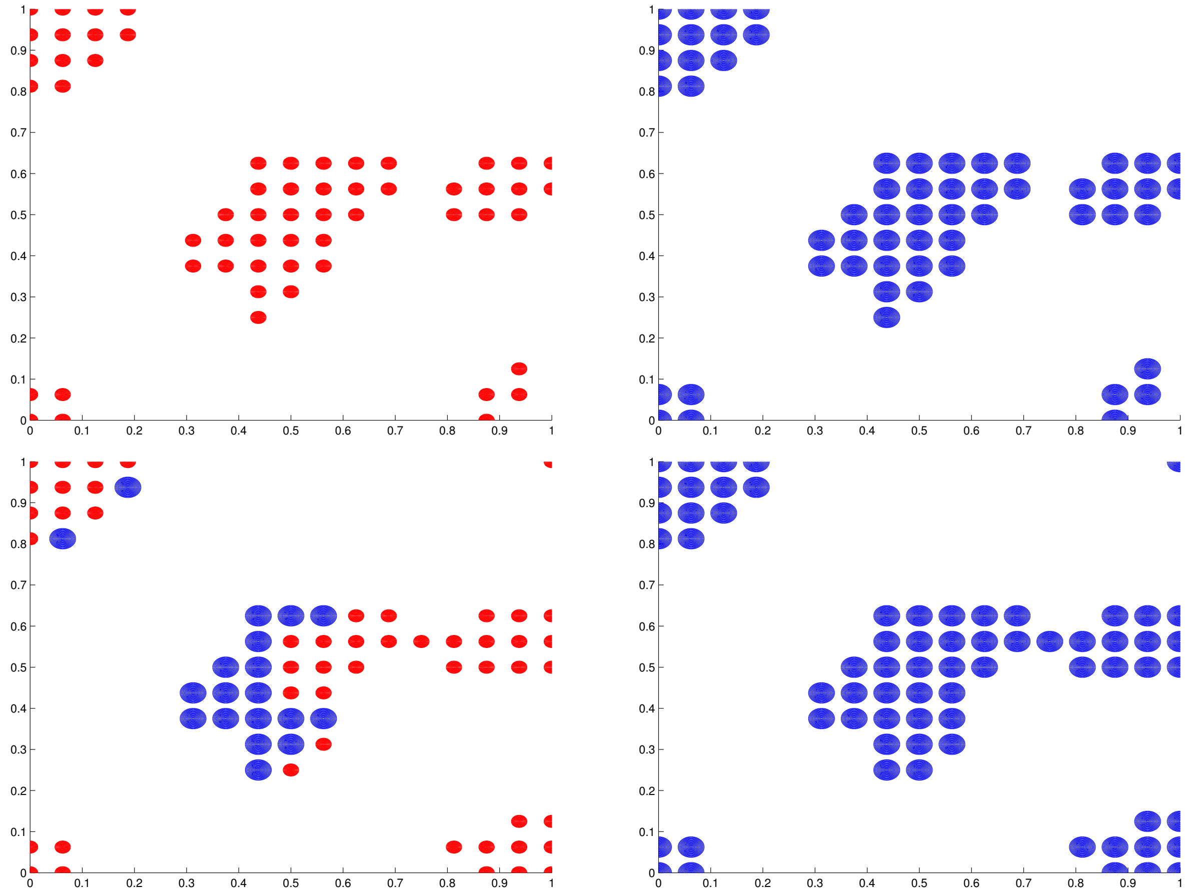
# Numerical Examples

We solve the problem by using the adaptive algorithm.



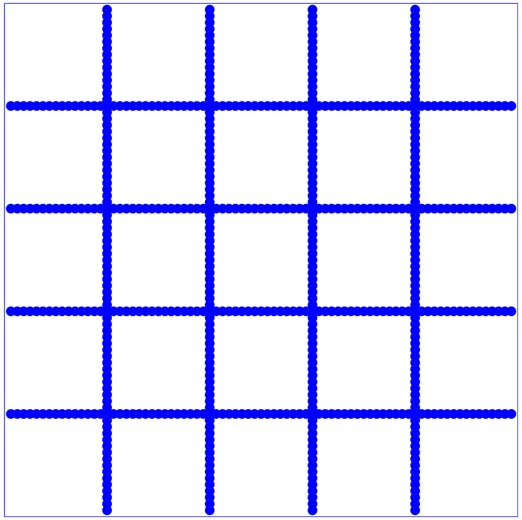

We plot the difference between our solution and a reference solution.

# Numerical Examples

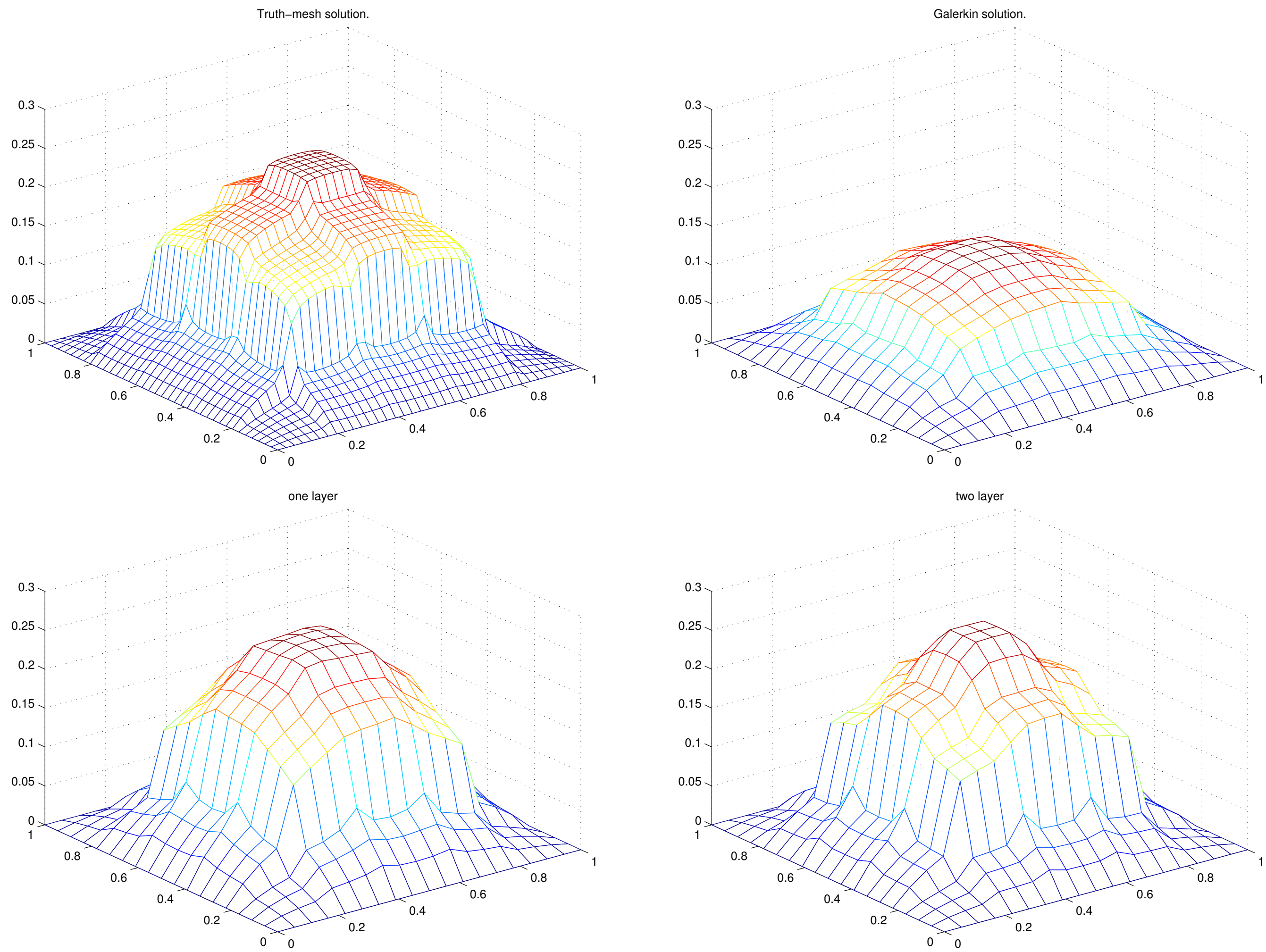


# Numerical Examples

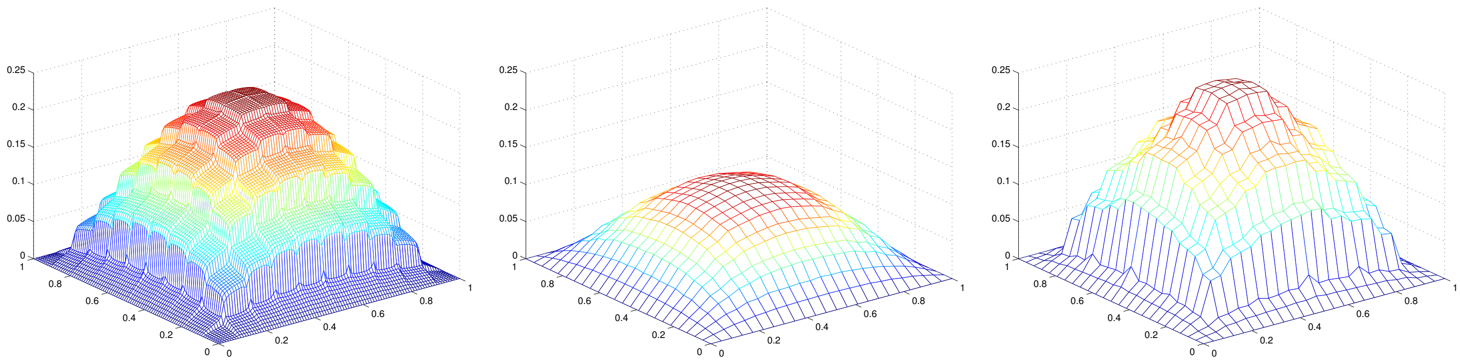
In this example we study a discontinuous coefficient $a$ in $-\nabla \cdot a\nabla u = f$. $a = 1$ (white) and $a = 0.05$ (blue).



# Numerical Examples



# Numerical Examples



The number of layers seems to depend on the fine scale structure rather that the domain size.

# Outlook

- Extended numerical tests in both 2D and 3D.
- Mixed formulation.
- Other equations (convection-diffusion, ...).
- More scales.
- Comparing results with classical Homogenization theory.