Numerical simulation of beam network models

Axel Målqvist

Morgan Görtz, Moritz Hauck, Fredrik Hellman, and Andreas Rupp

Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg
Fraunhofer Chalmers Centre

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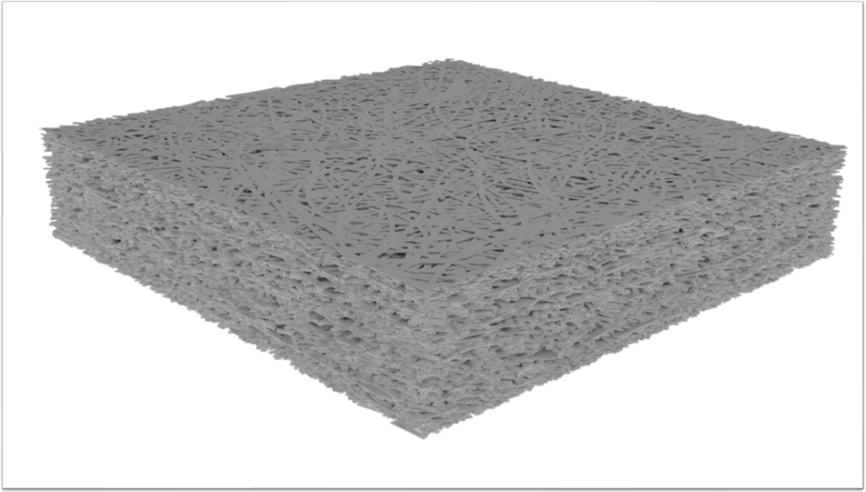
Motivation: Simulation of fibre based materials



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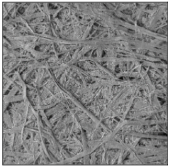
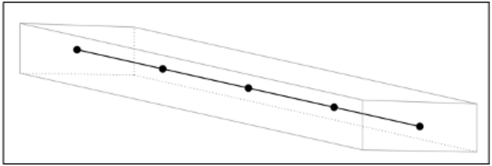




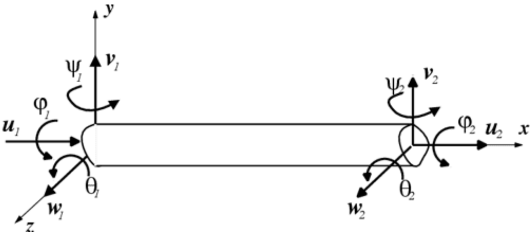
- Wood fibres $\Rightarrow$ flattened cylinders $\Rightarrow$ Timoshenko beams
- Discretize the beams and consider small deformations $\Rightarrow$ displacement/rotation solves $Az = F$
- A is SPD, sparse but large and ill-conditioned
- Direct methods are often used in practise (memory intense)

Main goal: derive and analyze robust iterative solvers

Outline

1. **The Timoshenko beam model**
2. Hybridized formulation
3. Iteration by subspace decomposition
4. Numerical examples
5. Conclusion and future work

The Timoshenko[1] beam model



- 1D model of the elastic deformation of a 3D beam
- Assumption: the cross sections remains plain after deformation
- Six degrees of freedom (centreline displacement and cross-section rotation)

[1] Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, London Edinburgh Philos. Mag. and J. Sci., 1921

Governing equation[2] (single beam)

$$-C_{\boldsymbol{n}}(\partial_x \boldsymbol{u}_\mathfrak{e} + \boldsymbol{i}_\mathfrak{e} \times \boldsymbol{r}_\mathfrak{e}) = \boldsymbol{n}_\mathfrak{e} \qquad -C_{\boldsymbol{m}}\partial_x \boldsymbol{r}_\mathfrak{e} = \boldsymbol{m}_\mathfrak{e}$$

$$\partial_x \boldsymbol{n}_\mathfrak{e} = \boldsymbol{f}_\mathfrak{e} \qquad \partial_x \boldsymbol{m}_\mathfrak{e} + \boldsymbol{i}_\mathfrak{e} \times \boldsymbol{n}_\mathfrak{e} = \boldsymbol{g}_\mathfrak{e}$$

- Unit vector in direction of $\mathfrak{e}$, $\boldsymbol{i}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$
- Centre line displacement, $\boldsymbol{u}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$
- Cross-section rotation, $\boldsymbol{r}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$
- Stress from normal and shear forces: $\boldsymbol{n}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$
- Moment from torsion and bending, $\boldsymbol{m}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$
- Material parameter, $C_{\boldsymbol{n}}$, $C_{\boldsymbol{m}}$ symmetric $\mathbb{R}^3 \times \mathbb{R}^3$ depending on Young's modulus, Shear modulus, and cross-section.
- Distributed force $\boldsymbol{f}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$ and moment $\boldsymbol{g}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$

[2] Carrera et. al., Beam Structures, Wiley 2011

Continuity and balance conditions[3]

The network is represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.

1. *Continuity of solution:* $\boldsymbol{u}_{\mathfrak{e}}(\mathfrak{n}) = \boldsymbol{u}_{\mathfrak{n}}$ and $\boldsymbol{r}_{\mathfrak{e}}(\mathfrak{n}) = \boldsymbol{r}_{\mathfrak{n}}$
2. *Dirichlet boundary nodes:* $\boldsymbol{u}_{\mathfrak{n}} = \boldsymbol{u}^D_{\mathfrak{n}}$ and $\boldsymbol{r}_{\mathfrak{n}} = \boldsymbol{r}^D_{\mathfrak{n}}$, $\mathfrak{n} \in \mathcal{N}_D$
3. *Balance equations:* Let $[\![\cdot]\!]_{\mathfrak{n}}$ be a summation at $\mathfrak{n}$ and $\nu_{\mathfrak{e}} = \pm 1$:

$$[\![\boldsymbol{n}_{\mathfrak{e}} \nu_{\mathfrak{e}}]\!]_{\mathfrak{n}} = \boldsymbol{f}_{\mathfrak{n}} \qquad [\![\boldsymbol{m}_{\mathfrak{e}} \nu_{\mathfrak{e}}]\!]_{\mathfrak{n}} = \boldsymbol{g}_{\mathfrak{n}}$$

[3] Lagnese et. at. Modeling, analysis and control of dynamic elastic multi-link structures, Birkhäuser Boston, 1994

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HDG[4] discretization

Aim: reduce dofs in global solve to dofs at joints

- *Primal variables:* $\bar{\boldsymbol{u}}_{\mathfrak{e}}, \bar{\boldsymbol{r}}_{\mathfrak{e}} \in V_{h,p}^{\mathfrak{e}} := (\mathbb{P}_{h,p}(\mathfrak{e}))^3 \subset (L^2(\mathfrak{e}))^3, \forall \mathfrak{e} \in \mathcal{E}$
- *Dual variables:* $\bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{m}}_{\mathfrak{e}} \in V_{h,p}^{\mathfrak{e}} \subset (H^1(\mathfrak{e}))^3, \forall \mathfrak{e} \in \mathcal{E}$
- *Hybrid variables:* $\bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{r}}_{\mathfrak{n}} \in \mathbb{R}^3, \forall \mathfrak{n} \in \mathcal{N} \setminus \mathcal{N}_D$

[4] Cockburn et. al. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems, SINUM, (2009)

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For all $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{w}} \in V^{\mathfrak{e}}_{h,p}$:

$$
\begin{aligned}
-(C_{\boldsymbol{n}}^{-1}\bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{p}})_{\mathfrak{e}} \qquad\qquad + (\bar{\boldsymbol{u}}_{\mathfrak{e}}, \partial_x \bar{\boldsymbol{p}})_{\mathfrak{e}} - (\boldsymbol{i}_{\mathfrak{e}} \times \bar{\boldsymbol{r}}_{\mathfrak{e}}, \boldsymbol{p})_{\mathfrak{e}} &= \langle \bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{p}} \nu_{\mathfrak{e}} \rangle_{\mathfrak{e}} \\
-(C_{\boldsymbol{m}}^{-1}\bar{\boldsymbol{m}}_{\mathfrak{e}}, \bar{\boldsymbol{q}})_{\mathfrak{e}} \qquad\qquad + (\bar{\boldsymbol{r}}_{\mathfrak{e}}, \partial_x \bar{\boldsymbol{q}})_{\mathfrak{e}} &= \langle \bar{\boldsymbol{r}}_{\mathfrak{n}}, \bar{\boldsymbol{q}} \nu_{\mathfrak{e}} \rangle_{\mathfrak{e}} \\
(\partial_x \bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{v}})_{\mathfrak{e}} &= (\boldsymbol{f}_{\mathfrak{e}}, \bar{\boldsymbol{v}})_{\mathfrak{e}} \\
(\boldsymbol{i}_{\mathfrak{e}} \times \bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}} + (\partial_x \bar{\boldsymbol{m}}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}} &= (\boldsymbol{g}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}}
\end{aligned}
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[4] Cockburn et. al. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems, SINUM, (2009)

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For all $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{w}} \in V_{h,p}^{\mathfrak{e}}$: (penalty parameter $\tau_{\mathfrak{e}} > 0$)

$$
\begin{aligned}
-(C_{\boldsymbol{n}}^{-1}\bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{p}})_{\mathfrak{e}} \quad &+ (\bar{\boldsymbol{u}}_{\mathfrak{e}}, \partial_x \bar{\boldsymbol{p}})_{\mathfrak{e}} - (\boldsymbol{i}_{\mathfrak{e}} \times \bar{\boldsymbol{r}}_{\mathfrak{e}}, \boldsymbol{p})_{\mathfrak{e}} &&= \langle \bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{p}}\nu_{\mathfrak{e}} \rangle_{\mathfrak{e}} \\
-(C_{\boldsymbol{m}}^{-1}\bar{\boldsymbol{m}}_{\mathfrak{e}}, \bar{\boldsymbol{q}})_{\mathfrak{e}} \quad &+ (\bar{\boldsymbol{r}}_{\mathfrak{e}}, \partial_x \bar{\boldsymbol{q}})_{\mathfrak{e}} &&= \langle \bar{\boldsymbol{r}}_{\mathfrak{n}}, \bar{\boldsymbol{q}}\nu_{\mathfrak{e}} \rangle_{\mathfrak{e}} \\
(\partial_x \bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{v}})_{\mathfrak{e}} \quad &+ \tau_{\mathfrak{e}} \langle \bar{\boldsymbol{u}}_{\mathfrak{e}}, \bar{\boldsymbol{v}} \rangle_{\mathfrak{e}} &&= (\boldsymbol{f}_{\mathfrak{e}}, \bar{\boldsymbol{v}})_{\mathfrak{e}} + \tau_{\mathfrak{e}} \langle \bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{v}} \rangle_{\mathfrak{e}} \\
(\boldsymbol{i}_{\mathfrak{e}} \times \bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}} + (\partial_x \bar{\boldsymbol{m}}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}} \quad &+ \tau_{\mathfrak{e}} \langle \bar{\boldsymbol{r}}_{\mathfrak{e}}, \bar{\boldsymbol{w}} \rangle_{\mathfrak{e}} &&= (\boldsymbol{g}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}} + \tau_{\mathfrak{e}} \langle \bar{\boldsymbol{r}}_{\mathfrak{n}}, \bar{\boldsymbol{w}} \rangle_{\mathfrak{e}}
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- *Hybrid variables:* $\bar{\boldsymbol{u}}_\mathfrak{n}, \bar{\boldsymbol{r}}_\mathfrak{n} \in \mathbb{R}^3$, $\forall \mathfrak{n} \in \mathcal{N} \setminus \mathcal{N}_D$

The discrete balance equations reads

$$[\![\bar{\boldsymbol{n}}_\mathfrak{e}\nu_\mathfrak{e} + \tau_\mathfrak{e}(\bar{\boldsymbol{u}}_\mathfrak{e} - \bar{\boldsymbol{u}}_\mathfrak{n})]\!]_n = \boldsymbol{f}_\mathfrak{n}, \qquad [\![\bar{\boldsymbol{m}}_\mathfrak{e}\nu_\mathfrak{e} + \tau_\mathfrak{e}(\bar{\boldsymbol{r}}_\mathfrak{e} - \bar{\boldsymbol{r}}_\mathfrak{n})]\!]_n = \boldsymbol{g}_\mathfrak{n}$$

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HDG discretization[5]

Global system:

$$A\bar{z}_h = F,$$

where $\bar{z}_h = (\bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{r}}_{\mathfrak{n}})$ with 6 dofs per joint.

- Independent local solves on edges are needed to form A and F
- F contains applied forces, moments and boundary data

[5] Rupp, Hauck, M., Arbitrary order approximations at constant cost for Timoshenko beam network models, arXiv:2407.14388

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Theorem (Spectral equivalence to graph Laplacian)

If $C_{\mathbf{n}}$ and $C_{\mathbf{m}}$ are edgewise constant. Then, for all $(\lambda, \phi) \in V \times V$:

$$\alpha\left(\lambda^\top L\lambda + \phi^\top L\phi\right) \leq (\lambda,\phi)^\top A(\lambda,\phi) \leq \beta\left(\lambda^\top L\lambda + \phi^\top L\phi\right),$$

where $v^\top Lv = \sum_{x\in\mathcal{N}} \frac{1}{2} \sum_{x\sim y} \frac{|v(x)-v(y)|^2}{|x-y|}$ and $0 < \alpha < \beta$ depend on the algebraic connectivity of the graph $\mathcal{G}$ and $C_{\mathbf{n}}$ and $C_{\mathbf{m}}$.

[5] Rupp, Hauck, M., Arbitrary order approximations at constant cost for Timoshenko beam network models, arXiv:2407.14388

A priori error bound[6][7]

Theorem (Convergence of HDG method)

If $\tau_\mathfrak{e} \sim h_\mathfrak{e}^s$ for some $s \in \{-1, 0, 1\}$ and $\boldsymbol{u}_\mathfrak{e}, \boldsymbol{r}_\mathfrak{e}, \boldsymbol{n}_\mathfrak{e}, \boldsymbol{m}_\mathfrak{e} \in H^{p+1}(\mathfrak{e})$ for all $\mathfrak{e} \in \mathcal{E}$, then it holds

$$\left[\sum_{\mathfrak{e}\in\mathcal{E}} \left[\|\boldsymbol{u}_\mathfrak{e} - \bar{\boldsymbol{u}}_\mathfrak{e}\|_\mathfrak{e}^2 + \|\boldsymbol{r}_\mathfrak{e} - \bar{\boldsymbol{r}}_\mathfrak{e}\|_\mathfrak{e}^2 \right] \right]^{1/2} \lesssim h^{p+1-s^+},$$

$$\left[\sum_{\mathfrak{e}\in\mathcal{E}} \left[\|\boldsymbol{n}_\mathfrak{e} - \bar{\boldsymbol{n}}_\mathfrak{e}\|_\mathfrak{e}^2 + \|\boldsymbol{m}_\mathfrak{e} - \bar{\boldsymbol{m}}_\mathfrak{e}\|_\mathfrak{e}^2 \right] \right]^{1/2} \lesssim h^{p+1-|s|},$$

where $s^+ := \max(s, 0)$.

[6] Celiker, Cockburn, Shi, Hybridizable DG methods for Timoshenko beams, JSC (2010)

[7] Rupp, Hauck, M., Arbitrary order approximations at constant cost for Timoshenko beam network models, arXiv:2407.14388

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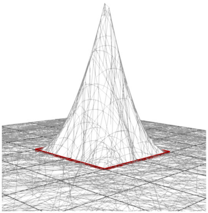
Subspace decomposition preconditioner[8]

Let $V\ (=\mathbb{R}^{3n}) = V_0 + V_1 + \cdots + V_m$ with

$$V_0 := V_H \quad (\text{Q1-FEM})^3 \text{ on mesh } \mathcal{T}_H$$

$$V_i := \{\boldsymbol{v} \in V : \ \operatorname{supp}(\boldsymbol{v}) \subset U_i\}$$

Define $P_i\colon\ V \times V \to V_i \times V_i$ such that



$$(AP_i v, w) = (Av, w)$$

for all w and form $P := P_0 + P_1 + \cdots + P_m$.

- $BAz = BF$, with preconditioner $P = BA$
- Preconditioned conjugate gradient method.
- Semi-iterative: direct method on decoupled problems

[8] Xu, Iterative methods by subspace decomposition and subspace correction, SIAM Review, 1992

Convergence analysis[9]

Lemma (Properties of the decomposition)

If $A \sim L$ then for $H \geq 2R_0$ (at least) one decomposition $v = \sum_{j=0}^{m} v_j$ satisfies:

$$\sum_{j=0}^{m} |v_j|_A^2 \leq C_1 |v|_A^2, \quad C_1 = C_d \beta \alpha^{-1} \sigma \mu^2$$

and every decomposition satisfies $|v|_A^2 \leq C_2 \sum_{j=0}^{m} |v_j|_A^2$ with $C_2 = C_d$.

Theorem (Convergence of PCG)

With $\kappa = C_1 C_2$, $H > 2R_0$, it holds

$$|z - z^{(\ell)}|_A \leq 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^{\ell} |z - z^{(0)}|_A.$$

[9] Görtz, Hellman, M., Iterative solution of spatial network models by subspace decomposition, Math. Comp. (2024)

Network homogeneity and connectivity $H > R_0$

1. *Homogeneity:* $\max_{x\in\Omega} |1|^2_{M,B_H(x)} \leq \sigma(R_0) \min_{x\in\Omega} |1|^2_{M,B_H(x)}$

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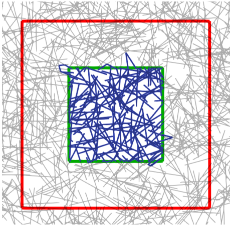
2. *Connectivity:* $\mu(R_0) < \infty$ if there for all $x \in \Omega$ exists $\mathcal{G}' \subset \mathcal{G}$ that



- contains all edges with one endpoint in $B_H(x)$
- only contains edges with endpoints contained in $B_{H+R_0}(x)$

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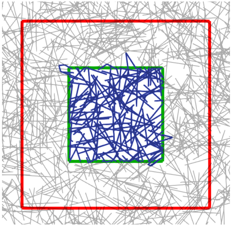

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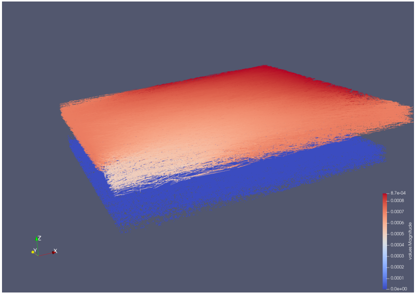
Consider $L'\phi = \lambda M'\phi$, M' is diagonal (mass). Then $\lambda_1 = 0$ and $\lambda_2 > 0$ measures algebraic connectivity.

If every $\mathcal{G}'$ fulfills an iso-perimetric inequality[10]: $\lambda_2^{-1/2} = \mu(R_0)H$ with moderate μ.

[10] Cheeger 1970, Fiedler 1973, Chung 1997 Spectral graph theory (AMS)

Outline

Example: Elastic deformation of paper



- 4 mm x 4 mm paper
- 615K edges and 424K nodes
- We study stretching of the paper caused by Dirichlet boundary conditions (upper right)
- HDG discretization with $p = 5$ and $\tau = 1$
- Preconditioner with $8 \times 8 \times 1$ element in coarse space

Example: Elastic deformation of paper

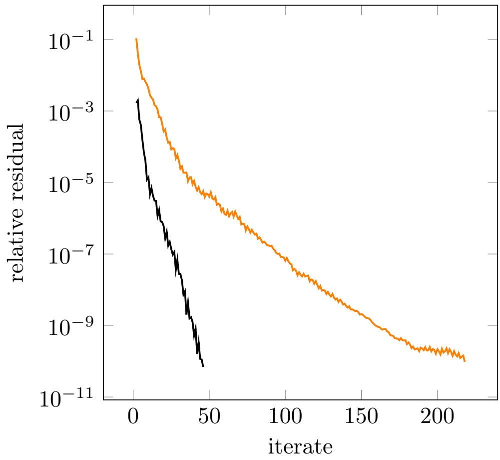


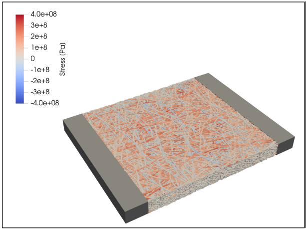
Figure: Convergence of PCG: constant material parameters (black) and realistic (orange).

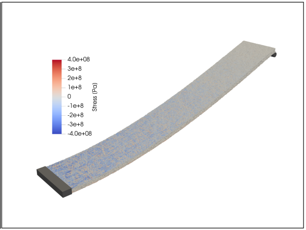
Outline

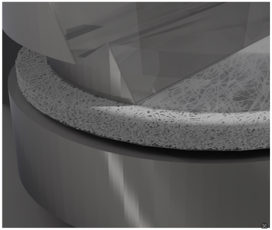
Ongoing and future work

Robust iterative approach to solve spatial network models

Engineering applications[11] in collaboration with Fraunhofer Chalmers Centre and recently packaging company Tetra Pak.







[11] Görtz et. al., Iterative method for large-scale Timoshenko beam models assessed on commercial-grade paperboard, Computational Mechanics (2025).