Finite element methods for the Joule heating problem in three spatial dimensions

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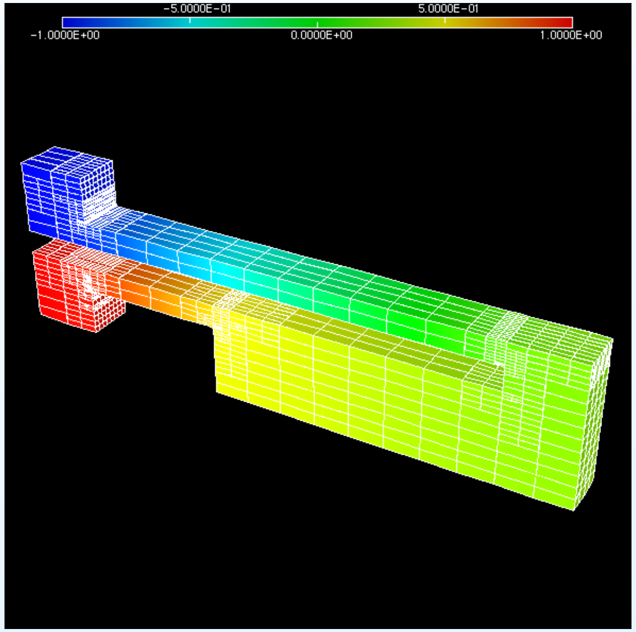
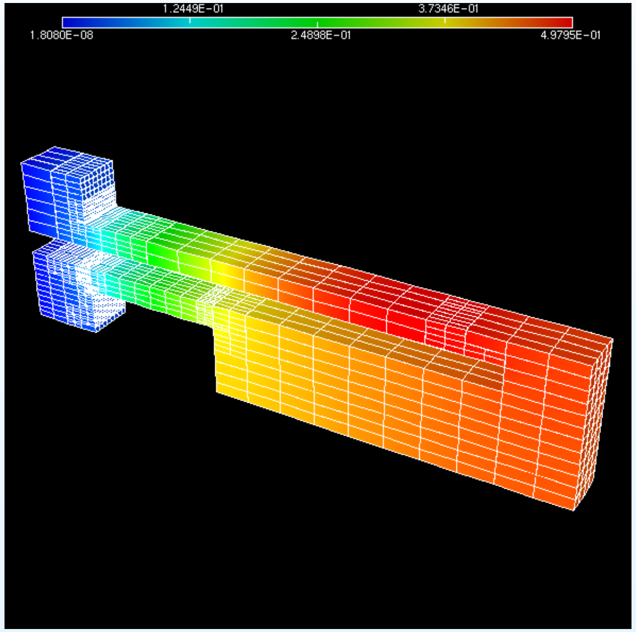
Application

A voltage is applied to the boundary of a device and current flows through it, the current flow produces Joule heating, which induces thermal stresses into the device.





Potential (left) and temperature (right). Microelectromechanical systems (MEMS) are e.g. used to position mirrors in sensors.

Mathematical formulation

We seek the electric potential ϕ satisfying:

$$\begin{cases} -\nabla \cdot (\sigma(u)\nabla\phi) = f & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\sigma = \sigma(x, u)$ is electric conductivity and $f = f(x)$ is a source.

The temperature u satisfies the stationary heat equation:

$$\begin{cases} -\Delta u = \sigma(u)|\nabla\phi|^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume polygonal domain, $f \in L^\infty(\Omega)$, and $\sigma(\cdot) \in L^\infty(\Omega)$ positive.

Goals and difficulties

- Show that standard finite element approximation converges to weak solution as $h \to 0$.
- It turns out, existence of weak solution in $H_0^1(\Omega) \times H_0^1(\Omega)$ is non trivial to prove since $\sigma(u)|\nabla\phi|^2 \in L^1(\Omega) \not\subset H^{-1}(\Omega)$ for arbitrary $\phi \in H_0^1(\Omega)$.
- Given existence of weak solution we want to prove existence of finite element approximation, it turns out we need $L^\infty(\Omega)$-norm estimates on ϕ_h. In 3D this leads to difficulties.
- Finally, we prove converges of subsequence under additional assumptions on the mesh.

Proposed solutions to $\sigma(u)|\nabla\phi|^2 \notin H^{-1}(\Omega)$

- Gallouët and Herbin, 1994, existence in $H^1(\Omega) \times W^{1,d/(d-1)-\epsilon}(\Omega)$, $d = 2, 3$, for homogenous Dirichlet bc using entropy condition for the temperature equation to get well defined map.
- Recently, in the case when $f \in L^\infty(\Omega)$ and therefore $\phi \in L^\infty(\Omega) \cap H_0^1(\Omega)$, the following was shown,

$$\begin{aligned}\sigma(u)|\nabla\phi|^2 &= (-\nabla\cdot\sigma(u)\nabla\phi)\phi + \nabla\cdot(\phi\sigma(u)\nabla\phi)\\ &= f\phi + \nabla\cdot(\phi\sigma(u)\nabla\phi) \in H^{-1}(\Omega),\end{aligned}$$

  which makes it possible to stay in $H_0^1(\Omega) \cap L^\infty(\Omega) \times H_0^1(\Omega)$.

Existence (Boccardo, Orsina, and Poretta)

Theorem: Under the assumption above there exist a weak solution to the Joule heating problem such that $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $u \in H_0^1(\Omega)$.

Proof. Remember,

$$-\nabla \cdot \sigma(u)\nabla\phi = f \quad \text{and} \quad -\Delta u = f\phi + \nabla \cdot (\phi\sigma(u)\nabla\phi).$$

A fixpoint map is constructed $\bar{u} = Tu$. It maps into $H_0^1(\Omega)$. The map can be proven to be compact and continuous. Schauder's fixed point theorem gives existence result.

Note that uniqueness is not proven.

The finite element method

We let V_h be spanned by cont. piecewise linear functions. Since we need $\phi_h \in L^\infty(\Omega)$ we assume that the stiffness matrix is an M-matrix i.e. the off-diagonal terms are negative and eigenvalues are positive.

Delaunay gives this in 2D and it is e.g. fulfilled in 3D using a mesh constructed according to a paper by Chen, Holst, and Xu, in SINUM, 2007.

The finite element method now reads: find $\phi_h \in V_h$ and $u_h \in V_h$ such that,

$$(\sigma(u_h)\nabla\phi_h, \nabla v) = (f, v), \quad \text{for all } v \in V_h,$$

$$(\nabla u_h, \nabla w) = (f\phi_h, w) - (\phi_h\sigma(u_h)\nabla\phi_h, \nabla w), \quad \text{for all } w \in V_h.$$

Existence of discrete solution

Theorem. Under the mesh assumption above there exists at least one solution $(\phi_h, u_h) \in V_h \times V_h$ to the finite element formulation.

Proof. The proof is similar to the continuous proof. Brouwder's fixed point theorem is used. The $L^\infty(\Omega)$ bound on ϕ_h is crucial.

Theorem. We have the following convergence of sub-sequences,

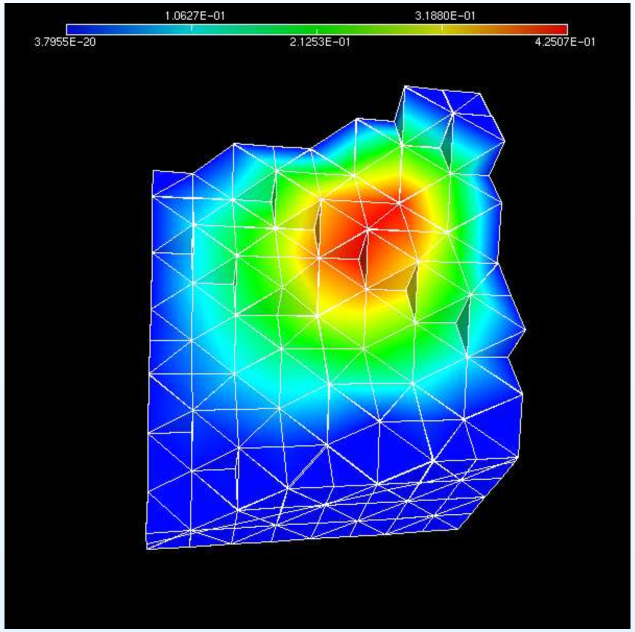
$$\phi_h \to \phi \text{ in } H_0^1(\Omega) \cap L^\infty(\Omega) \text{ as } h \to 0,$$

$$u_h \to u \text{ in } H_0^1(\Omega) \text{ as } h \to 0.$$

Proof. Bounded sequences has weakly convergent sub-sequences. We prove that the sub-sequences are actually strongly convergent using Dominated Convergence Theorem. Finally, we prove that the limit fulfills the original weak form.

Numerical experiments (Model problem)

We let $f = 10xyze^{-\left((x-0.5)^2+(y-0.5)^2+(z-0.5)^2\right)}$ and $\sigma(u) = 0.05 + \frac{0.15}{\pi}\left(\frac{\pi}{2} + \arctan\left(\frac{u+0.05}{0.05}\right)\right)$. We use a Gauss-Seidel type iteration between the equations.



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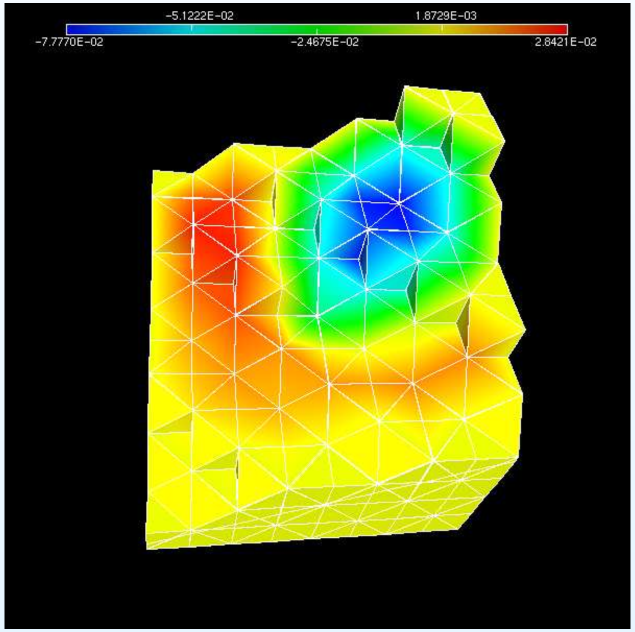
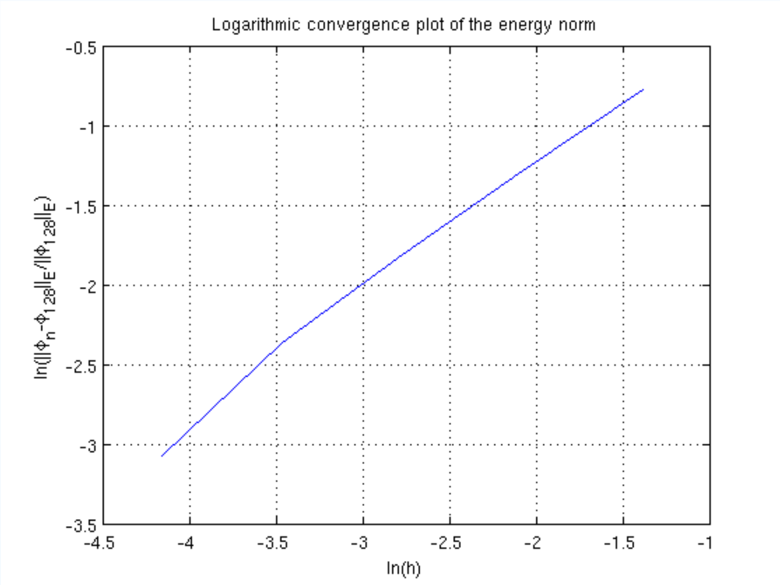


We see ϕ to the left and u to the right.

Numerical experiments (Convergence study)

A logarithmic plot of the relative error (compared to a reference solution) in the energy norm as a function of h.



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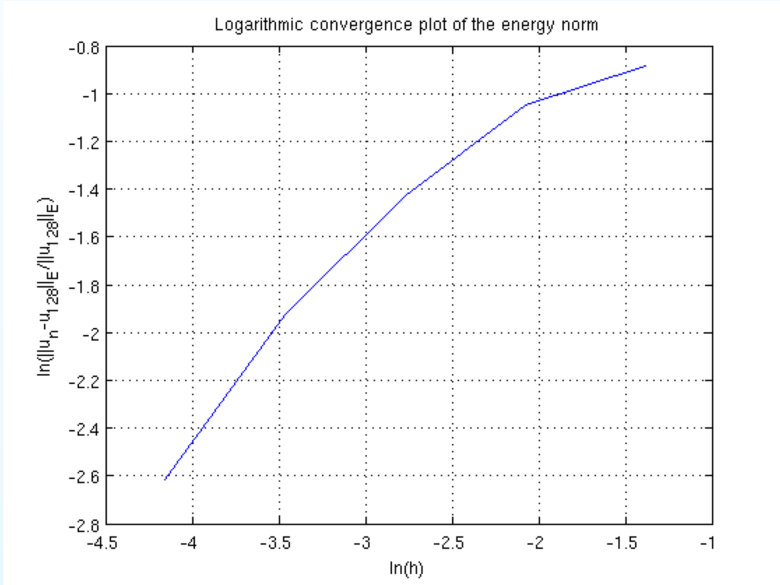
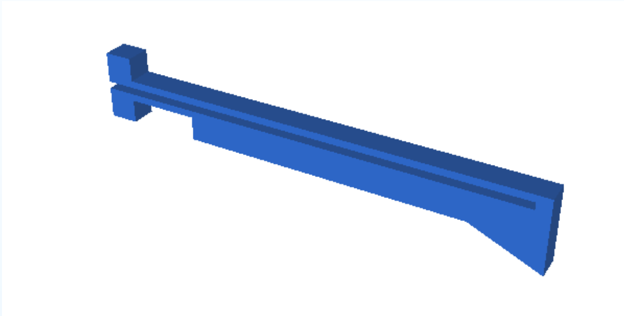


We see convergence in ϕ to the left and u to the right. Using convex domain and smooth data leads to higher regularity then just H^1.

Engineering application (Problem formulation)

We study the Joule heating problem for the geometry,



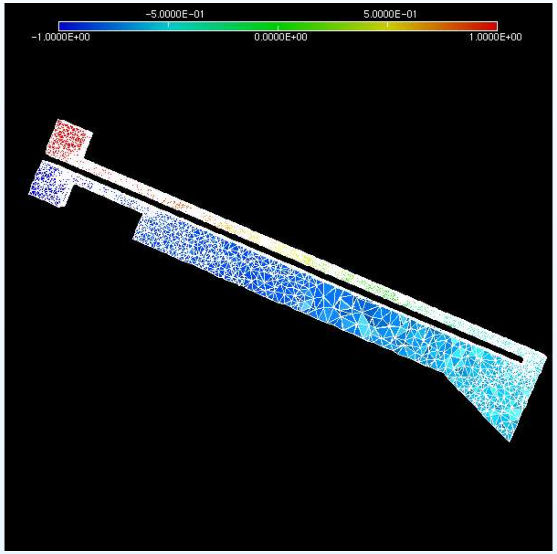
with modified boundary conditions.

We apply a given voltage at the boundary of the pads to the left and use homogeneous Neumann conditions on the rest of the boundary.

Similarly we set the temperature on the pads to zero and use homogeneous Neumann conditions on the rest of the boundary.

Engineering application (Simulation)

We apply voltage at the boundary. This give rise to a current and heat development.



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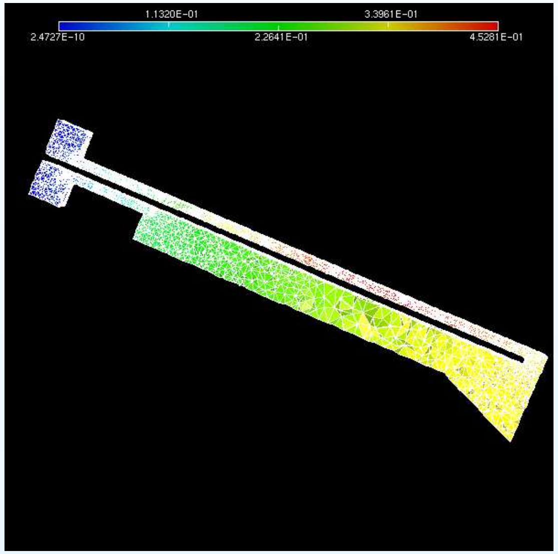

High temperature in narrow part leads to bending making it possible to close a circuit.

Note that thermal expansion is not included in the model problem, σ is still $arctan$ based.

Future work

- Adaptivity is the next step. A posteriori error estimates of the type,

$$\|\phi - \phi_h\|_X + \|u - u_h\|_X \leq C \sum_K \rho(\phi_h, u_h).$$

  The constant C will among other thing depend on a bound of the inverse of the linearized operator. We will also need to prove Lipschitz continuity for the linearized operator.
- Study more realistic boundary conditions, see Holst and Målqvist.
- Time dependent heat equation.
- Study convergence criteria for the Gauss-Seidel type iteration between the equations.