# Iterative methods for Timoshenko beam network models

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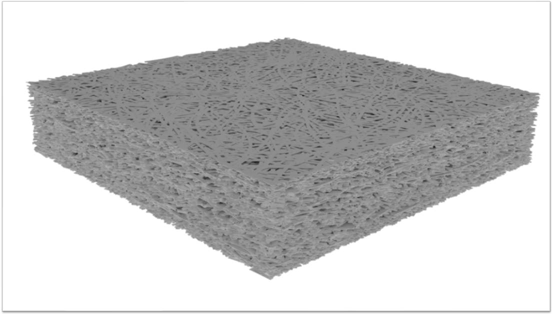
# Motivation: Simulation of paper/paperboard



- Collaboration with Fraunhofer Chalmers Centre (FCC), paper making industry (Stora Enso)[1,2] and packaging (Tetra Pak)
- Simulation of mechanical properties (tensile/bending strength)

[1] Kettil, Multiscale methods for simulation of paper making, PhD thesis, 2019

[2] Görtz, Numerical homogenization of network models and micro-mechanical simulation of paperboard, PhD thesis, 2024

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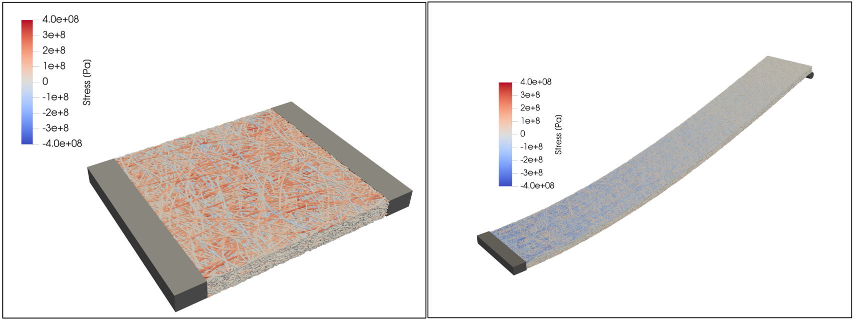


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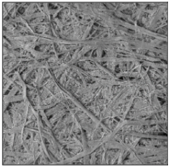
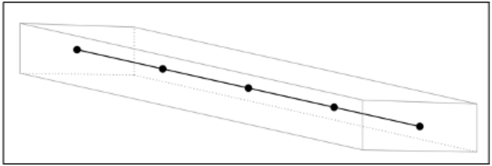




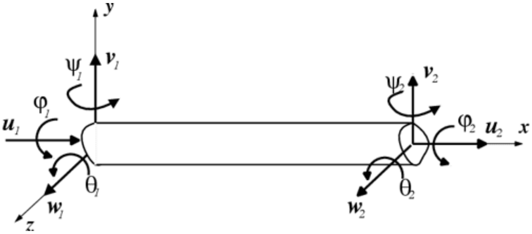
- Paper fibres are hollow flattened slender cylinders
- Model: Timoshenko beams with rigid joints
- The displacement solves a linear system of equations $Au = F$
- $A$ is SPD, sparse but large and ill-conditioned
- Direct methods are used (FCC)

Main goal: derive and analyze an efficient iterative method

# Outline

1. **The Timoshenko beam model**
2. Hybridized formulation
3. Iteration by subspace decomposition
4. Numerical examples
5. Conclusion and future work

# The Timoshenko[3] beam model




- 1D model of the elastic deformation of a 3D beam
- Assumption: the cross sections remains plain after deformation
- Six degrees of freedom (centreline displacement and cross-section rotation)

[3] Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, London Edinburgh Philos. Mag. and J. Sci., 1921

# Governing equation[4] (single beam)

$$-C_{\boldsymbol{n}}(\partial_x \boldsymbol{u}_\mathfrak{e} + \boldsymbol{i}_\mathfrak{e} \times \boldsymbol{r}_\mathfrak{e}) = \boldsymbol{n}_\mathfrak{e} \qquad -C_{\boldsymbol{m}}\partial_x \boldsymbol{r}_\mathfrak{e} = \boldsymbol{m}_\mathfrak{e}$$

$$\partial_x \boldsymbol{n}_\mathfrak{e} = \boldsymbol{f}_\mathfrak{e} \qquad \partial_x \boldsymbol{m}_\mathfrak{e} + \boldsymbol{i}_\mathfrak{e} \times \boldsymbol{n}_\mathfrak{e} = \boldsymbol{g}_\mathfrak{e}$$

- Unit vector in direction of $\mathfrak{e}$, $\boldsymbol{i}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$
- Centre line displacement, $\boldsymbol{u}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$
- Cross-section rotation, $\boldsymbol{r}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$
- Stress from normal and shear forces: $\boldsymbol{n}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$
- Moment from torsion and bending, $\boldsymbol{m}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$
- Material parameter, $C_{\boldsymbol{n}}$, $C_{\boldsymbol{m}}$ symmetric $\mathbb{R}^3 \times \mathbb{R}^3$ depending on Young's modulus, Shear modulus, and cross-section.
- Distributed force $\boldsymbol{f}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$ and moment $\boldsymbol{g}_\mathfrak{e} : \mathfrak{e} \to \mathbb{R}^3$

[4] Carrera et. al., Beam Structures, Wiley 2011

# Weak formulation (single beam)

For edge $\mathfrak{e}$ and for all $\boldsymbol{p}, \boldsymbol{q} \in V_{\boldsymbol{m}}^{\mathfrak{e}} = (H^1(\mathfrak{e}))^3$ and $\boldsymbol{v}, \boldsymbol{w} \in V_{\boldsymbol{u}}^{\mathfrak{e}} = (L^2(\mathfrak{e}))^3$ it holds:

$$
\begin{aligned}
-(C_{\boldsymbol{n}}^{-1}\boldsymbol{n}_{\mathfrak{e}}, \boldsymbol{p})_{\mathfrak{e}} \qquad\qquad &+ (\boldsymbol{u}_{\mathfrak{e}}, \partial_x \boldsymbol{p})_{\mathfrak{e}} - (\boldsymbol{i}_{\mathfrak{e}} \times \boldsymbol{r}_{\mathfrak{e}}, \boldsymbol{p})_{\mathfrak{e}} &&= \langle \boldsymbol{u}_{\mathfrak{n}}, \boldsymbol{p}\nu_{\mathfrak{e}} \rangle_{\mathfrak{e}} \\
-(C_{\boldsymbol{m}}^{-1}\boldsymbol{m}_{\mathfrak{e}}, \boldsymbol{q})_{\mathfrak{e}} \qquad &+ (\boldsymbol{r}_{\mathfrak{e}}, \partial_x \boldsymbol{q})_{\mathfrak{e}} &&= \langle \boldsymbol{r}_{\mathfrak{n}}, \boldsymbol{q}\nu_{\mathfrak{e}} \rangle_{\mathfrak{e}} \\
(\partial_x \boldsymbol{n}_{\mathfrak{e}}, \boldsymbol{v})_{\mathfrak{e}} & &&= (\boldsymbol{f}_{\mathfrak{e}}, \boldsymbol{v})_{\mathfrak{e}} \\
(\boldsymbol{i}_{\mathfrak{e}} \times \boldsymbol{n}_{\mathfrak{e}}, \boldsymbol{w})_{\mathfrak{e}} + (\partial_x \boldsymbol{m}_{\mathfrak{e}}, \boldsymbol{w})_{\mathfrak{e}} & &&= (\boldsymbol{g}_{\mathfrak{e}}, \boldsymbol{w})_{\mathfrak{e}}
\end{aligned}
$$

where $(\boldsymbol{v}, \boldsymbol{w})_{\mathfrak{e}} := \int_{\mathfrak{e}} \boldsymbol{v} \cdot \boldsymbol{w}$ and $\langle \boldsymbol{p}, \boldsymbol{q} \rangle_{\mathfrak{e}} := \sum_{\mathfrak{n} \sim \mathfrak{e}} \boldsymbol{p}(\mathfrak{n}) \cdot \boldsymbol{q}(\mathfrak{n})$. The unit normals are denoted $\nu_{\mathfrak{e}}$ with $\nu_{\mathfrak{e}}(\mathfrak{n}_1) = -1$ and $\nu_{\mathfrak{e}}(\mathfrak{n}_2) = 1$.

Given $\boldsymbol{u}_{\mathfrak{n}} \in \mathbb{R}^3$ and $\boldsymbol{r}_{\mathfrak{n}} \in \mathbb{R}^3$ the weak form has unique solution $\boldsymbol{m}_{\mathfrak{e}}, \boldsymbol{n}_{\mathfrak{e}} \in V_{\boldsymbol{m}}^{\mathfrak{e}}$ and $\boldsymbol{u}_{\mathfrak{e}}, \boldsymbol{r}_{\mathfrak{e}} \in V_{\boldsymbol{u}}^{\mathfrak{e}}$.

# Continuity and balance conditions[5]

The network is represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.

1. *Continuity of solution:* $\boldsymbol{u}_{\mathfrak{e}}(\mathfrak{n}) = \boldsymbol{u}_{\mathfrak{n}}$ and $\boldsymbol{r}_{\mathfrak{e}}(\mathfrak{n}) = \boldsymbol{r}_{\mathfrak{n}}$
2. *Dirichlet boundary nodes:* $\boldsymbol{u}_{\mathfrak{n}} = \boldsymbol{u}_{\mathfrak{n}}^{D}$ and $\boldsymbol{r}_{\mathfrak{n}} = \boldsymbol{r}_{\mathfrak{n}}^{D}$, $\mathfrak{n} \in \mathcal{N}_D$
3. *Balance equations:* Let $[\![\cdot]\!]_{\mathfrak{n}}$ be a summation operator and $\nu_{\mathfrak{e}}$ the normal:

$$[\![\boldsymbol{n}_{\mathfrak{e}}\nu_{\mathfrak{e}}]\!]_{\mathfrak{n}} = \boldsymbol{f}_{\mathfrak{n}} \qquad [\![\boldsymbol{m}_{\mathfrak{e}}\nu_{\mathfrak{e}}]\!]_{\mathfrak{n}} = \boldsymbol{g}_{\mathfrak{n}},$$

[5] Lagnese et. at. Modeling, analysis and control of dynamic elastic multi-link structures, Birkhäuser Boston, 1994

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# Balance equation on weak form

For each edge $\mathfrak{e} \in \mathcal{E}$ we introduce the following maps:

$$\boldsymbol{N}_{\mathfrak{e}} : (\boldsymbol{u}_{\mathfrak{n}}, \boldsymbol{r}_{\mathfrak{n}}, \boldsymbol{f}_{\mathfrak{e}}, \boldsymbol{g}_{\mathfrak{e}}) \mapsto \boldsymbol{n}_{\mathfrak{e}}, \qquad \boldsymbol{M}_{\mathfrak{e}} : (\boldsymbol{u}_{\mathfrak{n}}, \boldsymbol{r}_{\mathfrak{n}}, \boldsymbol{f}_{\mathfrak{e}}, \boldsymbol{g}_{\mathfrak{e}}) \mapsto \boldsymbol{m}_{\mathfrak{e}},$$
$$\boldsymbol{U}_{\mathfrak{e}} : (\boldsymbol{u}_{\mathfrak{n}}, \boldsymbol{r}_{\mathfrak{n}}, \boldsymbol{f}_{\mathfrak{e}}, \boldsymbol{g}_{\mathfrak{e}}) \mapsto \boldsymbol{u}_{\mathfrak{e}}, \qquad \boldsymbol{R}_{\mathfrak{e}} : (\boldsymbol{u}_{\mathfrak{n}}, \boldsymbol{r}_{\mathfrak{n}}, \boldsymbol{f}_{\mathfrak{e}}, \boldsymbol{g}_{\mathfrak{e}}) \mapsto \boldsymbol{r}_{\mathfrak{e}}.$$

The balance equations in the nodes then reads

$$[\![\boldsymbol{N}_{\mathfrak{e}}(\boldsymbol{u}_{\mathfrak{n}}, \boldsymbol{r}_{\mathfrak{n}}, \boldsymbol{f}_{\mathfrak{e}}, \boldsymbol{g}_{\mathfrak{e}})\nu_{\mathfrak{e}}]\!]_{\mathfrak{n}} = \boldsymbol{f}_{\mathfrak{n}} \qquad [\![\boldsymbol{M}_{\mathfrak{e}}(\boldsymbol{u}_{\mathfrak{n}}, \boldsymbol{r}_{\mathfrak{n}}, \boldsymbol{f}_{\mathfrak{e}}, \boldsymbol{g}_{\mathfrak{e}})\nu_{\mathfrak{e}}]\!]_{\mathfrak{n}} = \boldsymbol{g}_{\mathfrak{n}},$$

Multiplication with test functions and summation over nodes yields

$$\sum_{\mathfrak{n} \in \mathcal{N} \setminus \mathcal{N}_D} \left( [\![\boldsymbol{N}_{\mathfrak{e}}(\boldsymbol{u}_{\mathfrak{n}}, \boldsymbol{r}_{\mathfrak{n}}, \boldsymbol{f}_{\mathfrak{e}}, \boldsymbol{g}_{\mathfrak{e}})\nu_{\mathfrak{e}}]\!]_{\mathfrak{n}} \cdot \boldsymbol{\mu}_{\mathfrak{n}} + [\![\boldsymbol{M}_{\mathfrak{e}}(\boldsymbol{u}_{\mathfrak{n}}, \boldsymbol{r}_{\mathfrak{n}}, \boldsymbol{f}_{\mathfrak{e}}, \boldsymbol{g}_{\mathfrak{e}})]\!]_{\mathfrak{n}} \cdot \boldsymbol{\psi}_{\mathfrak{n}} \right)$$
$$= \sum_{\mathfrak{n} \in \mathcal{N} \setminus \mathcal{N}_D} \left( \boldsymbol{f}_{\mathfrak{n}} \cdot \boldsymbol{\mu}_{\mathfrak{n}} + \boldsymbol{g}_{\mathfrak{n}} \cdot \boldsymbol{\psi}_{\mathfrak{n}} \right)$$

# Hybridized formulation

Let $V_\lambda$ be the space of vector valued functions defined on the nodes $\mathcal{N}$ fulfilling homogeneous Dirichlet boundary conditions.

Find $\boldsymbol{u}_\mathfrak{n} = \lambda_\mathfrak{n} + \boldsymbol{u}_\mathfrak{n}^D$ and $\boldsymbol{r}_\mathfrak{n} = \boldsymbol{\phi}_\mathfrak{n} + \boldsymbol{r}_\mathfrak{n}^D$, $(\lambda_\mathfrak{n}, \boldsymbol{\phi}_\mathfrak{n}) \in V_\lambda \times V_\lambda$, such that

$$A((\lambda_\mathfrak{n}, \boldsymbol{\phi}_\mathfrak{n}), (\boldsymbol{\mu}, \boldsymbol{\psi})) = F((\boldsymbol{\mu}, \boldsymbol{\psi})), \quad (\boldsymbol{\mu}, \boldsymbol{\psi}) \in V_\lambda \times V_\lambda$$

where

$$A((\lambda_\mathfrak{n}, \boldsymbol{\phi}_\mathfrak{n}), (\boldsymbol{\mu}, \boldsymbol{\psi})) := -\sum_{\mathfrak{n} \in \mathcal{N} \setminus \mathcal{N}_D} \left( [\![ \boldsymbol{N}_\mathfrak{e}(\lambda_\mathfrak{n}, \boldsymbol{\phi}_\mathfrak{n}) \nu_\mathfrak{e} ]\!]_\mathfrak{n} \cdot \boldsymbol{\mu}_\mathfrak{n} + [\![ \boldsymbol{M}_\mathfrak{e}(\lambda_\mathfrak{n}, \boldsymbol{\phi}_\mathfrak{n}) \nu_\mathfrak{e} ]\!]_\mathfrak{n} \cdot \boldsymbol{\psi}_\mathfrak{n} \right)$$

$$\begin{aligned} F((\boldsymbol{\mu}, \boldsymbol{\psi})) := \sum_{\mathfrak{n} \in \mathcal{N} \setminus \mathcal{N}_D} & \left( [\![ \boldsymbol{N}_\mathfrak{e}(\boldsymbol{u}_\mathfrak{n}^D, \boldsymbol{r}_\mathfrak{n}^D, \boldsymbol{f}_\mathfrak{e}, \boldsymbol{g}_\mathfrak{e}) \nu_\mathfrak{e} ]\!]_\mathfrak{n} - \boldsymbol{f}_\mathfrak{n} \right) \cdot \boldsymbol{\mu}_\mathfrak{n} \\ & + \left( [\![ \boldsymbol{M}_\mathfrak{e}(\boldsymbol{u}_\mathfrak{n}^D, \boldsymbol{r}_\mathfrak{n}^D, \boldsymbol{f}_\mathfrak{e}, \boldsymbol{g}_\mathfrak{e}) \nu_\mathfrak{e} ]\!]_\mathfrak{n} - \boldsymbol{g}_\mathfrak{n} \right) \cdot \boldsymbol{\psi}_\mathfrak{n}. \end{aligned}$$

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A hybrid formulation where the unknowns $(\lambda_\mathfrak{n}, \boldsymbol{\phi}_\mathfrak{n})$ sits on nodes connecting the subdomains (beams).

- *Primal variables:* $\boldsymbol{u}_\mathfrak{e}, \boldsymbol{r}_\mathfrak{e} \in V_{\boldsymbol{u}}^\mathfrak{e}$ for all $\mathfrak{e} \in \mathcal{E}$
- *Dual variables:* $\boldsymbol{m}_\mathfrak{e}, \boldsymbol{n}_\mathfrak{e} \in V_{\boldsymbol{m}}^\mathfrak{e}$ for all $\mathfrak{e} \in \mathcal{E}$
- *Hybrid variables:* $\boldsymbol{u}_\mathfrak{n}, \boldsymbol{r}_\mathfrak{n} \in \mathbb{R}^3$ for all $\mathfrak{n} \in \mathcal{N}$

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$$A((\lambda_\mathfrak{n}, \boldsymbol{\phi}_\mathfrak{n}), (\mu, \psi)) = F((\mu, \psi)), \quad (\mu, \psi) \in V_\lambda \times V_\lambda$$

- One global problem with $6|\mathcal{N} \setminus \mathcal{N}_D|$ dofs plus independent local problems on all edges $\mathfrak{e} \in \mathcal{E}$.
- $A$ is symmetric and coercive, consequently the weak form is well posed
- Only local solver need to be discretized
- With constant data local problems can be solved analytically[6]

[6] Kufner et. al. Simulation and structural optimization of 3D Timoshenko beam networks based on fully analytical network solutions, M2AN, (2018)

# HDG[7] formulation

The space of polynomials of degree at most $p$ is denoted $\mathbb{P}_p(\mathfrak{e})$ and we let $V_p^{\mathfrak{e}} := (\mathbb{P}_p(\mathfrak{e}))^3$. The discrete method seeks

- $\bar{\boldsymbol{u}}_{\mathfrak{e}}, \bar{\boldsymbol{r}}_{\mathfrak{e}} \in V_p^{\mathfrak{e}}$ for all edges $\mathfrak{e} \in \mathcal{E}$
- $\bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{m}}_{\mathfrak{e}} \in V_p^{\mathfrak{e}}$ for all edges $\mathfrak{e} \in \mathcal{E}$
- $\bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{r}}_{\mathfrak{n}} \in \mathbb{R}^3$ for all $\mathfrak{n} \in \mathcal{N} \setminus \mathcal{N}_D$ (continuity imposed weakly)

such that the discrete balance equations hold

$$[\![\bar{\boldsymbol{n}}_{\mathfrak{e}} \nu_{\mathfrak{e}} + \tau_{\mathfrak{e}}(\bar{\boldsymbol{u}}_{\mathfrak{e}} - \bar{\boldsymbol{u}}_{\mathfrak{n}})]\!]_n = f_{\mathfrak{n}}, \qquad [\![\bar{\boldsymbol{m}}_{\mathfrak{e}} \nu_{\mathfrak{e}} + \tau_{\mathfrak{e}}(\bar{\boldsymbol{r}}_{\mathfrak{e}} - \bar{\boldsymbol{r}}_{\mathfrak{n}})]\!]_n = g_{\mathfrak{n}},$$

with penalty parameter $\tau_{\mathfrak{e}} > 0$, and the local equations...

[7] Rupp et. al. PDEs on hypergraphs and networks of surfaces, M2AN, (2022)

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- $\bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{r}}_{\mathfrak{n}} \in \mathbb{R}^3$ for all $\mathfrak{n} \in \mathcal{N} \setminus \mathcal{N}_D$ (continuity imposed weakly)

For all $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{w}} \in V_p^{\mathfrak{e}}$:

$$\begin{aligned}
-(C_{\boldsymbol{n}}^{-1}\bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{p}})_{\mathfrak{e}} \quad &+ (\bar{\boldsymbol{u}}_{\mathfrak{e}}, \partial_x \bar{\boldsymbol{p}})_{\mathfrak{e}} - (\boldsymbol{i}_{\mathfrak{e}} \times \bar{\boldsymbol{r}}_{\mathfrak{e}}, \boldsymbol{p})_{\mathfrak{e}} &&= \langle \bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{p}}\nu_{\mathfrak{e}}\rangle_{\mathfrak{e}}, \\
-(C_{\boldsymbol{m}}^{-1}\bar{\boldsymbol{m}}_{\mathfrak{e}}, \bar{\boldsymbol{q}})_{\mathfrak{e}} \quad &+ (\bar{\boldsymbol{r}}_{\mathfrak{e}}, \partial_x \bar{\boldsymbol{q}})_{\mathfrak{e}} &&= \langle \bar{\boldsymbol{r}}_{\mathfrak{n}}, \bar{\boldsymbol{q}}\nu_{\mathfrak{e}}\rangle_{\mathfrak{e}}, \\
(\partial_x \bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{v}})_{\mathfrak{e}} \quad &+ \tau_{\mathfrak{e}}\langle \bar{\boldsymbol{u}}_{\mathfrak{e}}, \bar{\boldsymbol{v}}\rangle_{\mathfrak{e}} &&= (\boldsymbol{f}_{\mathfrak{e}}, \bar{\boldsymbol{v}})_{\mathfrak{e}} + \tau_{\mathfrak{e}}\langle \bar{\boldsymbol{u}}_{\mathfrak{n}}, \bar{\boldsymbol{v}}\rangle_{\mathfrak{e}}, \\
(\boldsymbol{i}_{\mathfrak{e}} \times \bar{\boldsymbol{n}}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}} + (\partial_x \bar{\boldsymbol{m}}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}} \quad &+ \tau_{\mathfrak{e}}\langle \bar{\boldsymbol{r}}_{\mathfrak{e}}, \bar{\boldsymbol{w}}\rangle_{\mathfrak{e}} &&= (\boldsymbol{g}_{\mathfrak{e}}, \bar{\boldsymbol{w}})_{\mathfrak{e}} + \tau_{\mathfrak{e}}\langle \bar{\boldsymbol{r}}_{\mathfrak{n}}, \bar{\boldsymbol{w}}\rangle_{\mathfrak{e}}.
\end{aligned}$$

The local solver is well posed for $\tau_{\mathfrak{e}} > 0$.

[7] Rupp et. al. PDEs on hypergraphs and networks of surfaces, M2AN, (2022)

# A priori error bound[8]

## Theorem (Convergence of HDG method)

*If* $\tau_\mathfrak{e} \sim h_\mathfrak{e}^s$ *for some* $s \in \{-1, 0, 1\}$ *and* $\boldsymbol{u}_\mathfrak{e}, \boldsymbol{r}_\mathfrak{e}, \boldsymbol{n}_\mathfrak{e}, \boldsymbol{m}_\mathfrak{e} \in H^{p+1}(\mathfrak{e})$ *for all* $\mathfrak{e} \in \mathcal{E}$*, then it holds*

$$\left[\sum_{\mathfrak{e}\in\mathcal{E}} \left[\|\boldsymbol{u}_\mathfrak{e} - \bar{\boldsymbol{u}}_\mathfrak{e}\|_\mathfrak{e}^2 + \|\boldsymbol{r}_\mathfrak{e} - \bar{\boldsymbol{r}}_\mathfrak{e}\|_\mathfrak{e}^2\right]\right]^{1/2} \lesssim h^{p+1-s^+},$$

$$\left[\sum_{\mathfrak{e}\in\mathcal{E}} \left[\|\boldsymbol{n}_\mathfrak{e} - \bar{\boldsymbol{n}}_\mathfrak{e}\|_\mathfrak{e}^2 + \|\boldsymbol{m}_\mathfrak{e} - \bar{\boldsymbol{m}}_\mathfrak{e}\|_\mathfrak{e}^2\right]\right]^{1/2} \lesssim h^{p+1-|s|},$$

*where* $s^+ := \max(s, 0)$.

[8] Rupp, Hauck, M., Arbitrary order approximations at constant cost for Timoshenko beam network models, arXiv:2407.14388

# Numerical example (convergence)

- Toy problem: 2D unit cross $(+)$ embedded in 3D,

$$\big([-1,1]\times\{0\}\cup\{0\}\times[-1,1]\big)\times\{0\}$$

- Four edges before refinement
- Dirichlet bc at the tips of the cross
- $C_{\boldsymbol{n}} = C_{\boldsymbol{m}} = I$
- Manufactured solution, forces chosen so that

$$\boldsymbol{u}(x,y,z) = \begin{pmatrix} 0 \\ \cos(\pi y) \\ \cos(\pi x) \end{pmatrix}, \qquad \boldsymbol{r}(x,y,z) = \begin{pmatrix} 0 \\ \sin(\pi x) \\ \sin(\pi y) \end{pmatrix}$$

# Numerical example (convergence)

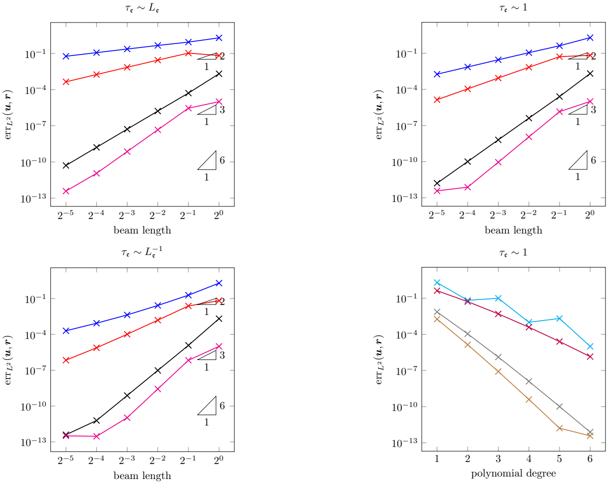


Figure: Poly. deg. 1 (blue), 2 (red), 5 (black), and 6 (magenta) . Bottom right: beam lengths 1 (cyan), $2^{-1}$ (purple), $2^{-4}$ (gray), and $2^{-5}$ (brown).

# Important property of the hybrid formulation

*Recall the formulation:*

Let $V_\lambda$ be the space of vector valued functions defined on the nodes $\mathcal{N}$ fulfilling homogeneous Dirichlet boundary conditions.

Find $\boldsymbol{u}_\mathfrak{n} = \lambda_\mathfrak{n} + \boldsymbol{u}_\mathfrak{n}^D$ and $\boldsymbol{r}_\mathfrak{n} = \boldsymbol{\phi}_\mathfrak{n} + \boldsymbol{r}_\mathfrak{n}^D$, $(\lambda_\mathfrak{n}, \boldsymbol{\phi}_\mathfrak{n}) \in V_\lambda \times V_\lambda$, such that

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A key observation in the convergence analysis is that:

$A$ is spectrally equivalent to a weighted graph Laplacian

# Graph Laplacian and norms

- Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a graph of nodes and edges, $x \in \Omega \subset \mathbb{R}^3$
- Let $\hat{V} : \mathcal{N} \to \mathbb{R}$ be scalar functions on $\mathcal{N}$. For $v, w \in \hat{V}$

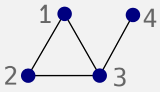
$$(v, w) = \sum_x v(x)w(x)$$

$$(L^g v, v) = \sum_{(x,y)\in\mathcal{E}} (v(x) - v(y))^2$$

$$(Lv, v) = \sum_{(x,y)\in\mathcal{E}} \frac{(v(x) - v(y))^2}{|x - y|}$$

$$|v|_L = (Lv, v)^{1/2}$$

Example:




$$L^g = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

# Graph Laplacian and norms

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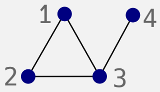


$$L^g = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

- Let $M$ be diagonal with $M_{xx} = \frac{1}{2}\sum_{y\sim x}|x - y|$, $|v|_M = (Mv, v)^{1/2}$
- For a P1-FEM function $v_h \in V_h$ on a mesh of $[a, b] \subset \mathbb{R}$ we have $|v_h|_{H^1(\Omega)} = |v_h|_L$. $M$ is the lumped mass matrix.

# Spectral equivalence

## Theorem (Spectral equivalence to graph Laplacian)

*Assume that the maximal edge length is sufficiently small and that the material coefficients $C_n$ and $C_m$ are edgewise constant. Then, there holds for all $(\lambda, \phi) \in V_\lambda \times V_\lambda$ that*

$$L(\lambda,\lambda) + L(\phi,\phi) \lesssim A((\lambda,\phi),(\lambda,\phi)) \lesssim L(\lambda,\lambda) + L(\phi,\phi),$$

*where the hidden constants depend material data and on the reciprocal of $\lambda_{\min} := \min_{\mu \in V_\lambda \setminus \{0\}} \frac{L(\mu,\mu)}{M(\mu,\mu)}$.*

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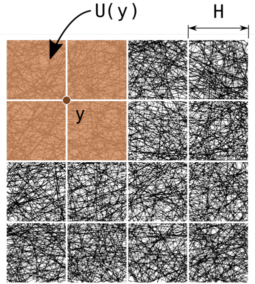
# Geometric coarsening[9]




- $\mathcal{T}_H$ is a mesh of boxes
- $\hat{V}_H$ is Q1-FEM with basis $\{\varphi_y\}_y$
- $V_H \subset \hat{V}_H$ satisfy the boundary conditions
- Clément type interpolation operator

$$\mathcal{I}_H v = \sum_{\text{free DoFs } y} \bar{v}_{U(y)} \varphi_y \in V_H$$

## Lemma (Stability and approximability of $\mathcal{I}_H$)

*For all $v \in V$ and for $H \geq R_0 > 0$,*

$$H^{-1} |v - \mathcal{I}_H v|_M + |\mathcal{I}_H v|_L \leq C |v|_L,$$

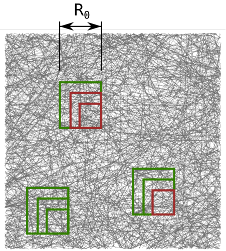
*where $C = C_d \mu \sqrt{\sigma}$.*

[9] Görtz, Hellman, M., Iterative solution of spatial network models by subspace decomposition, Math. Comp. (2024)

# Network homogeneity

The network must resemble a homogeneous material on coarse scales $H \geq R_0$.

1. *Homogeneity:* Let $B_H(x)$ be a box at $x$ of side length $2H$, with $H \geq R_0$. We assume limited mass variation

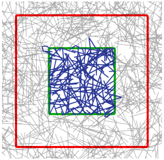



$$1 \leq \frac{\max_x |1|^2_{M,B_H(x)}}{\min_x |1|^2_{M,B_H(x)}} \leq \sigma(R_0)$$

Limited density variation on scales larger than $R_0$.

# Network connectivity

2. *Connectivity:* For all $H > R_0$ and $x \in \Omega$ there is a connected subgraph $\mathcal{G}'$ that contains



- all edges with one endpoint in $B_H(x)$
- only edges with endpoints contained in $B_{H+R_0}(x)$

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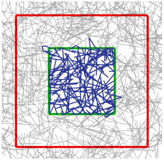


- all edges with one endpoint in $B_H(x)$
- only edges with endpoints contained in $B_{H+R_0}(x)$

Consider $L'\phi = \lambda M'\phi$, $\lambda_1 = 0$, $\lambda_2 > 0$ (Algebraic connectivity[10]):

$$|v - \bar{v}|_{M,B_H} \leq |v - \bar{v}|_{M'} \leq \lambda_2^{-1/2}|v - \bar{v}|_{L'} \leq \lambda_2^{-1/2}|v|_{L,B_{H+R_0}}$$

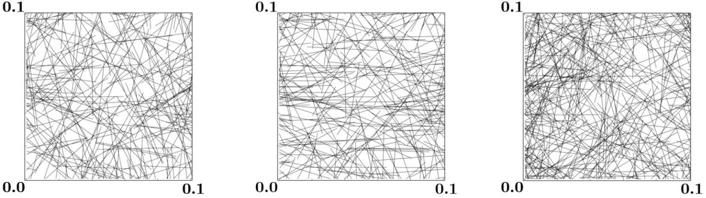
If $\mathcal{G}'$ fulfills an iso-perimetric inequality $\lambda_2 \sim H^{-2}$ and therefore

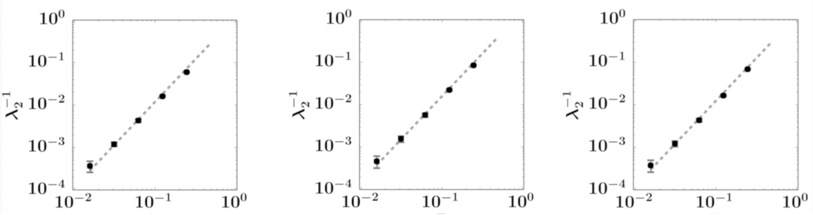
$$\lambda_2^{-1/2} = \mu(R_0)H$$

[10] Chung, Spectral graph theory, AMS, 1997

# Example: Connectivity $\lambda_2^{-1/2} \approx \mu H$

Finite length fibers $r = 0.05$ and $|1|_M^2 = 1000$, $\Omega = [0, 1]^2$



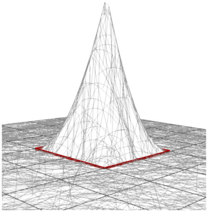

$H$ varies from $2^{-2}$ to $2^{-6}$. Here $R_0 \sim 2^{-6}$.

# Subspace decomposition preconditioner[11]

Let $V_\lambda = V_{\lambda,0} + V_{\lambda,1} + \cdots + V_{\lambda,m}$ with

$$V_{\lambda,0} := V_H \times V_H \times V_H$$

$$V_{\lambda,i} := \{\boldsymbol{v} \in V_\lambda : \ \text{supp}(\boldsymbol{v}) \subset U_i\}$$



Define $P_i : \ V_\lambda \times V_\lambda \to V_{\lambda,i} \times V_{\lambda,i}$ such that

$$(AP_i(\lambda, \phi), (\mu, \psi)) = (A(\lambda, \phi), (\mu, \psi))$$

for all $(\mu, \psi)$ and form $P := P_0 + P_1 + \cdots + P_m$.

- $BAz = BF$, with preconditioner $P = BA$ and $z = (\lambda, \phi)$
- Preconditioned conjugate gradient method.
- Semi-iterative: direct method on decoupled problems

[11] Kornhuber & Yserentant, Numerical homogenization of elliptic multiscale problmes by subspace decomposition, MMS, 2016

# Convergence analysis

## Lemma (Properties of the decomposition)

*If the interpolation bound holds and $A$ is spectrally equivalent to the weighted Graph Laplacian with constants $\alpha$ and $\beta$, then for $H > 2R_0$*

*At least one decomposition $v = \sum_{j=0}^{m} v_j$ satisfies: $\sum_{j=0}^{m} |v_j|_A^2 \leq C_1 |v|_A^2$*

*Every decomposition satisfies: $|v|_A^2 \leq C_2 \sum_{j=0}^{m} |v_j|_A^2$*

*The constants are $C_1 = C_d \beta \alpha^{-1} \sigma \mu^2$ and $C_2 = C_d$.*

## Theorem (Convergence of PCG)

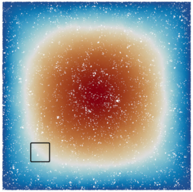
*With $\kappa = C_1 C_2$, $H > 2R_0$, and $z = (\lambda, \phi)$ it holds*

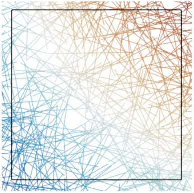
$$|z - z^{(\ell)}|_A \leq 2 \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^{\ell} |z - z^{(0)}|_A .$$

# Outline

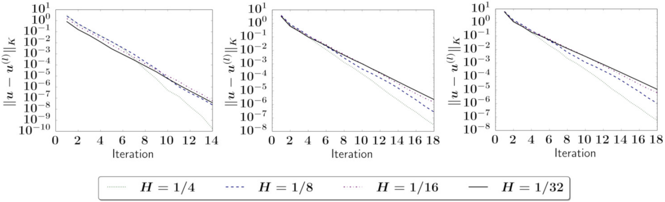
# Example: Convergence graph Laplacian

$Ku = M1$, $(Kv, v) = \sum_{x \sim y} \gamma_{xy} \frac{(v(x)-v(y))^2}{|x-y|}$ , $u|_{\partial\Omega} = 0$ , $|1|_M^2 = 1000.$

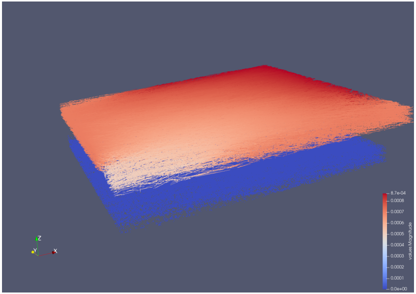




Grid $\gamma = 1$ (left), rand $\gamma = 1$ (center), rand $\gamma \in U([0.1, 1])$ (right)



# Example: Elastic deformation of paper




- 4 mm x 4 mm paper
- 615K edges and 424K nodes
- We study stretching of the paper caused by Dirichlet boundary conditions (upper right)
- HDG discretization with $p = 5$ and $\tau = 1$
- Preconditioner with $8 \times 8 \times 1$ element in coarse space

# Example: Elastic deformation of paper

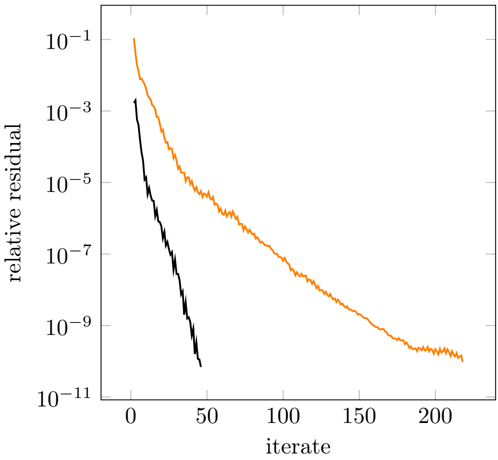
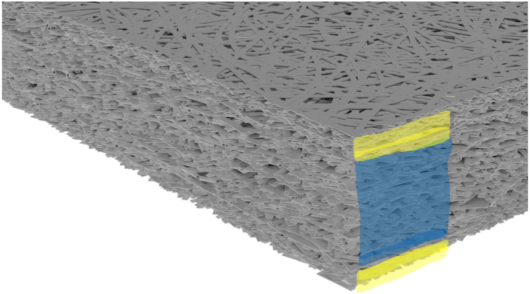


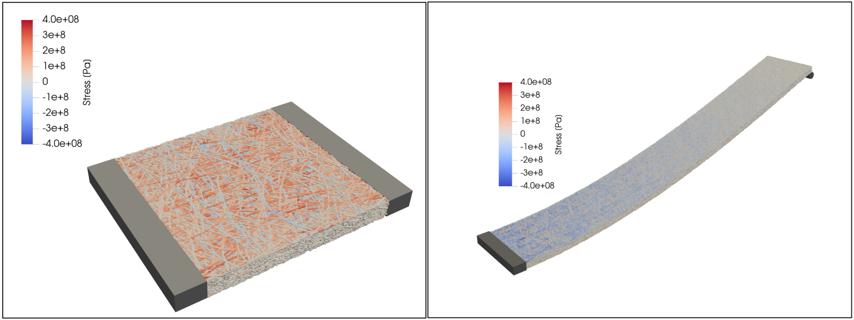

Figure: Convergence of PCG: constant material parameters (black) and realistic (orange).

# Engineering application (FCC/Stora Enso)



- Three-ply paperboard
- Grammage: $400g/m^2$
- Measure: (tensile) 4mm $\times$ 4mm (bending) 50mm $\times$ 4mm
- Dofs: (tensile) 16M (bending) 200M

# Engineering application (FCC/Stora Enso)




- Solver converges in 60 iterations (practical purposes)
- Validated on various commercial paperboards
- Results consistent with experimental data[12]

[12] Görtz et. al., Iterative method for large-scale Timoshenko beam models assessed on commercial-grade paperboard, Computational Mechanics (2024)

# Outline

# Conclusions and future works

Robust iterative approach to solve spatial network models with applications in the paper industry

1. Görtz, Hellman, M., *Iterative solution of spatial network models by subspace decomposition,* Math. Comp. (2024)
2. Rupp, Hauck, M., *Arbitrary order approximations at constant cost for Timoshenko beam network models,* arXiv:2407.14388
3. Görtz et. al., *Iterative method for large-scale Timoshenko beam models assessed on commercial-grade paperboard,* Computational Mechanics (2024)

# Conclusions and future works

Robust iterative approach to solve spatial network models with applications in the paper industry

Future work:

- $\delta$-overlap in DD
- Algebraic coarsening
- Multilevel preconditioner
- Elastic wave propagation
- Large deformation, non-linear models