Generalized Finite Element Methods

Axel Målqvist[1]
Daniel Peterseim and Fredrik Hellman

16th European Finite Element Fair

University of Heidelberg

2018-06-08

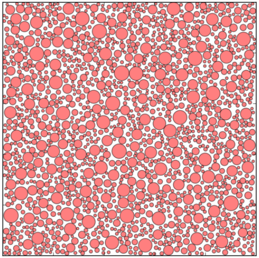
[1] Chalmers University of Technology and University of Gothenburg

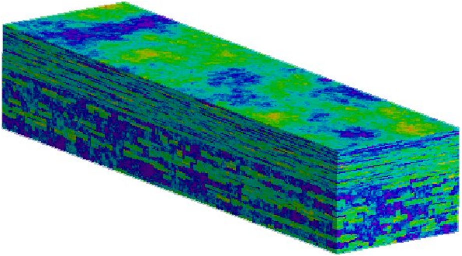
Elliptic model problem

The Poisson equation

$$-\nabla \cdot (A \nabla u) = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

with data $0 < \alpha \leq A \leq \beta < \infty$ and $f \in L^2(\Omega)$.





Elliptic model problem

FEM: $u_h \in V_h \subset V$ such that

$$a(u_h, v) := \int_\Omega (A \nabla u_h) \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx \quad \text{for all } v \in V_h$$

with data $0 < \alpha \leq A \leq \beta < \infty$ and $f \in L^2(\Omega)$.

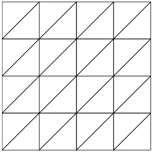
Numerical error (piecewise linear continuous FE approximation)

- For solution $u \in H^2(\Omega)$ we have for ϵ-periodic $A = A(x/\epsilon)$

$$|||u - u_h||| := \|A^{1/2} \nabla (u - u_h)\|_{L^2(\Omega)} \sim C(\alpha, \beta) \frac{h}{\epsilon},$$

Can we do better?

Decomposition of scales



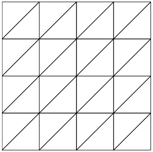
- (coarse) P1-FE space $V_H \subset V$ so $H > h$
- $\mathfrak{I}_H : V \to V_H$ some FE interpolation operator

Decomposition

$$V = V_H \oplus V^{\mathrm{f}} \quad \text{with } V^{\mathrm{f}} := \text{kernel}\, \mathfrak{I}_H = \{v \in V \mid \mathfrak{I}_H v = 0\}$$

Decomposition of scales

- (coarse) P1-FE space $V_H \subset V$ so $H > h$
- $\mathfrak{I}_H : V \to V_H$ some FE interpolation operator



Decomposition

$$V = V_H \oplus V^{\mathrm{f}} \quad \text{with } V^{\mathrm{f}} := \text{kernel}\, \mathfrak{I}_H = \{v \in V \mid \mathfrak{I}_H v = 0\}$$

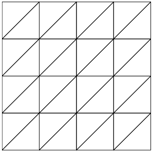
Let $\mathcal{R}_H : V \to V_H$ (Ritz projection) and $\mathcal{R}^{\mathrm{f}} : V \to V^{\mathrm{f}}$ fulfill

$$a(\mathcal{R}_H u, v) = a(u, v) \quad \forall v \in V_H,$$
$$a(\mathcal{R}^{\mathrm{f}} u, v) = a(u, v) \quad \forall v \in V^{\mathrm{f}}.$$

$a(V_H, V^{\mathrm{f}}) \neq 0$ but $a(V - \mathcal{R}^{\mathrm{f}} V, V^{\mathrm{f}}) = a(V_H - \mathcal{R}^{\mathrm{f}} V_H, V^{\mathrm{f}}) = 0$

Decomposition of scales

- (coarse) P1-FE space $V_H \subset V$ so $H > h$
- $\mathfrak{I}_H : V \to V_H$ some FE interpolation operator



a-Orthogonal Decomposition

$$V = V_H^{\text{ms}} \oplus V^{\text{f}} \quad \text{with } V_H^{\text{ms}} := (V_H - \mathcal{R}^f V_H)$$

Let $\mathcal{R}_H^{\text{ms}} : V \to V_H^{\text{ms}}$ (multiscale projection) and $\mathcal{R}^f : V \to V^{\text{f}}$ fulfill

$$a(\mathcal{R}_H^{\text{ms}} u, v) = a(u, v) \quad \forall v \in V_H^{\text{ms}},$$
$$a(\mathcal{R}^f u, v) = a(u, v) \quad \forall v \in V^{\text{f}}.$$

$a(V_H^{\text{ms}}, V^{\text{f}}) = 0 : u = \mathcal{R}_H^{\text{ms}} u + \mathcal{R}^f u$ and therefore $\mathfrak{I}_H(u - \mathcal{R}_H^{\text{ms}} u) = 0$.

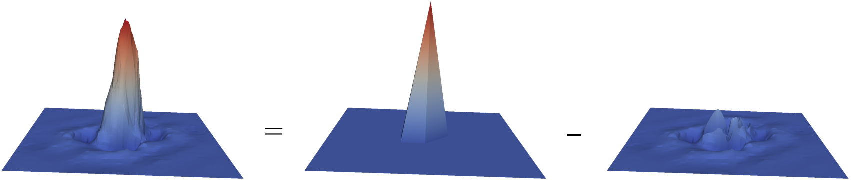
Generalized Finite Elements

- $\phi_x \in V_H$ denotes classical nodal basis function
- $\mathcal{R}^f \phi_x \in V^f$ denotes the finescale correction of ϕ_x

Generalized FE space

$$V_H^{ms} = \text{span}\left\{\phi_x - \mathcal{R}^f \phi_x\right\}$$

Example



$\phi_x - \mathcal{R}^f \phi_x \in V_H^{ms}$ $=$ $\phi_x \in V_H$ $-$ $\mathcal{R}^f \phi_x \in V^f$

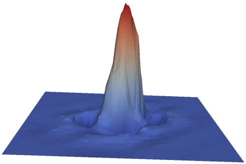
Generalized Finite Elements

- $\phi_x \in V_H$ denotes classical nodal basis function
- $\mathcal{R}^f \phi_x \in V^f$ denotes the finescale correction of ϕ_x

Generalized FE space

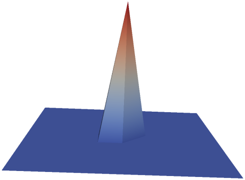
$$V_H^{ms} = \text{span}\left\{\phi_x - \mathcal{R}^f \phi_x\right\}$$

Example



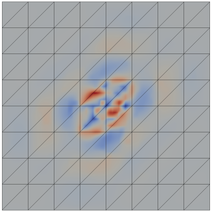
$\phi_x - \mathcal{R}^f \phi_x \in V_H^{ms}$

=



$\phi_x \in V_H$

−



$\mathcal{R}^f \phi_x \in V^f$

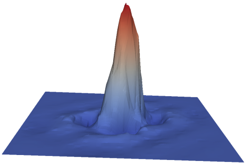
Generalized Finite Elements

- $\phi_x \in V_H$ denotes classical nodal basis function
- $\mathcal{R}^{\mathrm{f}}\phi_x \in V^{\mathrm{f}}$ denotes the finescale correction of ϕ_x

Generalized FE space

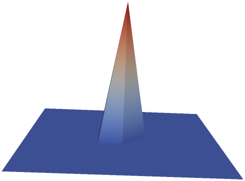
$$V_{H,k}^{\mathrm{ms}} = \mathrm{span}\left\{\phi_x - \mathcal{R}_k^{f}\phi_x\right\}$$

Example



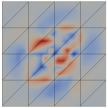
$\phi_x - \mathcal{R}^{f}\phi_x \in V_H^{\mathrm{ms}}$

$=$



$\phi_x \in V_H$

$-$



$\mathcal{R}_k^{f}\phi_x \in V^{\mathrm{f}}$

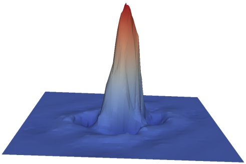
Generalized Finite Elements

- $\phi_x \in V_H$ denotes classical nodal basis function
- $\mathcal{R}^f \phi_x \in V^f$ denotes the finescale correction of ϕ_x

Generalized FE space

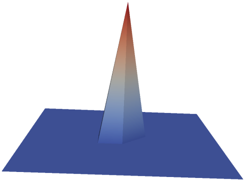
$$V_{H,k}^{\text{ms},h} = \text{span}\left\{\phi_x - \mathcal{R}^f_{k,h}\phi_x\right\}$$

Example



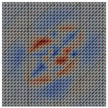
$\phi_x - \mathcal{R}^f\phi_x \in V_H^{\text{ms}}$

$=$



$\phi_x \in V_H$

$-$



$\mathcal{R}^f_{k,h}\phi_x \in V^f$

Generalized Finite Elements

- $\phi_x \in V_H$ denotes classical nodal basis function
- $\mathcal{R}^f \phi_x \in V^f$ denotes the finescale correction of ϕ_x

Generalized FE space

$$V_{H,k}^{\mathrm{ms},h} = \mathrm{span}\left\{\phi_x - \mathcal{R}_{k,h}^f \phi_x\right\}$$

Localized Orthogonal Decomposition

Find $u_{H,k}^{\mathrm{ms},h} \in V_{H,k}^{\mathrm{ms},h}$ such that $a(u_{H,k}^{\mathrm{ms},h}, v) = (f, v), \quad$ for all $v \in V_{H,k}^{\mathrm{ms},h}$

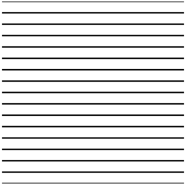
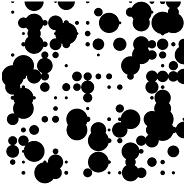
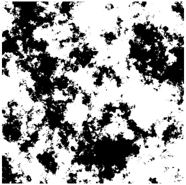
A priori bound:

$$|||u_h - u_{H,k}^{\mathrm{ms},h}||| \leq C(\alpha,\beta)H$$

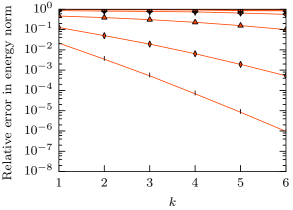
where $k = C_1(\beta/\alpha)\log(H^{-1})$ and C independent of A'.

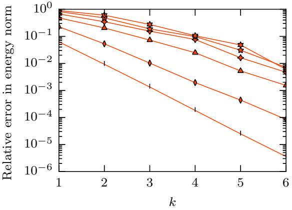
The high contrast problem

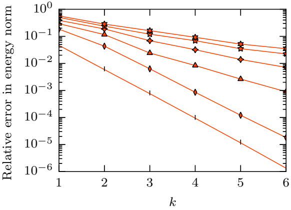
Three examples, black $A = 1$ white $A = \alpha$: $H = 2^{-4}$, $h = 2^{-10}$.

Let $\alpha = 10^{-1}, \ldots, 10^{-6}$ and plot $|||u_h - u_{H,k}^{\mathrm{ms},h}|||$ vs. k, with $\mathfrak{I}_H^{\omega}$,

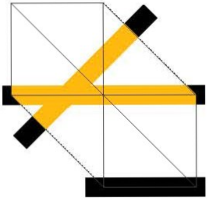






Geometry dependent interpolation

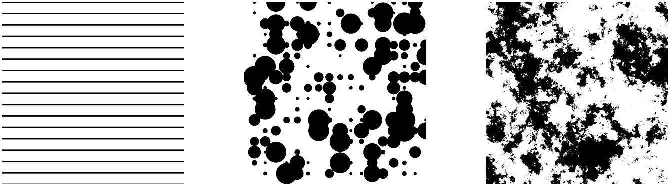
- The interpolant $\Im_H v = \sum_{x \in \mathcal{N}} \bar{v}_{\sigma_x} \phi_x$ defines V_f and V_H^{ms}.
- We need to force correctors to be small in the channels!



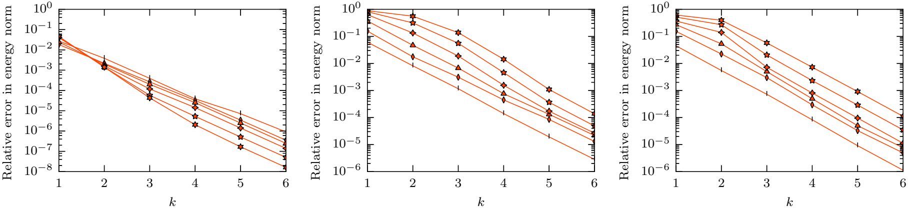
1. If $x \in \Omega_\alpha$ let $\sigma_x = \omega_x$, vertex patch
2. If $x \in \Omega_1$ let $\sigma_x \subset \omega_x \cap \Omega_1$, connected
3. We need sufficiently many nodes in Ω_1 (separation $\sim H$)

Numerical example: High contrast

High contrast data Three examples: $H = 2^{-4}$, $h = 2^{-10}$,



We let $\alpha = 10^{-1}, \dots, 10^{-6}$ and plot $|||u_h - u_{H,k}^{\mathrm{ms},h}|||$ vs. k with $\mathfrak{I}_H^{\sigma}$,



Concluding remarks

- Orthogonal subspaces treats rapidly varying data[2].
- High contrast channels is challenging, geometry depended interpolation is a way forward[3].

Thank you for your attention!

[2] M. & Peterseim, *Localization of elliptic multiscale prob.*, Math. Comp. 2014

[3] Hellman & M., *Contrast independent localization of multiscale problems,* SIAM MMS 2017. (Related work done by Peterseim Scheichl 2016)