On convergence of a multiscale method

Axel Målqvist

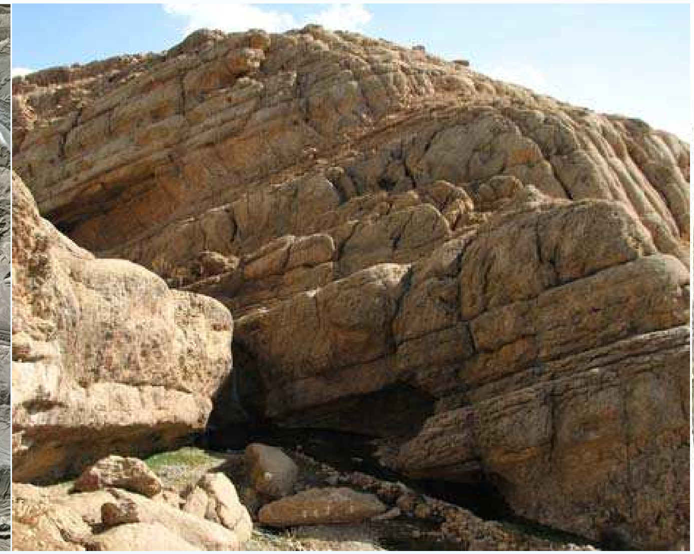
axel.malqvist@it.uu.se

Division of Scientific Computing

Uppsala University

Sweden

Motivating example: secondary oil recovery


Find pressure p and water concentration s such that:

$$-\nabla \cdot k\lambda(s)\nabla p = q, \quad \dot{s} - \nabla \cdot [f(s)\lambda(s)k\nabla p] = g, \quad \text{in } \Omega,$$

where k is permeability, $\lambda(s)$ the total mobility, f fractional flow, and g, q sink and source terms.

Outline and papers

Outline

- Model problem, elliptic
- Previous work
- Derivation of a multiscale method
- Convergence analysis, with numerical examples
- Adaptivity, with numerical examples
- Conclusions and future work

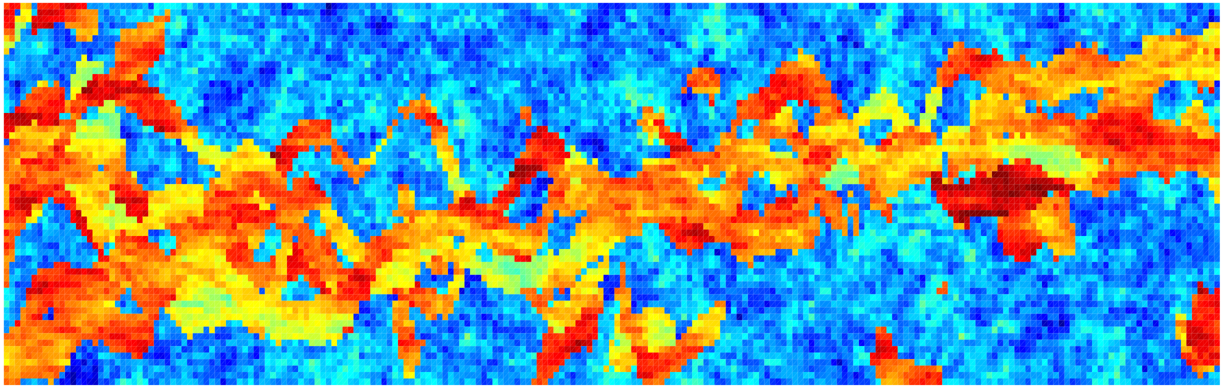
Papers

- M. G. Larson and A. Målqvist, *Adaptive variational multiscale methods based on a posteriori error estimation: energy norm estimates for elliptic problems*, CMAME 2007
- A. Målqvist, *A priori error analysis of a multiscale method*, preprint

Thanks

- M. G. Larson, Umeå, G. Tsogtgerel, McGill, D. Elfverson Uppsala, and D. Peterseim Humboldt.

Model problem



We consider the strong form:

$$-\nabla \cdot \alpha \nabla u = f, \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

The weak form reads: find $u \in \mathcal{V} := H_0^1(\Omega)$ such that,

$$a(u,v) := \int_\Omega \alpha \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx := l(v), \quad \text{for all } v \in \mathcal{V}.$$

We assume $f \in L^2(\Omega)$ and $0 < \alpha_0 \leq \alpha \in L^\infty(\Omega)$.

Why do we need to resolve the coefficients?

Example: Consider the Poisson equation

$$-\nabla \cdot \alpha \nabla u = f,$$

with coefficient α, with oscillations down to a scale ϵ, solved using the finite element method (mesh size H),

$$\|\sqrt{\alpha}\nabla(u - u_H)\|_{L^2(\Omega)} \lesssim H\|\Delta u\|_{L^2(\Omega)} \lesssim \epsilon^{-1} H\|f\|_{L^2(\Omega)},$$

- $\epsilon < H$ will give unreliable results even with exact quadrature.
- $\epsilon > H$ will be too computationally expensive to solve on a single mesh.

From now on we assume nothing on the coefficients, more than what is needed to guarantee existence and uniqueness.

Some previous works and related methods

- Upscaling techniques: Durlofsky et al. 98, Nielsen et al. 98
- Variational multiscale method: Hughes et al. 95, Arbogast 04, Larson-Målqvist 05, Nolen et al. 08, Nordbotten 09
- Multiscale finite element method: Hou-Wu 96, Efendiev-Ginting 04, Aarnes-Lie 06
- Multiscale finite volume method: Jenny et al. 03
- Heterogeneous multiscale method: Engquist-E 03, E-Ming-Zhang 04
- Equation free: Kevrekidis et al. 05
- ...

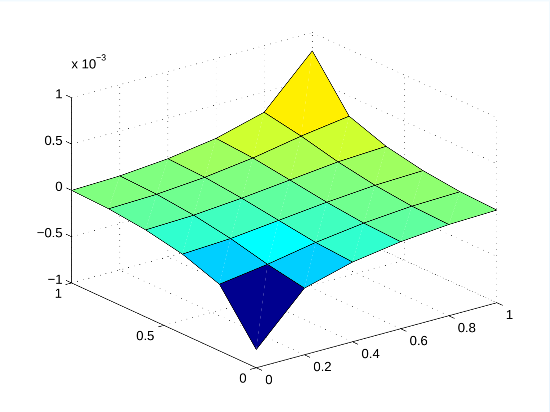
Local approximations (in parallel) on a fine scale are used to modified a coarse scale equation.

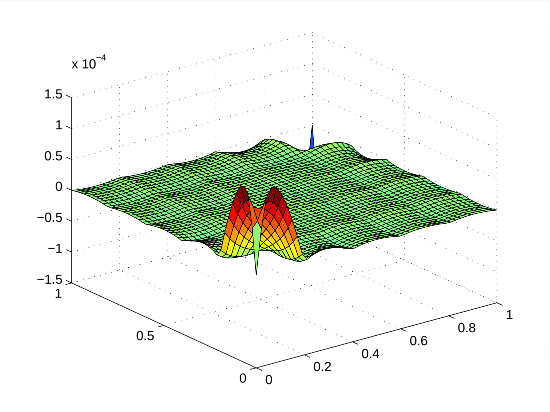
The variational multiscale method (VMS)

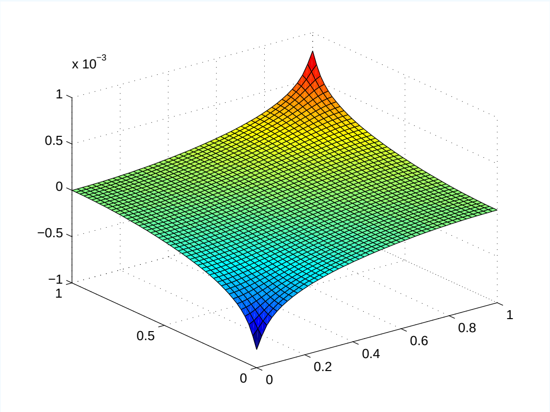
The weak form reads: find $u \in \mathcal{V}$ such that,

$$a(u,v) = l(v), \quad \forall v \in \mathcal{V}.$$

Now let $\mathcal{V}_c \oplus \mathcal{V}_f = \mathcal{V}$.







- $\mathcal{V}_c$ is a finite dimensional approximation of $\mathcal{V}$. (FE space)
- $\mathcal{V}_f$ can be chosen e.g. as hierarchical basis, $L^2(\Omega)$-orthogonal to $\mathcal{V}_c$, or wavelet modified hierarchical basis.

The variational multiscale method

Using the split we have: find $u_c \in \mathcal{V}_c$ and $u_f \in \mathcal{V}_f$ such that,

$$a(u_c, v_c) + a(u_f, v_c) = l(v_c), \quad \forall v_c \in \mathcal{V}_c$$

$$a(u_f, v_f) = l(v_f) - a(u_c, v_f) := (f - \mathcal{L}u_c, v_f), \quad \forall v_f \in \mathcal{V}_f.$$

The fine scale problem is then decoupled on each element K,

$$a(u_c, v_c) + a(u_f, v_c) = l(v_c), \quad \forall v_c \in \mathcal{V}_c$$

$$a(u_{f,K}, v_f) = l(v_f) - a(u_c, v_f) := (f - \mathcal{L}u_c, v_f), \quad \forall v_f \in \mathcal{V}_f \cap H_0^1(K).$$

The solutions $u_{f,K} \in H_0^1(K)$ are approximated using the element Green's function. The coarse equations reads:

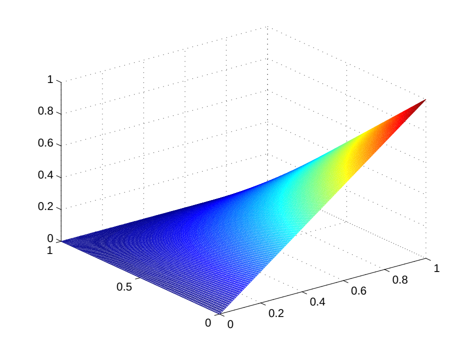
$$a(u_c^{\text{vms}}, v_c) + a(M(f - \mathcal{L}u_c^{\text{vms}}), v_c) = l(v_c), \quad \forall v_c \in \mathcal{V}_c,$$

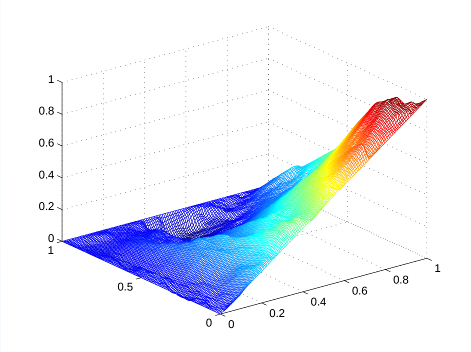
where M is the approximate fine scale solution operator.

The multiscale finite element method (MsFEM)

On each element we let $\mathcal{T}\phi_i \in H_0^1(K)$ solve,

$$a(\phi_i + \mathcal{T}\phi_i, v) = 0, \quad \forall v \in H_0^1(K), \quad K \in \mathcal{K}, \quad i \in \mathcal{N}.$$





We then get the multiscale finite element solution by solving: find $u_c^{\text{msfem}} \in \mathcal{V}_c^{\text{ms}} = \text{span}(\{\phi_i + \mathcal{T}\phi_i\}_{i\in\mathcal{N}})$ such that,

$$a(u_c^{\text{msfem}}, v) = l(v), \quad \forall v \in \mathcal{V}_c^{\text{ms}}.$$

The modified basis functions $\mathcal{T}\phi_i$ are computed on subgrids of the individual coarse elements K.

Comments

- VMS gives a non-symmetric coarse scale equation even if the original problem is symmetric.
- VMS can be used to derive stabilized methods (GLS, SUPG, ...).
- MsFEM is based on ideas from homogenization theory where periodic problems can be solved using a homogenized (coarse scale) equation derived by solving a fine scale cell problem.
- The error analysis available for MsFEM is also based on ideas from homogenization theory an can only be applied for very special coefficients, such as periodic coefficients.
- A convergence proof for general L^∞ coefficient is still an open problem.

Properties we seek

We want to:

- use the split into a coarse scale where we can do computations using standard techniques and a fine scale where computations are totally decoupled.
- use the space $\mathcal{V}_f = \{v \in \mathcal{V} : \pi_c v = 0\}$ since it has very nice properties (condition number, decay of solution to elliptic problems).
- symmetric method based on modification of basis functions.
- use adaptivity to focus computational resources on local problems that contribute most to the error.
- be able to prove convergence for arbitrary L^∞ coefficients.

The proposed method

Let $\mathcal{K}$ be a (coarse) mesh with mesh parameter H and FE space $\mathcal{V}_c = \text{span}(\{\phi_i\}_{i\in\mathcal{N}})$. Further let $\mathcal{V}_f = \{v \in \mathcal{V} : \pi_c v = 0\}$ where $\pi_c : C(\Omega) \cap \mathcal{V} \to \mathcal{V}_c$ is an interpolant.

We let $\mathcal{T} : \mathcal{V}_c \to \mathcal{V}_f$ and $u_{l,i} \in \mathcal{V}_f$ solve (fine scale equations),

$$a(\phi_i + \mathcal{T}\phi_i, v) = 0, \quad \forall v \in \mathcal{V}_f,$$
$$a(u_{l,i}, v) = l(\phi_i v), \quad \forall v \in \mathcal{V}_f.$$

We let $\sum_{i\in\mathcal{N}} \left(\beta_i(\phi_i + \mathcal{T}\phi_i) + u_{l,i}\right)$ solve (coarse scale equation),

$$\sum_{i\in\mathcal{N}} \beta_i a(\phi_i + \mathcal{T}\phi_i, \phi_j + \mathcal{T}\phi_j) = l(\phi_j + \mathcal{T}\phi_j) - \sum_{i\in\mathcal{N}} a(u_{l,i}, \phi_j + \mathcal{T}\phi_j), \forall j$$

We note that $u = \sum_{i\in\mathcal{N}} \left(\beta_i(\phi_i + \mathcal{T}\phi_i) + u_{l,i}\right)$, why?

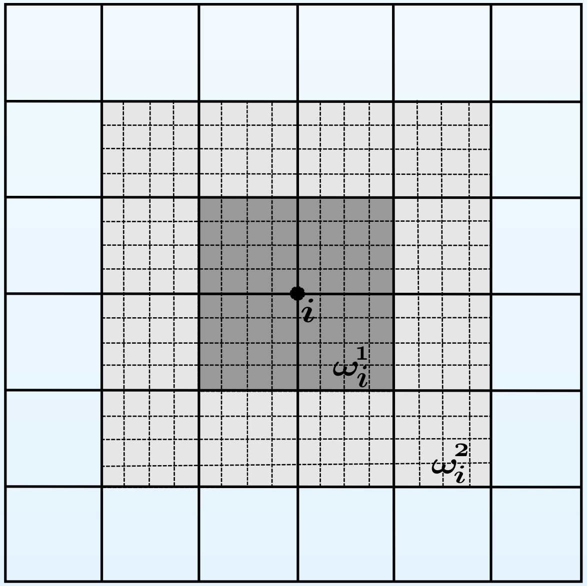
Approximation of $\mathcal{T}\phi_i$ and $u_{l,i}$

Spatial approximation $\mathcal{V}_f^h \subset \mathcal{V}_f$ gives,

$$a(\phi_i + \mathcal{T}^h\phi_i, v) = 0, \quad \forall v \in \mathcal{V}_f^h,$$

$$a(u_{l,i}^h, v) = l(\phi_i v), \quad \forall v \in \mathcal{V}_f^h.$$

Localization: introduce a patch ω_i^k around $\mathrm{supp}(\phi_i)$,



Let $\mathcal{V}_f^h(\omega_i^k) = \{v \in \mathcal{V}_f : v$ piecewise polynomial, $\mathrm{supp}(v) \subset \omega_i^k\}$.

Discrete version

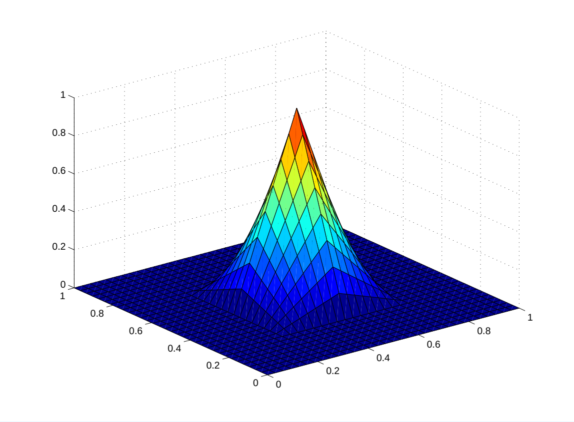
Let $\mathcal{T}^{h,k}\phi_i \in \mathcal{V}_f^h(\omega_i^k)$ and $u_{l,i}^{h,k} \in \mathcal{V}_f^h(\omega_i^k)$ be given by,

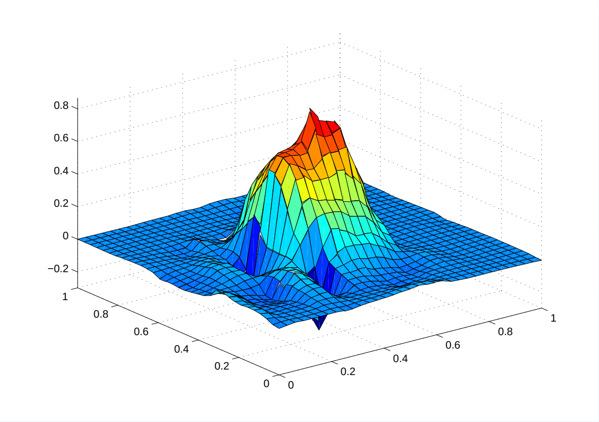
$$a(\phi_i + \mathcal{T}^{h,k}\phi_i, v) = 0, \quad \forall v \in \mathcal{V}_f^h(\omega_i^k),$$

$$a(u_{l,i}^{h,k}, v) = l(\phi_i v), \quad \forall v \in \mathcal{V}_f^h(\omega_i^k).$$

The method reads: Find $u_c^{h,k} \in \mathcal{V}_c$ such that

$$a(u_c^{h,k}+\mathcal{T}^{h,k}u_c^{h,k}, v+\mathcal{T}^{h,k}v) = l(v+\mathcal{T}^{h,k}v)-a(u_l^{h,k}, v+\mathcal{T}^{h,k}v), \; \forall v \in \mathcal{V}_c.$$

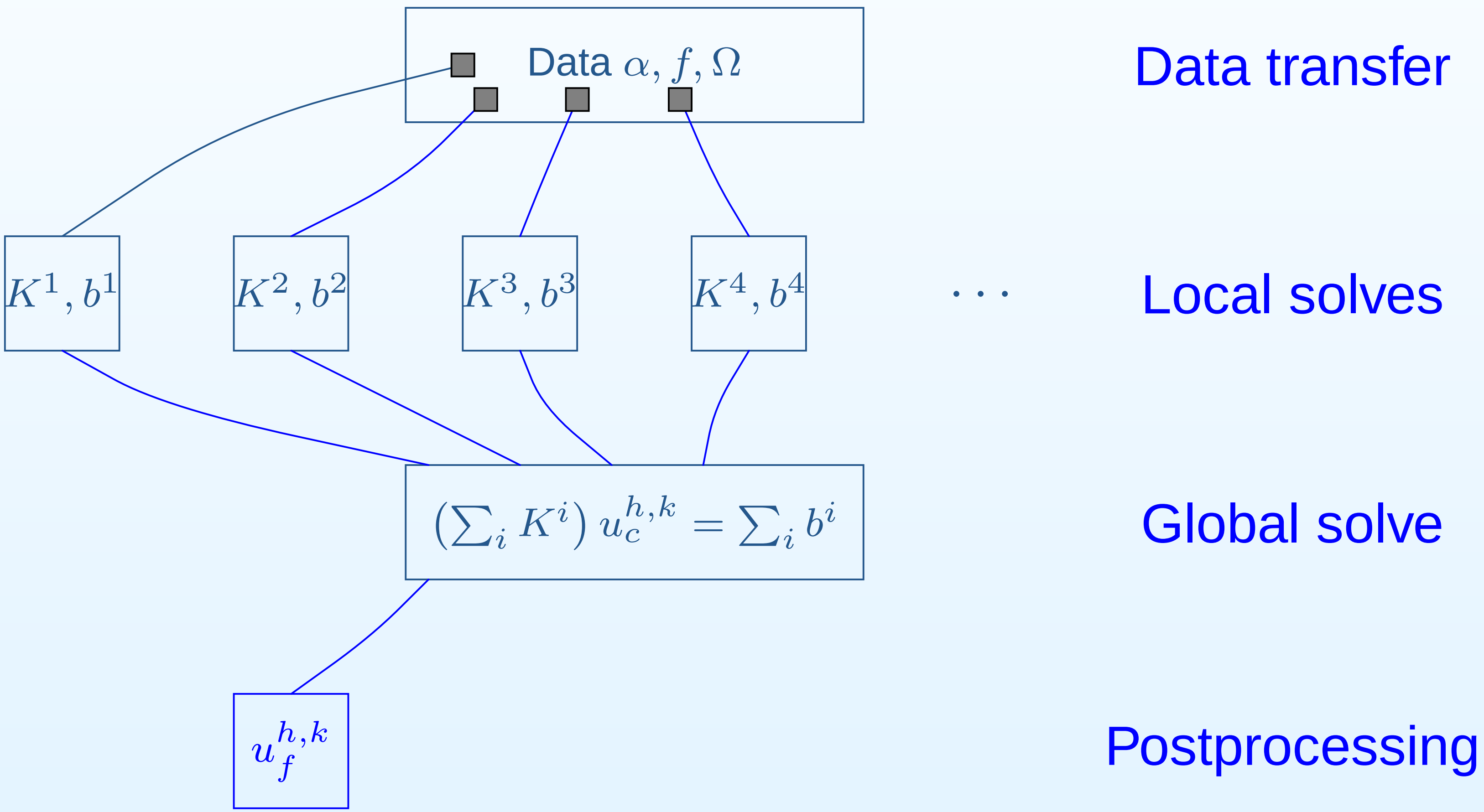




$u^{h,k} = u_c^{h,k} + u_f^{h,k}$, where $u_f^{h,k} = \mathcal{T}^{h,k}u_c^{h,k} + u_l^{h,k} \in \mathcal{V}_f$.

Sketch of algorithm

One local problem for each coarse dof, minimal communication.

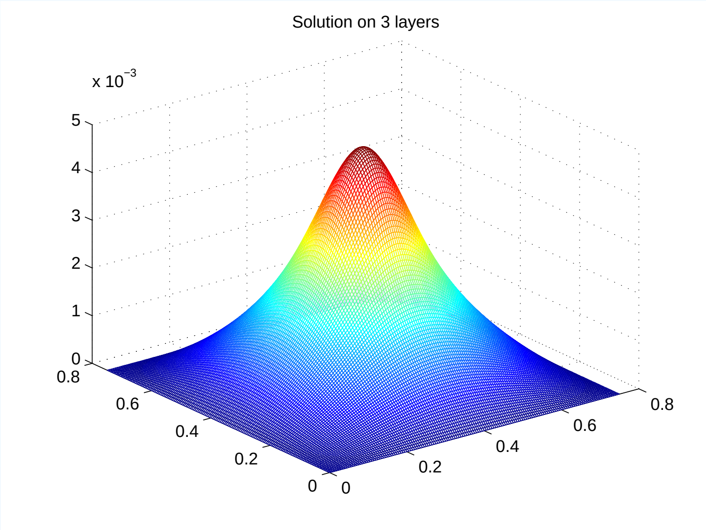


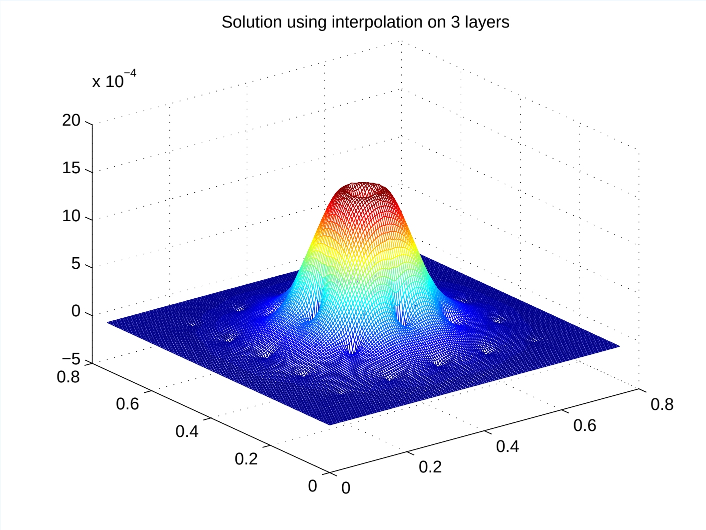
Observation about decay in $\mathcal{V}_f$ (Fourier)

Consider the Poisson equation,

$$-\Delta u = \phi_i \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

where ϕ_i has local support in Ω. The weak form reads: find $u \in \mathcal{W}$ such that, $a(u, v) = (\phi_i, v)$ for all $v \in \mathcal{W}$.





To the left $\mathcal{W} = \mathcal{V}$ (log decay) and right $\mathcal{W} = \mathcal{V}_f$ (exp decay).

Constraints are realized using Lagrangian multipliers.

Summary so far

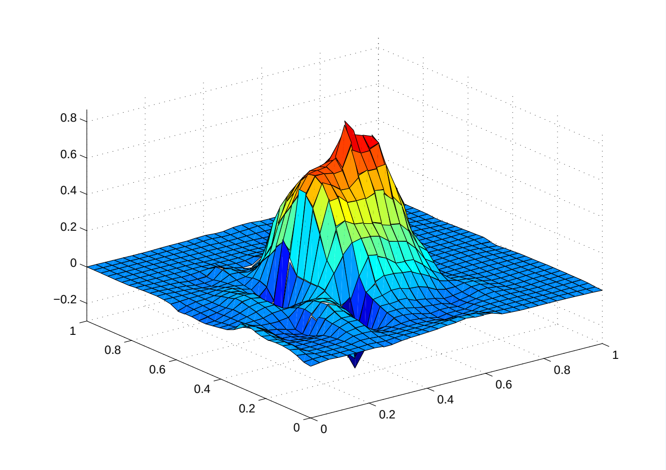
- We have derived a multiscale method using modified basis functions a-orthogonal to the kernal of the coarse scale interpolant π_c.
- The approximation is computed by solving totally decoupled local fine scale problems on patches using homogeneous (Dirichlet) boundary conditions.
- The quality of the solution will directly depend on the decay of these localized fine scale solutions.
- Experiments indicate exponential decay (in terms of k) for fix H.
- The main goal is to prove this result theoretically.

A priori error analysis

Let $\mathcal{V}^h$ be the FE space resulting from J uniform refinements of $\mathcal{V}_c$. Further let u^h solve $a(u^h, v) = l(v)$ for all $v \in \mathcal{V}^h$.

We show $\|u^h - u^{h,k}\| \lesssim \rho^k$, for some $0 \le \rho(J) < 1$, in two steps

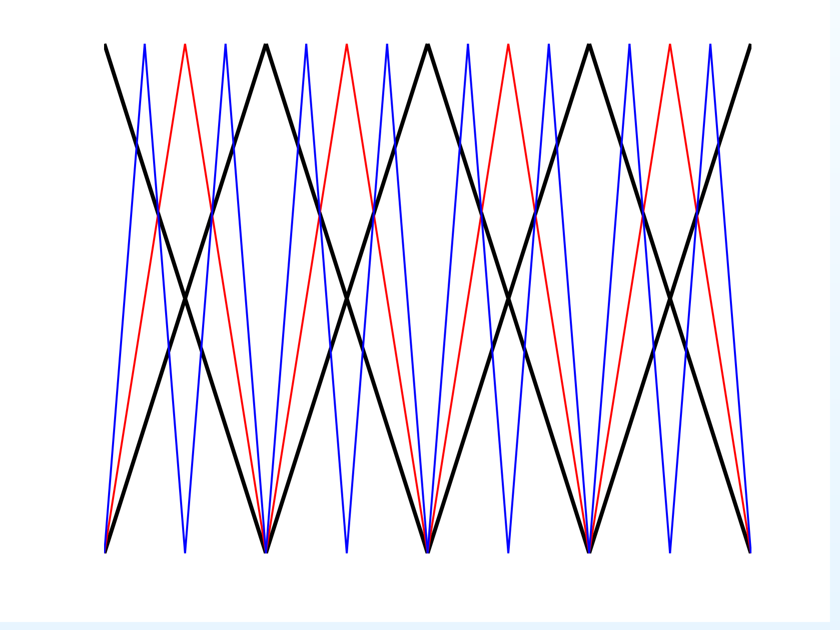
1. First we prove $\|\mathcal{T}^h \phi_i - \mathcal{T}^{h,k} \phi_i\| \lesssim \rho^k \|\mathcal{T}^h \phi_i\|$, (and $\|u^h_{l,i} - u^{h,k}_{l,i}\| \lesssim \rho^k \|u^h_{l,i}\|$),



2. and then, $\|u^h - u^{h,k}\| \lesssim \rho^k$ using the bound of the basis functions $\phi_i + \mathcal{T}^h \phi_i$ and the right hand side $-a(u^h_{l,i}, \cdot)$.

Proof of step 1: condition number of stiffness matrix

Let $\mathcal{V}_f^h = \{v \in \mathcal{V}^h : \pi_c v = 0\}$. Further let $\{\chi_j\}_{j\in\mathcal{M}}$ be a hierarchical basis of $\mathcal{V}_f^h$, and $A_{ij} = a(\chi_j, \chi_i)$.



Then $\kappa(A) = J^2$ in 2D and $\kappa(A) = 2^{2J}$ in 3D, Marion & Xu 1995, but independent of the coarse mesh size H.

Proof of step 1: decay of basis function

Let $\mathcal{T}^h \phi_i = \sum_{j\in\mathcal{M}} \beta_j \chi_j$. We use CG with $\hat{\beta}_0 = 0$ and right hand side $b_j = -a(\phi_i, \chi_j)$, which has support on a 1-ring. We have,

$$|\beta - \hat{\beta}^m|_A \leq 2\left(\frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1}\right)^m |\beta|_A := 2\rho^m|\beta|_A, \text{ where } |v|_A^2 = v^T A v.$$

HB only spreads information within ω_i^k in $2k$ iterations,

$$|\beta_{\Omega\backslash\omega^k}|^2 = \sum_{j\in\mathcal{M}(\Omega\backslash\omega_i^k)} |\beta_j|^2 = \sum_{j\in\mathcal{M}(\Omega\backslash\omega_i^k)} |\beta_j - \hat{\beta}_j^{2k}|^2 \leq |\beta - \hat{\beta}^{2k}|^2,$$

where $\beta_{\Omega\backslash\omega^k}$ only contains the node values outside ω_i^k.

Furthermore $|\beta_{\Omega\backslash\omega^k}|_A^2 \leq C_\kappa|\beta - \hat{\beta}^{2k}|_A^2 \leq C_\kappa\rho^{4k}|\beta|_A^2$ which means that the coefficients in β decays away from node i and more precisely $|||\mathcal{T}^h\phi_i|_{\Omega\backslash\omega_i^k}||| \leq C\rho^{2k}|||\mathcal{T}^h\phi_i|||$, with $|||v|||_\omega^2 = \langle v, v\rangle_\omega$.

Proof of step 1: error in basis function

We have $\|\mathcal{T}^h\phi_i|_{\Omega\setminus\omega_i^k}\| \leq C\rho^{2k}\|\mathcal{T}^h\phi_i\|$, where C depends on the condition number of A.

We use Galerkin Orthogonality to conclude,

$$a(\mathcal{T}^h\phi_i - \mathcal{T}^{h,k}\phi_i, \mathcal{T}^h\phi_i - \mathcal{T}^h\phi_i|_{\Omega\setminus\omega_i^k} - \mathcal{T}^{h,k}\phi_i) = 0,$$

i.e.

$$\begin{aligned}\|\mathcal{T}^h\phi_i - \mathcal{T}^{h,k}\phi_i\|^2 &= a(\mathcal{T}^h\phi_i - \mathcal{T}^{h,k}\phi_i, \mathcal{T}^h\phi_i|_{\Omega\setminus\omega_i^k}) \\ &\leq \|\mathcal{T}^h\phi_i - \mathcal{T}^{h,k}\phi_i\|\|\mathcal{T}^h\phi_i|_{\Omega\setminus\omega_i^k}\|\end{aligned}$$

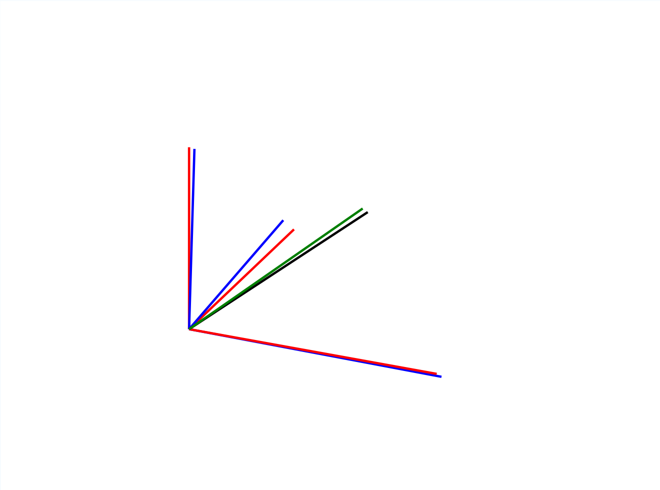
We conclude,

$$\|\mathcal{T}^h\phi_i - \mathcal{T}^{h,k}\phi_i\| \leq C\rho^{2k}\|\mathcal{T}^h\phi_i\|,$$

$$\|u_{l,i}^h - u_{l,i}^{h,k}\| \leq C\rho^{2k}\|u_{l,i}^h\|.$$

Step 2: global error bound

$\mathcal{V}_c^h = \text{span}(\{\phi_i + \mathcal{T}^h \phi_i\})$ (blue) $\mathcal{V}_c^{h,k} = \text{span}(\{\phi_i + \mathcal{T}^{h,k} \phi_i\})$ (red).



We compute u_c^h (black) and $u_c^{h,k}$ (green) as projections:

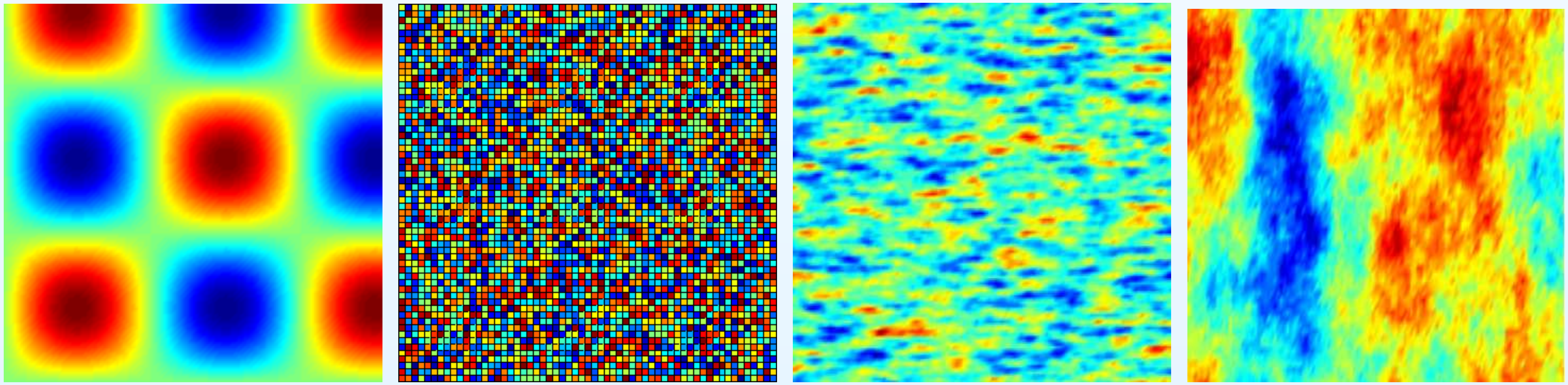
Theorem 1 *Let u^h be the reference solution and $u^{h,k} = (1 + \mathcal{T}^{h,k})u_c^{h,k} + u_l^{h,k}$ the multiscale approximation. Then,*

$$\|u^h - u^{h,k}\| \leq C \left(\|u^h\|_{L^\infty(\Omega)}/H + \|f\|_{L^2(\Omega)} \right) \rho^{2k},$$

where $\rho = \frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1}$ and $\sqrt{\kappa(A)} \sim J$ in 2D and $\sqrt{\kappa(A)} \sim 2^J$ in 3D.

Numerical examples

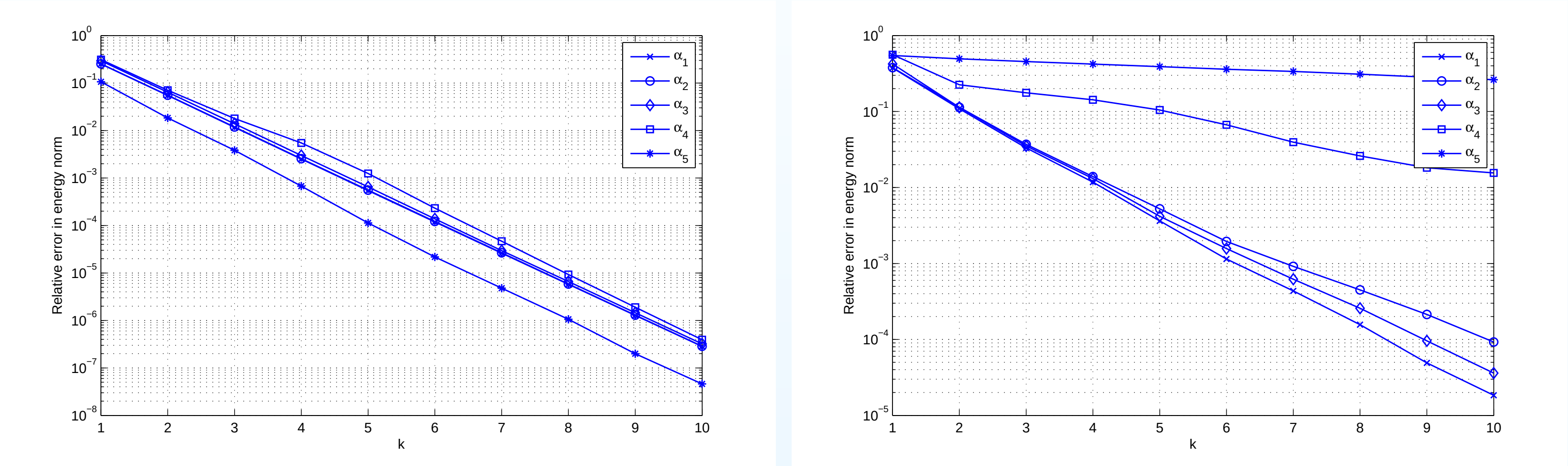
$$\begin{cases} \alpha_1(x,y) = 1, \\ \alpha_2(x,y) = 1 + 0.5 \cdot sin(8x) sin(8y), \\ \alpha_3(x,y) = 0.1 + 0.9 * \texttt{rand}, \quad (x,y) \in \tau, \quad \forall \tau \in \mathcal{T}_1, \\ \alpha_4(x,y) = \alpha_{\mathsf{GSLIB}}(i,j), \text{ for } \frac{i-1}{120} \leq x < \frac{i}{120}, \ \frac{j-1}{120} \leq y < \frac{j}{120}, \\ \alpha_5(x,y) = \alpha_{\mathsf{SPE}}(i,j), \text{ for } \frac{i-1}{120} \leq x < \frac{i}{120}, \ \frac{j-1}{120} \leq y < \frac{j}{120}, \end{cases}$$



We let $f = \chi_{\text{inj}} - \chi_{\text{prod}}$, with supp$(\chi_{\text{inj}}) = [0, 1/60] \times [0, 1/60]$, and supp$(\chi_{\text{prod}}) = [1 - 1/60, 1] \times [1 - 1/60, 1]$.

Convergence of local solution $\mathcal{T}^{h,k}\phi_i$

$i = 435$, $h = H2^{-J}$, $J = 3$, $H = 1/30$, using rectangular mesh:



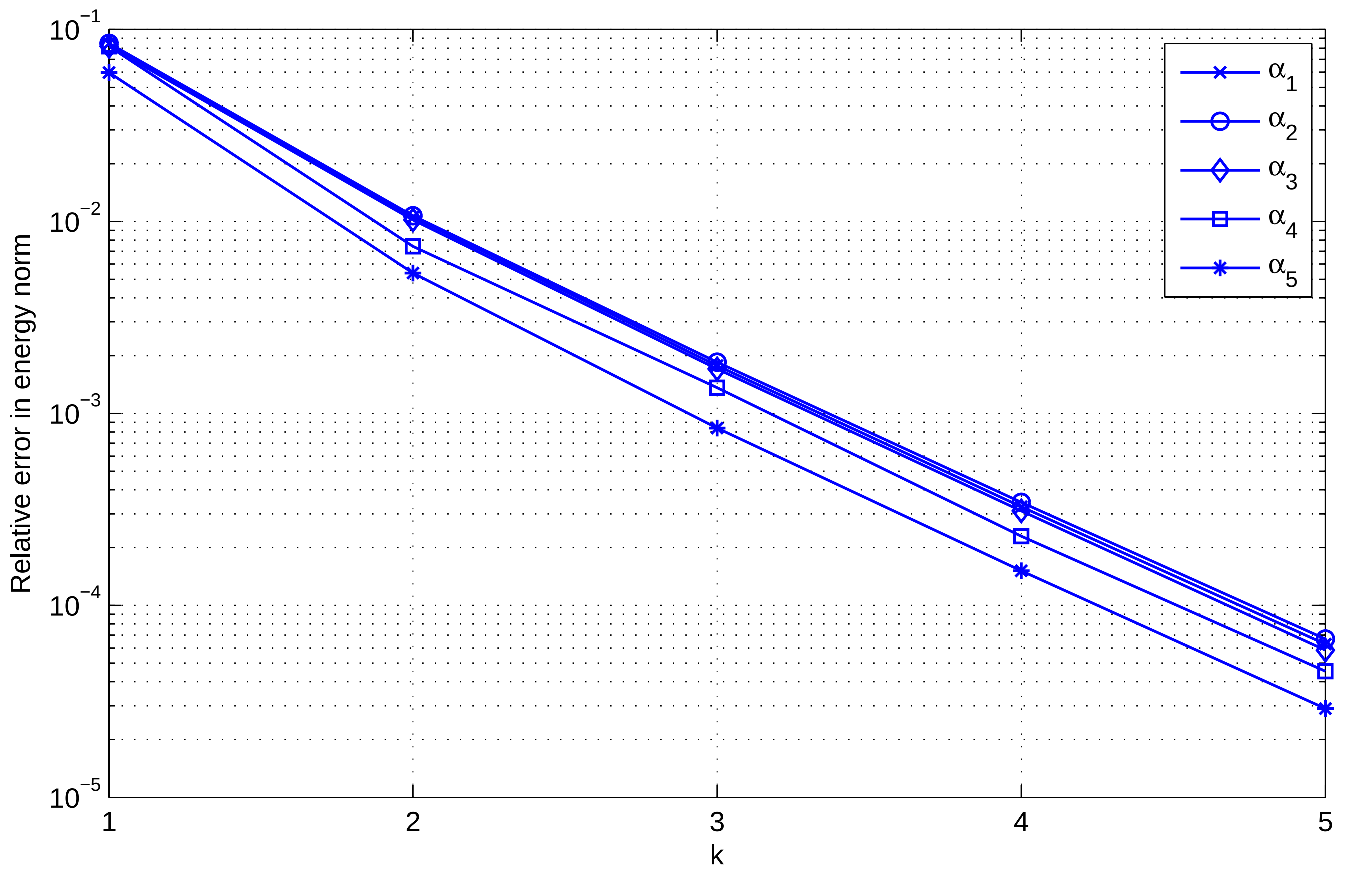
Relative error in energy norm (left). We get exponential convergence in k.

Corresponding error using $2k$ cg iterations (right) $\Rightarrow$ slower convergence for high aspect ratio.

Preconditioner that works in the argument?

Convergence of global solution $u^{h,k}$

Again $J = 3$ and $H = 1/30$. We plot the error $u^h - u^{h,k}$ in energy norm (relative).

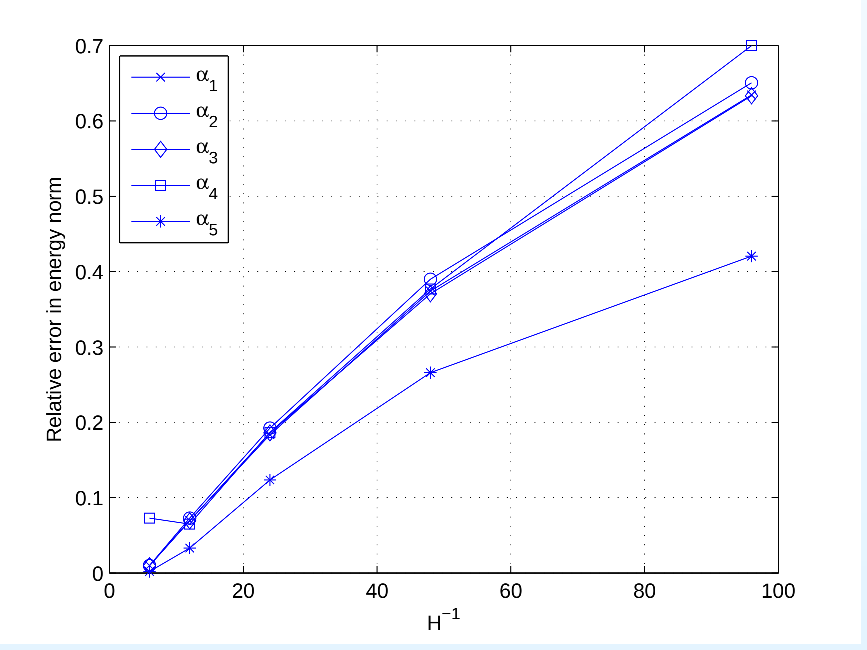


How does the error depend on H?

Remember

$$\|u^h - u^{h,k}\| \le C \left(\|u^h\|_{L^\infty(\Omega)}/H + \|f\|_{L^2(\Omega)} \right) \rho^{2k},$$

We let $J = 2$ and $k = 3$.



The bound is not sharp in terms of H.

Recent improvements

Using the modified basis $\{\phi_i + \mathcal{T}^k \phi_i\}_{i \in \mathcal{N}}$ it has been proven that,

$$\|u - u^k\| \leq C(\|Hf\|_{L^2(\Omega)} + \gamma^k),$$

where $0 \leq \gamma < 1$ is computable and only dependent on shape regularity constant and $\max \alpha / \min \alpha$. Classical a priori error analysis then gives a bound for $\|u^k - u^{h,k}\|$.

Note that the condition number is not present at all, analytical techniques using cut off functions are instead used.

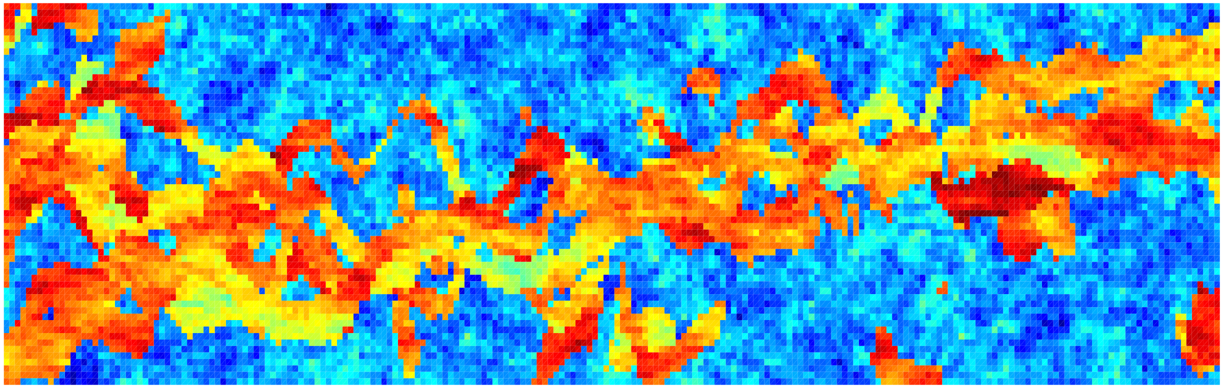
The constant C depends on $\max_T H / \min_T H$.

M. & Peterseim, *Localization of elliptic multiscale problems*, preprint arXiv

A posteriori error estimation and adaptivity

Motivation:

- The method we propose will have overlapping patches, which, especially in 3D, is expensive.
- The problems we consider often includes channels so the solution is typically localized in space.
- The size of the patches and the refinement level is difficult to predict a priori, we therefore need error indicators to tune these parameters automatically.



A posteriori error estimate

Let $\rho^2(g;v)=\sum_{K\in\mathcal{K}_h} h_K^2\|g+\nabla\cdot\alpha\nabla v\|^2_{L^2(K)}+h_K\|[n\cdot\alpha\nabla v]\|^2_{L^2(\partial K)}$

Theorem 2 *Let $u^{h,k}=(1+\mathcal{T}^{h,k})u_c^{h,k}+u_l^{h,k}$ be the multiscale approximation. Then,*

$$\|u-u^{h,k}\|^2\lesssim\sum_{i\in\mathcal{N}}\rho(-\nabla\cdot\alpha\nabla\phi_i;\mathcal{T}^{h,k}\phi_i)+\rho(f\phi_i;u_{l,i}^{h,k})$$
$$+\sum_{i\in\mathcal{N}}H\left(\|n\cdot\alpha\nabla\mathcal{T}^{h,k}\phi_i\|^2_{L^2(\partial\omega_i^k)}+\|n\cdot\alpha\nabla u_{l,i}^{h,k}\|^2_{L^2(\partial\omega_i^k)}\right)$$

- A standard element indicator on each patch measuring the effect of decreasing fine scale mesh size h.
- A new indicator on the boundary of each patch $\partial\omega_i^k$. The a priori analysis shows that $\mathcal{T}^{h,k}\phi_i$ and $u_{l,i}^{h,k}$ decays exponentially in k.

Adaptive algorithm

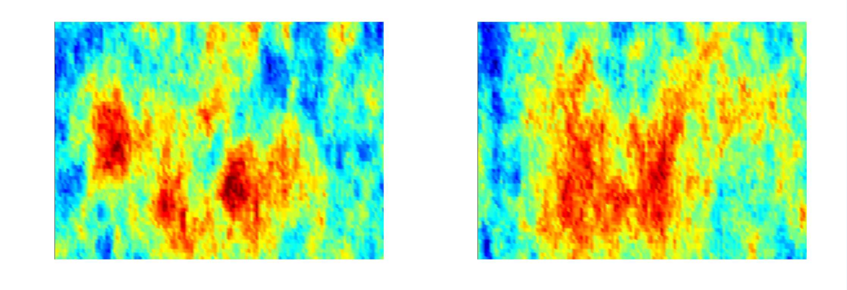
Given the bound

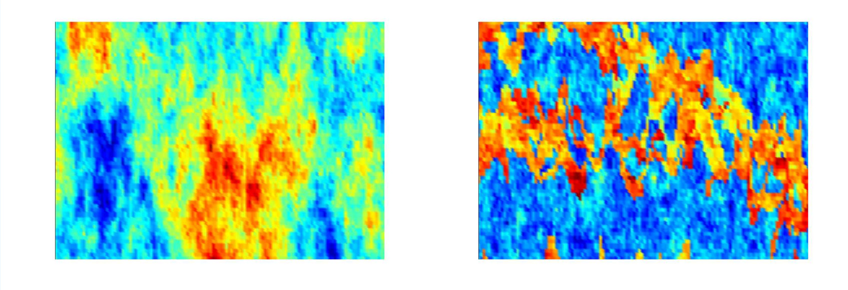
$$\|u - u^{h,k}\|^2 \lesssim \sum_{i\in\mathcal{N}} \rho(-\nabla\cdot\alpha\nabla\phi_i; \mathcal{T}^{h,k}\phi_i) + \rho(f\phi_i; u_{l,i}^{h,k})$$
$$+ \sum_{i\in\mathcal{N}} H\big(\|n\cdot\alpha\nabla\mathcal{T}^{h,k}\phi_i\|^2_{L^2(\partial\omega_i^k)} + \|n\cdot\alpha\nabla u_{l,i}^{h,k}\|^2_{L^2(\partial\omega_i^k)}\big).$$

1. Compute multiscale approximation.
2. Compute indicators.
3. If the error is small enough break.
4. Otherwise, decrease h locally if interior indicator is large and increase k locally is boundary indicator is large.
5. Go back to 1.

Numerical example

- Let the coarse mesh consist of 32×32 elements.
- Let the fine reference mesh consist of 256×256 elements.
- $f = -1$ in lower left corner ($0 \leq x, y \leq 1/128$) and $f = 1$ in upper right corner, otherwise $f = 0$.
- We consider four layers of the SPE data set:

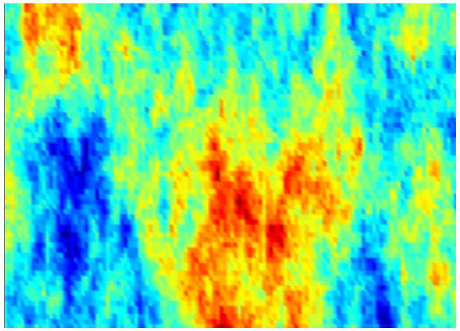




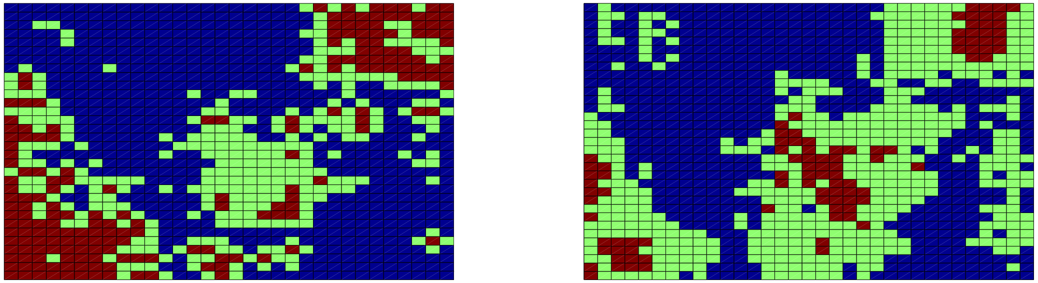
We use a symmetric Discontinuous Galerkin method as base for the multiscale method. Local problems are solved using Neumann boundary conditions, hanging nodes are allowed, there is a common reference mesh for the local problems.

Numerical example

We start with one refinement and two layers in each local problem. In each iteration we refine and increase the size of 30% of the patches (possibly different patches).

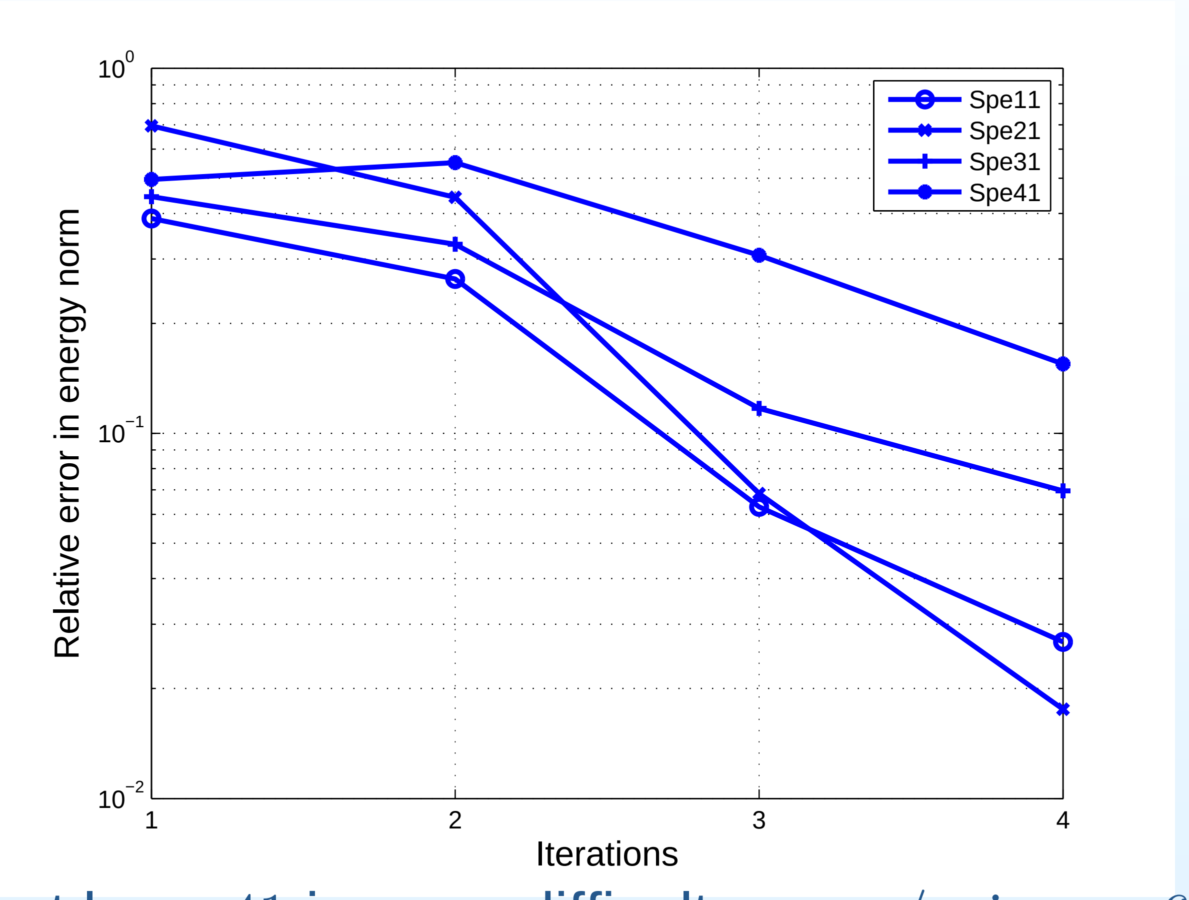


We plot refinements and layers for layer 31 after three iterations.



Numerical example

Convergence of relative error vs. number of iterations.



We note that layer 41 is more difficult, $\max a / \min a \approx 6 \cdot 10^6$ instead of $6 \cdot 10^5$.

Why DG?

Advantage:

- It allows for Neumann conditions on the patches since discontinuous fine scale solutions are not a problem.
- One can use adaptively refined local subgrid and still have a global reference grid by using hanging nodes.
- Construction of a conservative flux, which is essential in the application area, is easy.

Disadvantage:

- Expensive.
- There is a penalty parameter which needs to be tuned.

Summary of this talk

1. We prove an *a priori* error bound and thereby convergence as $k \to \infty$ for the proposed method, for fix H and $h = H2^{-J}$.
2. The bound reveals that for fix H and J we get *exponential decay* in the number of layers k.
3. Numerical experiments confirms this and furthermore reveals that a very small value $k \sim 2$ is needed for 2D examples in practise.
4. There are still improvements needed in the analysis in the case when $\frac{\max_x \alpha(x)}{\min_x \alpha(x)}$ or J is large. Preconditioner and/or wavelet basis might resolve this. Different split may also prove useful.
5. We show an *a posteriori* error bound and numerical examples with adaptively refined local problems.

Future directions

- Improving the convergence result with Peterseim (quasi-uniform, $\max a / \min a$)
- Convergence results for different equations, such as convection-diffusion
- Convergence of the adaptive algorithm
- Multiscale in time
- Implementation on parallel machines, 3D
- Solving the coupled system of hyperbolic and elliptic arising in porous media flow