On convergence of multiscale methods

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Outline and Papers

Outline

- Motivating example
- Previous work
- Derivation of multiscale methods
- Convergence analysis
- Numerical examples
- Conclusions and future work

Papers

- M.G. Larson and A. Målqvist, *Adaptive variational multiscale methods based on a posteriori error estimation: energy norm estimates for elliptic problems*, CMAME 2007
- A. Målqvist, *A priori error analysis of a multiscale method*, submitted

Thanks

- M. G. Larson, Umeå University and G. Tsogtgerel, McGill University

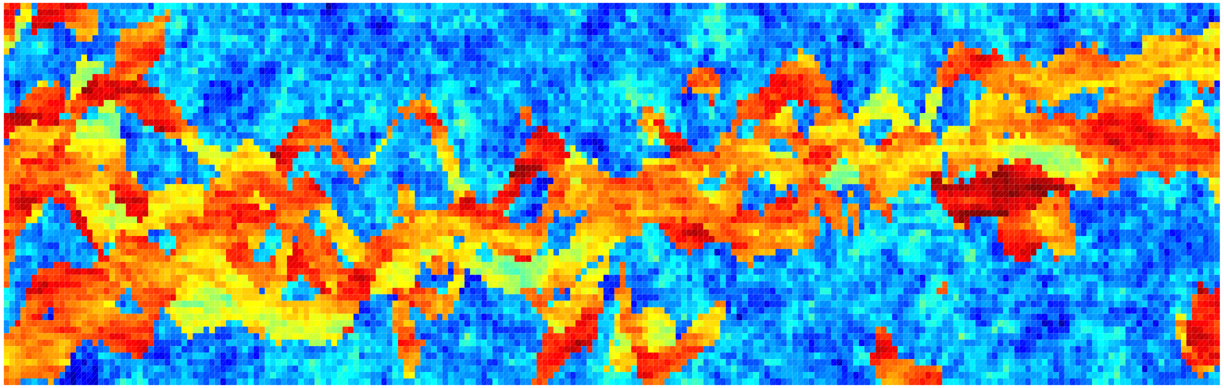
Motivating example: Secondary oil recovery

Find pressure p and water concentration s such that:

$$-\nabla \cdot k\lambda(s)\nabla p = q, \quad \dot{s} - \nabla \cdot [f(s)\lambda(s)k\nabla p] = g, \quad \text{in } \Omega,$$

where k is permeability, $\lambda(s)$ the total mobility, f fractional flow, and g, q sink and source terms.

Model problem



We consider the strong form:

$$-\nabla \cdot \alpha \nabla u = f, \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

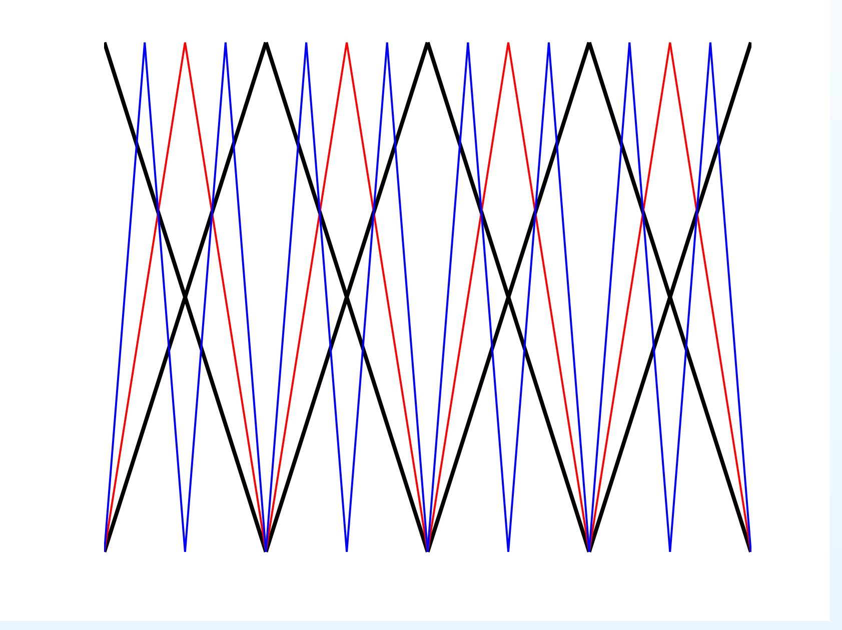
The weak form reads: find $u \in \mathcal{V} := H_0^1(\Omega)$ such that,

$$\langle u, v \rangle := \int_\Omega \alpha \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx := l(v), \quad \text{for all } v \in \mathcal{V}.$$

We assume $f \in L^2(\Omega)$ and $0 < \alpha_0 \leq \alpha \in L^\infty(\Omega)$.

Derivation of multiscale methods

We let $\mathcal{T}_0$ be a (coarse) mesh of Ω and $\mathcal{T}_J$ be the mesh after J refinements. Let $\mathcal{V}_0 \subset \mathcal{V}_J \subset \mathcal{V}$ be corresponding FE spaces.



We let $\pi_0 : C(\Omega) \cap \mathcal{V} \to \mathcal{V}_0$ and $\mathcal{W}_J = \{w \in \mathcal{V}_J : \pi_0 w = 0\}$. Let $\{\chi_i\}$ be hierarchical basis for $\mathcal{W}_J$ and $\{\phi_i\}$ a basis for $\mathcal{V}_0$.

Reference solution $u_J \in \mathcal{V}_J$ fulfills $\langle u_J, w \rangle = l(w)$ for all $w \in \mathcal{V}_J$.

Orthogonal split of scales

We introduce an a-orthogonal map $I + T_J$ with $T_J : \mathcal{V}_0 \to \mathcal{W}_J$:

$$\langle v_0 + T_J v_0, w \rangle = 0, \quad \text{for all } v_0 \in \mathcal{V}_0, w \in \mathcal{W}_J.$$

The map T_J exists for each $v_0 \in \mathcal{V}_0$ and is unique (Lax-Milgram).

Let $u_0 = \pi_0 u_J$. Then there exists a $u_{l,J} = u_J - u_0 - T_J u_0 \in \mathcal{W}_J$ such that,

$$\langle u_{l,J}, w \rangle = l(w), \quad \text{for all } w \in \mathcal{W}_J.$$

If we now write $u_J = u_0 + T_J u_0 + u_{l,J}$ we get the coarse scale equation:

$$\boxed{\text{Find } u_0 \in \mathcal{V}_0 \text{ s.t. } \langle u_0 + T_J u_0, v_0 \rangle = l(v_0) - \langle u_{l,J}, v_0 \rangle, \quad \text{for all } v_0 \in \mathcal{V}_0}$$

$$\Big(\langle u_0 + T_J u_0, v_0 {\color{red}+ T_J v_0} \rangle = l(v_0 {\color{red}+ T_J v_0}) - \langle u_{l,J}, v_0 {\color{red}+ T_J v_0} \rangle \Big)$$

Three multiscale methods

VMS:
$$\langle u_0 + T_J^{\text{vms}} u_0, v_0 \rangle = l(v_0) - \langle u_{l,J}^{\text{vms}}, v_0 \rangle,$$
$$\langle v_0 + T_J^{\text{vms}} v_0, v \rangle \approx 0,$$
$$\langle u_{l,J}^{\text{vms}}, v \rangle \approx l(v),$$

MsFEM:
$$\langle u_0 + T_J^{\text{mfem}} u_0, v_0 + T_J^{\text{mfem}} v_0 \rangle = l(v_0 + T_J^{\text{mfem}} v_0),$$
$$\langle v_0 + T_J^{\text{mfem}} v_0, v \rangle \approx 0,$$

Sym-AVMS:
$$\langle u_0 + T_J^{k} u_0, v_0 + T_J^{k} v_0 \rangle = l(v_0 + T_J^{k} v_0) - \langle u_{l,J}^{k}, v_0 + T_J^{k} v_0 \rangle,$$
$$\langle v_0 + T_J^{k} v_0, v \rangle \approx 0,$$
$$\langle u_{l,J}^{k}, v \rangle \approx l(v),$$

for all $v_0 \in \mathcal{V}_0$ and $v \in \mathcal{W}_J$. Note that $\langle v_0 + T_J v_0, w_J \rangle = 0$.

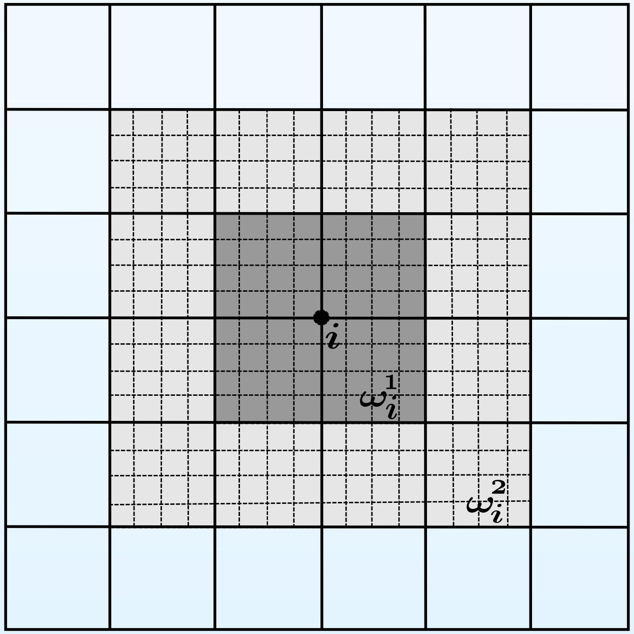
Approximation of T_J and $u_{l,J}$ in Sym-AVMS

We localize the fine scale equations. Let $\mathcal{V}_0 = \text{span}(\{\phi_i\})$ and,

$$\langle \phi_i + T_J \phi_i, v \rangle = 0, \quad \text{for all } w \in \mathcal{W}_J,$$

$$\langle u_{l,J,i}, v \rangle = l(\phi_i v), \quad \text{for all } w \in \mathcal{W}_J,$$

We introduce a patch ω_i^k around $\text{supp}(\phi_i)$:



Now let $\mathcal{W}_J(\omega_i^k) = \{v \in \mathcal{W}_J : \text{supp}(v) \subset \omega_i^k\}$.

Sym-AVMS

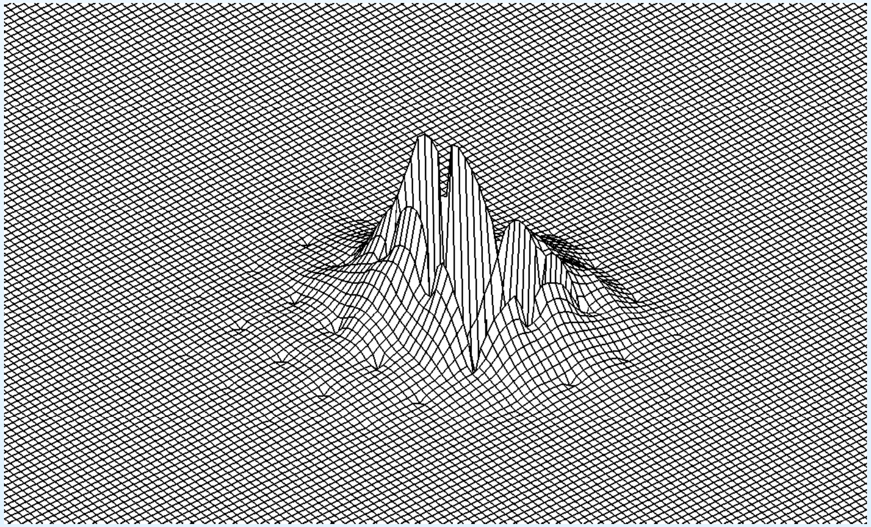
Let $T_J^k \phi_i \in \mathcal{W}_J(\omega_i^k)$ and $u_{l,J}^k \in \mathcal{W}_J(\omega_i^k)$ be given by,

$$\langle \phi_i + T_J^k \phi_i, v \rangle = 0, \quad \text{for all } w \in \mathcal{W}_J(\omega_i^k),$$

$$\langle u_{l,J,i}^k, v \rangle = l(\phi_i v), \quad \text{for all } w \in \mathcal{W}_J(\omega_i^k).$$

The method reads: Find $u_0^k \in \mathcal{V}_0$ such that

$$\langle u_0^k + T_J^k u_0^k, v_0 + T_J^k v_0 \rangle = l(v_0 + T_J^k v_0) - \langle u_{l,J}^k, v_0 + T_J^k v_0 \rangle, \ \forall v_0 \in \mathcal{V}_0.$$

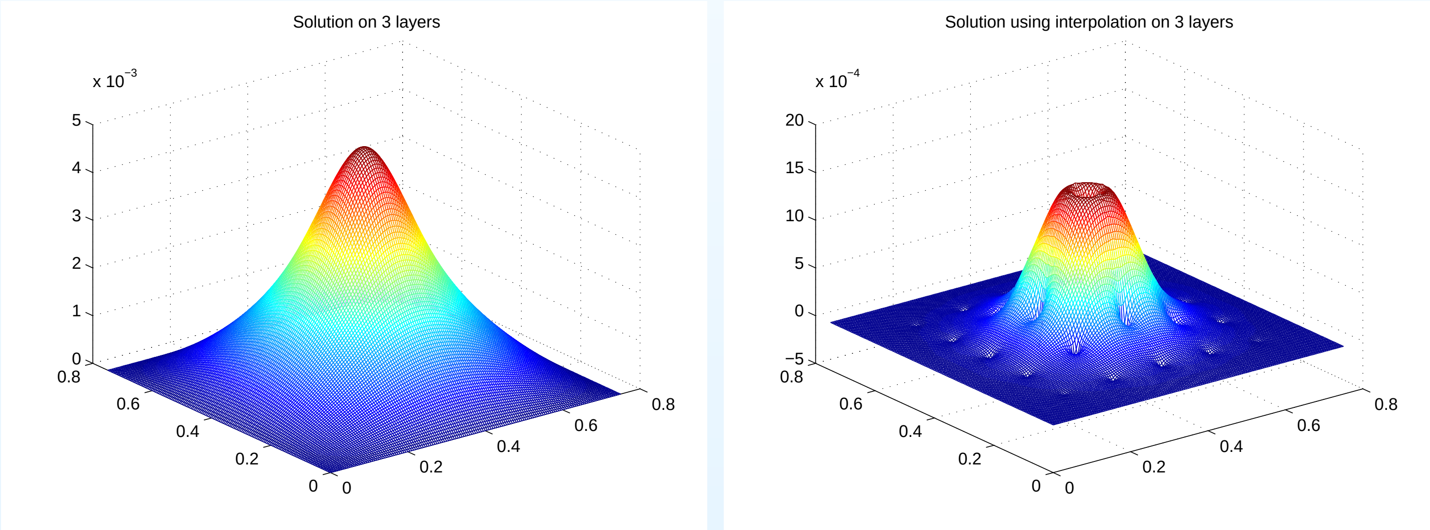


Observation about decay in $\mathcal{W}$ (Fourier)

Consider the Poisson equation,

$$-\Delta u = \phi_i \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

where ϕ_i has local support in Ω. The weak form reads: find $u \in \mathcal{Z}$ such that, $\langle u, v \rangle = (\phi_i, v)$ for all $v \in \mathcal{Z}$.

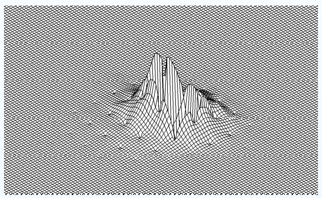


To the left $\mathcal{Z} = \mathcal{V}$ (log decay) and right $\mathcal{Z} = \mathcal{W}$ (exp decay).

Constraints are realized using Lagrangian multipliers.

Convergence analysis: basis functions $T_J\phi_i$

We sketch the convergence proof below. We start with the decay of $T_J\phi_i$.



Let $\{\chi_j\}_{j\in\mathcal{M}_J}$ be a hierarchical basis for $\mathcal{W}_J$. Let $\hat{A} = \langle\chi_l, \chi_j\rangle$, $l, j \in \mathcal{M}_J$. Further let $T_J\phi_i = \sum_{j\in\mathcal{M}_J} \alpha_j\chi_j$. We use CG with $\hat{\alpha}_0 = 0$ and right hand side $b_j = -\langle\phi_i, \chi_j\rangle$. We have,

$$|\alpha - \hat{\alpha}^m|_{\hat{A}} \leq 2\left(\frac{\sqrt{\kappa(\hat{A})} - 1}{\sqrt{\kappa(\hat{A})} + 1}\right)^m |\alpha|_{\hat{A}} := 2\rho^m|\alpha|_{\hat{A}}, \text{ where } |v|_A^2 = v^TAv.$$

Note that $\sqrt{\kappa(\hat{A})} \sim J$ in 2D and $\sqrt{\kappa(\hat{A})} \sim 2^J$ in 3D.

Convergence analysis: local solutions $T_J\phi_i$

We have $T_J\phi_i = \sum_{j\in\mathcal{M}_J} \alpha_j\chi_j$, with corresponding vector α, where $\mathcal{M}_J$ is the set non-coarse nodes on level J.

Since b_j has support on a coarse 1-ring and the HB only spreads information within ω_i^k in $2k$ iterations we have,

$$|\alpha_{\Omega\backslash\omega^k}|^2 = \sum_{j\in\mathcal{M}_{\mathcal{J}}(\Omega\backslash\omega_i^k)} |\alpha_j|^2 = \sum_{j\in\mathcal{M}_{\mathcal{J}}(\Omega\backslash\omega_i^k)} |\alpha_j - \hat{\alpha}_j^{2k}|^2 \leq |\alpha - \hat{\alpha}^{2k}|^2,$$

where $\alpha_{\Omega\backslash\omega^k}$ only contains the node values outside ω_i^k.

Furthermore $|\alpha_{\Omega\backslash\omega^k}|_{\hat{A}}^2 \leq C|\alpha - \hat{\alpha}^{2k}|_{\hat{A}}^2 \leq C\rho^{4k}|\alpha|_{\hat{A}}^2$ which means that the coefficients in α decays away from node i and more precisely $\|T_J\phi_i\|_{\Omega\backslash\omega_i^k} \leq C\rho^{2k}\|T_J\phi_i\|$, with $\|v\|_\omega^2 = \langle v, v\rangle_\omega$.

Convergence analysis: local solutions $T_J^k \phi_i \to T_J \phi_i$

Now let $T_J^k \phi_i = \sum_{j \in \mathcal{M}_J(\omega_i^k)} \alpha_j^k \chi_j$.

We have $\langle T_J \phi_i - T_J^k \phi_i, w \rangle = 0$ for all $w \in \mathcal{W}_J(\omega_i^k)$.

Now let $w = \sum_{j \in \mathcal{M}_J(\omega_i^k)} (\alpha_j - \alpha_j^k) \chi_k \in \mathcal{W}_J(\omega_i^k)$, with corresponding vectors α_{ω^k} and α^k. We get,

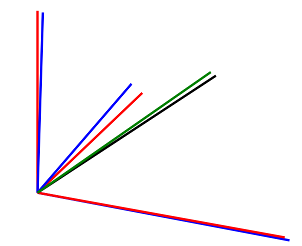
$$\begin{aligned} |\alpha - \alpha^k|_{\hat{A}}^2 &= (\alpha - \alpha_{\omega_k})^T \hat{A} (\alpha - \alpha^k) \\ &= \alpha_{\Omega \setminus \omega^k}^T \hat{A} (\alpha - \alpha^k) \\ &\leq |\alpha_{\Omega \setminus \omega^k}|_{\hat{A}} |\alpha - \alpha^k|_{\hat{A}}, \end{aligned}$$

But now $|\alpha - \alpha^k|_{\hat{A}} \leq C \rho^{2k} |\alpha|_{\hat{A}}$ or,

$\| T_J \phi_i - T_J^k \phi_i \| \leq C \rho^{2k} \| T_J \phi_i \|$ and $\| u_{l,J,i} - u_{l,J,i}^k \| \leq C \rho^{2k} \| u_{l,J,i} \|$.

Convergence analysis: system

$\mathcal{V}_{0,J} = \text{span}(\{\phi_i + T_J\phi_i\})$ (blue) $\mathcal{V}_{0,J}^k = \text{span}(\{\phi_i + T_J^k\phi_i\})$ (red).



We compute u_0 (black) and u_0^k (green) as projections:

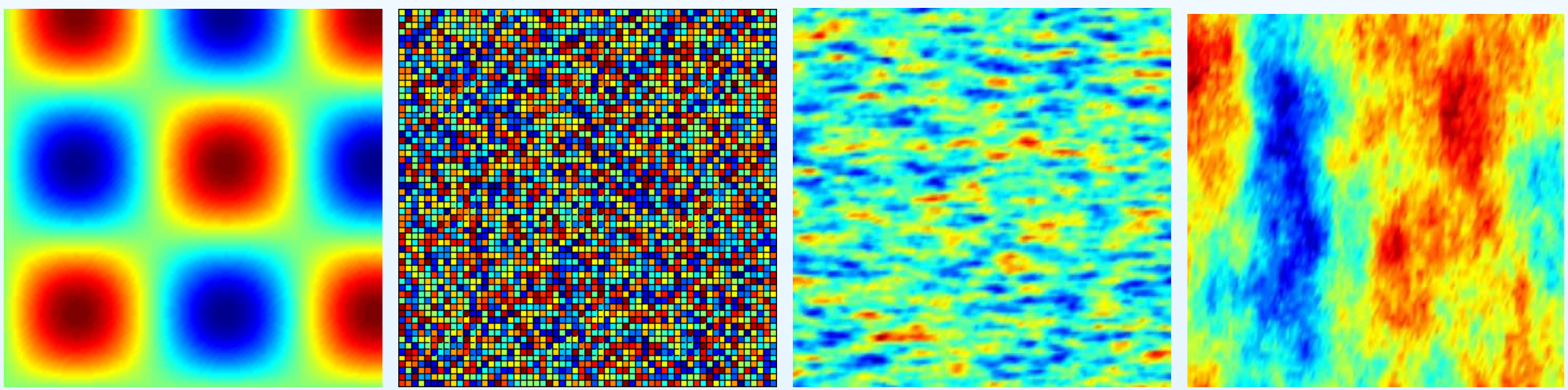
Theorem 1 *Let u_J be the reference solution and u_J^k the Sym-AVMS approximation. Then,*

$$|||u_J - u_J^k||| \leq C \left(\|u_J\|_{L^\infty(\Omega)}/h_0 + \|f\|_{L^2(\Omega)} \right) \rho^{2k},$$

where $\rho = \frac{\sqrt{\kappa(\hat{A})}-1}{\sqrt{\kappa(\hat{A})}+1}$ and $\sqrt{\kappa(\hat{A})} \sim J$ in 2D and $\sqrt{\kappa(\hat{A})} \sim 2^J$ in 3D.

Numerical examples

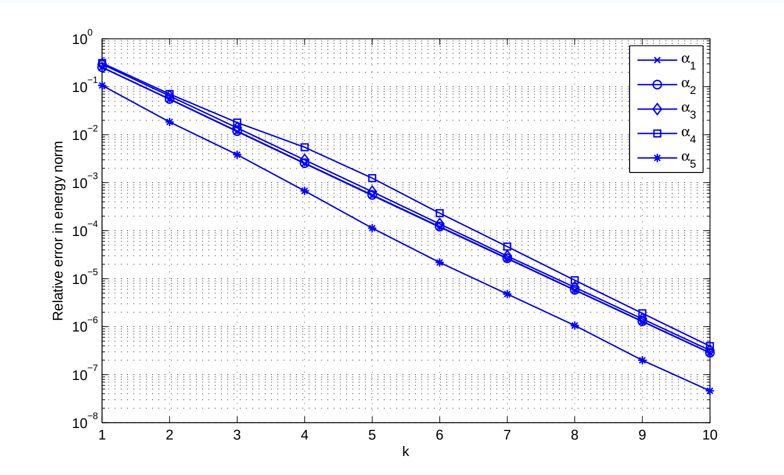
$$\begin{cases} \alpha_1(x,y) = 1, \\ \alpha_2(x,y) = 1 + 0.5 \cdot sin(8x)sin(8y), \\ \alpha_3(x,y) = 0.1 + 0.9 * \texttt{rand}, \quad (x,y) \in \tau, \text{ for all } \tau \in \mathcal{T}_1, \\ \alpha_4(x,y) = a_{\text{GSLIB}}(i,j), \text{ for } \frac{i-1}{120} \le x < \frac{i}{120}, \frac{j-1}{120} \le y < \frac{j}{120}, \\ \alpha_5(x,y) = a_{\text{SPE}}(i,j), \text{ for } \frac{i-1}{120} \le x < \frac{i}{120}, \frac{j-1}{120} \le y < \frac{j}{120}, \end{cases}$$

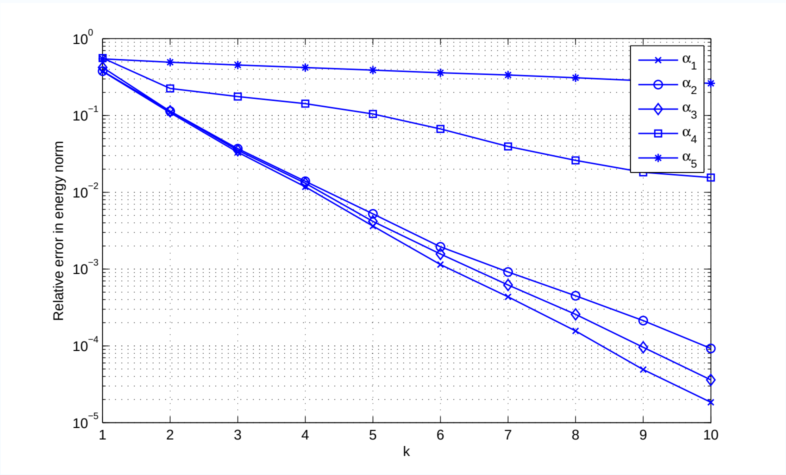


We let $f = \chi_{\text{inj}} - \chi_{\text{prod}}$, with supp$(\chi_{\text{inj}}) = [0, 1/60] \times [0, 1/60]$, and supp$(\chi_{\text{prod}}) = [1 - 1/60, 1] \times [1 - 1/60, 1]$.

Convergence of local solution $T_J^k \phi_i$

We let $i = 435$, $J = 3$, and $h_0 = 1/30$, using rectangular mesh.





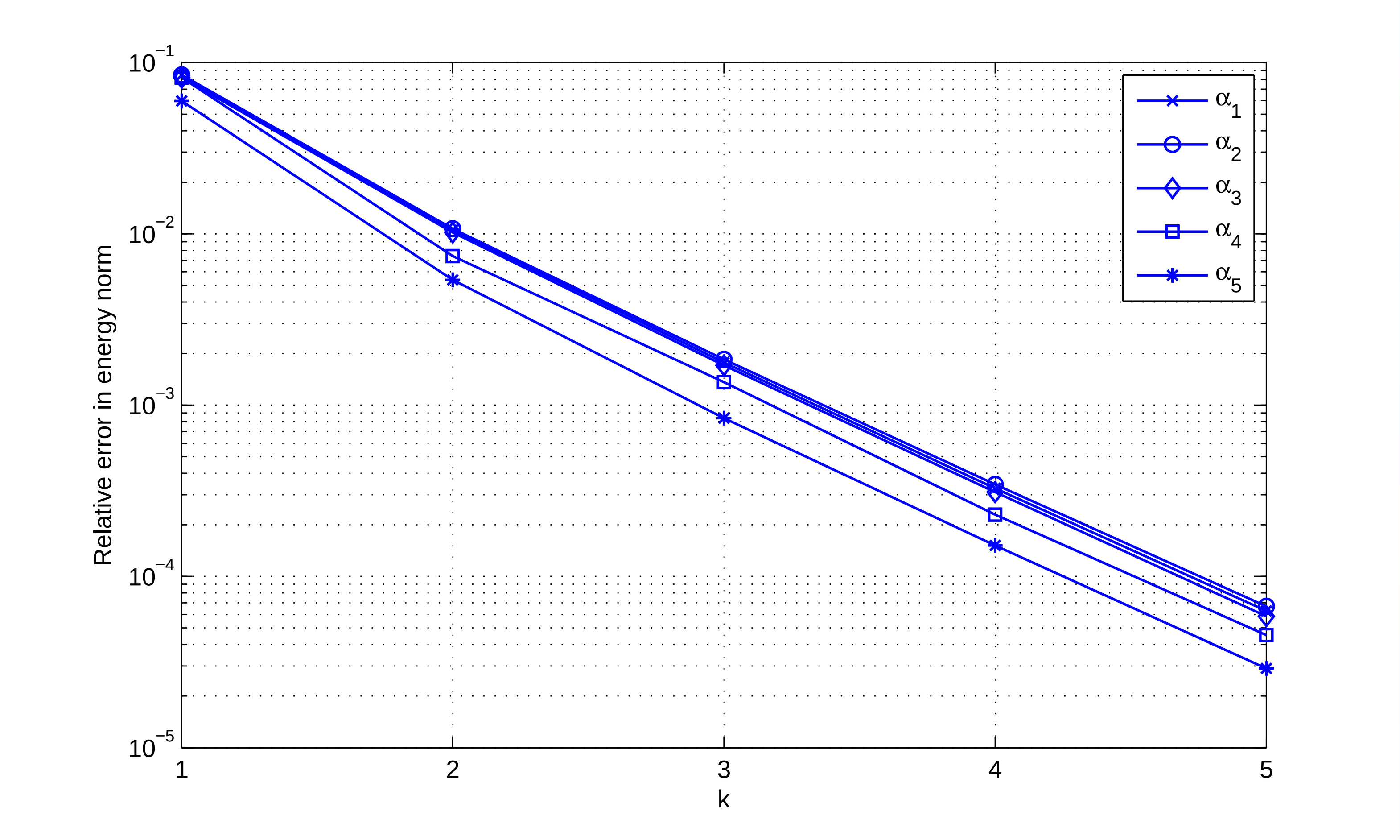
Relative error in energy norm (left). We get exponential convergence in k.

Corresponding error using $2k$ cg iterations (right) $\Rightarrow$ slower convergence for high condition numbers.

Preconditioner that works in the argument?

Convergence of global solution

Again $J = 3$ and $h_0 = 1/30$. We plot the error $u_J - u_J^k$ in energy norm (relative).

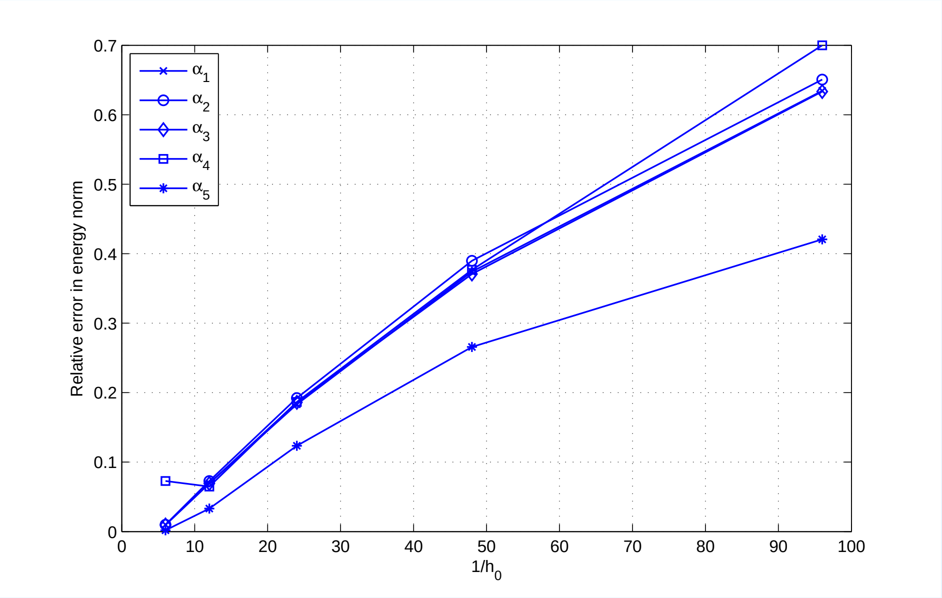


How does the error depend on h_0?

Remember

$$\|u_J - u_J^k\| \leq C \left(\|u_J\|_{L^\infty(\Omega)}/h_0 + \|f\|_{L^2(\Omega)} \right) \rho^{2k},$$

We let $J = 2$ and $k = 3$.



The bound is probably not sharp in terms of h_0.

Summary of this paper

1. We prove an *a priori* error bound and thereby convergence as $k \to \infty$ for Sym-AVMS, for fix h_0 and J.
2. The bound reveals that for fix h_0 and J we get *exponential decay* in the number of layers k.
3. Numerics experiments confirms this and furthermore reveals that a very small value $k \sim 2$ is needed for 2D examples in practise.
4. There are still improvements needed in the analysis in the case when $\frac{\max_x \alpha(x)}{\min_x \alpha(x)}$ is large and in the dependency on h_0. Preconditioner and/or wavelet basis might resolve this.

Other recent results and future directions

We have also studied

- multiscale methods for convection dominated stationary and hyperbolic problems
- a posteriori error estimation for Poisson equation, CG, DG, RT
- adaptive algorithms for local mesh/patch size refinement

Future projects include

- improving the convergence result
- adaptive algorithm for hyperbolic problems
- convergence of adaptive algorithms
- solving the coupled system using RT and DG
- multiscale in time
- implement AMVS on parallel machines, 3D