

Computation of eigenvalues using multiscale techniques

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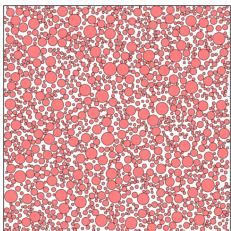
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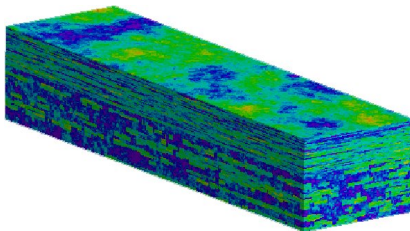
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Multiscale problems

Applications such as



▷ composite materials



▷ flow in a porous medium

require numerical solution of partial differential equations with rough data (module of elasticity, conductivity, or permeability).

Major challenge: Features on **multiple non-separated scales**.

Multiscale methods

Let A be rapidly varying data and consider a differential equation and its corresponding numerical approximation,

$$\mathcal{L}(A)u = f \qquad \mathcal{L}_h(A)u_h = f_h.$$

Classical finite element methods typically give

$$\|u - u_h\| \leq C(A, A')h^\gamma.$$

Multiscale methods seek an upscaled representation

$$\mathcal{L}_H(A)u_H = f_H$$

fulfilling $\|u_h - u_H\| \leq C(A)H^\gamma$ with C independent of A' .

How well is the spectrum of $\mathcal{L}$ preserved?

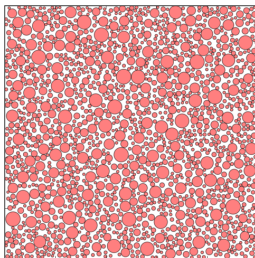
- 1 **Model problem**
- 2 Upscaling technique and error analysis
- 3 Numerical experiments
- 4 Applications to non-linear eigenvalue problems
- 5 Conclusions

Model multiscale eigenvalue problem

Prototypical self-adjoint eigenvalue problem

$$-\nabla \cdot \mathbf{A} \nabla u = \lambda u \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial\Omega$$

with data $0 < \alpha \leq A \leq \beta < \infty$

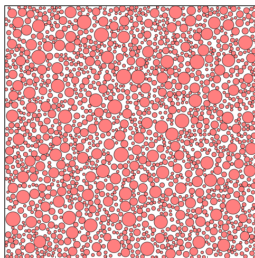


Model multiscale eigenvalue problem

Prototypical self-adjoint eigenvalue problem (variational form): find $u \in V := H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$a(u, v) := \int_{\Omega} (\mathbf{A} \nabla u) \cdot \nabla v \, dx = \lambda \int_{\Omega} u \cdot v \, dx \quad \text{for all } v \in V$$

with data $0 < \alpha \leq A \leq \beta < \infty$

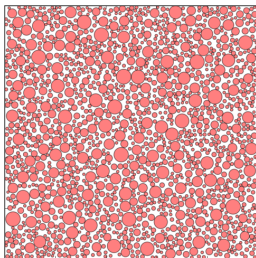


Model multiscale eigenvalue problem

Prototypical self-adjoint eigenvalue problem (FE approximation):
 $u_h \in V_h \subset V$ and $\lambda_h \in \mathbb{R}$ such that

$$a(u_h, v) := \int_{\Omega} (\mathbf{A} \nabla u_h) \cdot \nabla v \, dx = \lambda_h \int_{\Omega} u_h \cdot v \, dx \quad \text{for all } v \in V_h$$

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Model multiscale eigenvalue problem

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Numerical error (piecewise linear continuous FE approximation)

- For an eigenpair $(u^{(k)}, \lambda^{(k)})$ with $u^{(k)} \in H^2(\Omega)$ it holds

$$\begin{aligned} \lambda^{(k)} &\leq \lambda_h^{(k)} \leq \lambda^{(k)} + C(A, A', k) h^2, \\ \|u^{(k)} - u_h^{(k)}\| &:= \|A^{1/2} \nabla(u^{(k)} - u_h^{(k)})\|_{L^2(\Omega)} \leq C(A, A', k) h. \end{aligned}$$

- The mesh size h has to resolve the variations in A , e.g. $h < \epsilon$ if A is periodic.

Objectives

Investigate how the Localized Orth. Decomposition (LOD) in



A. Målqvist and D. Peterseim.

Localization of Elliptic Multiscale Problems.

Mathematics of Computation 2014

preserves the (low part of the) spectrum of $-\nabla \cdot A \nabla$.

Without assumptions on scales (A') or regularity (u):

$$\lambda_h \leq \lambda_H^{\text{ms}} \leq \lambda_h + CH^4,$$

$$|||u_h - u_H^{\text{ms}}||| \leq CH^2.$$



A. Målqvist and D. Peterseim.

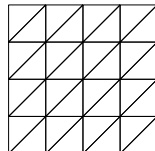
Computation of eigenvalues by numerical upscaling.

Numerische Mathematik 2015

- 1 Model problem
- 2 **Upscaling technique and error analysis**
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Multiscale decomposition

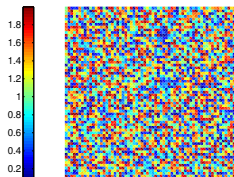
- (coarse) FE mesh $\mathcal{T}$ with parameter H
- P1-FE space $V_H := \{v \in V \mid \forall T \in \mathcal{T}, v|_T \in P_1(T)\}$
- $\mathfrak{I}_{\mathcal{T}} : V \rightarrow V_H$ a Clément interpolation operator



Decomposition

$$V = V_H \oplus V^f \quad \text{with } V^f := \text{kernel } \mathfrak{I}_{\mathcal{T}} = \{v \in V \mid \mathfrak{I}_{\mathcal{T}} v = 0\}$$

Example:



rough coefficient

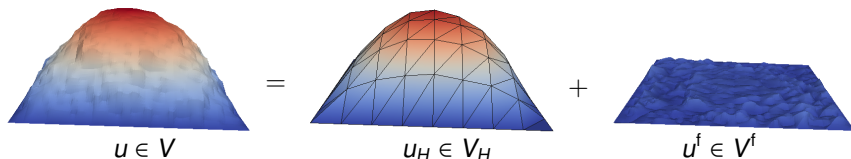
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Example:



Orthogonalization

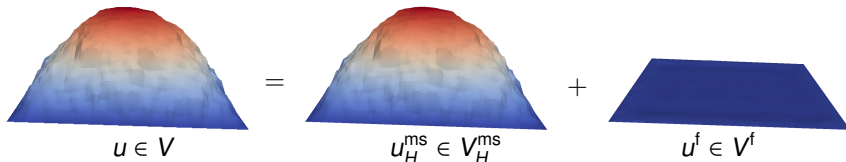
- For each $v \in V_H$ define finescale projection $\mathfrak{F}v \in V^f$ by

$$a(\mathfrak{F}v, w) = a(v, w) \quad \text{for all } w \in V^f$$

Orthogonal Decomposition

$$V = V_H^{\text{ms}} \oplus V^f \quad \text{with } V_H^{\text{ms}} := (V_H - \mathfrak{F}V_H)$$

Example:



Ideal multiscale representation

Given the space V_H^{ms} we construct a Galerkin approximation:

Ideal method

Find $u_H^{\text{ms}} \in V_H^{\text{ms}}$, $\lambda_H^{\text{ms}} \in \mathbb{R}$ such that

$$a(u_H^{\text{ms}}, v) = \lambda_H^{\text{ms}}(u_H^{\text{ms}}, v), \quad \forall v \in V_H^{\text{ms}}.$$

- We note that $\dim(V_H^{\text{ms}}) = \dim(V_H)$.
- For V_H^{ms} to be useful we need a discrete localized basis.

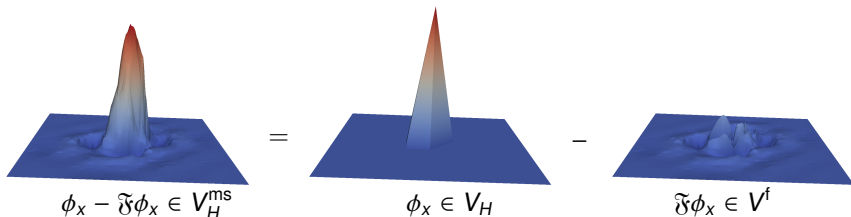
Modified nodal basis

- $\mathcal{N}$ denotes set of interior vertices of $\mathcal{T}$
- $\phi_x \in V_H$ denotes classical nodal basis function ($x \in \mathcal{N}$)
- $\mathfrak{F}\phi_x \in V^f$ denotes finescale correction of ϕ_x ($x \in \mathcal{N}$)

Ideal multiscale FE space

$$V_H^{\text{ms}} = \text{span} \{ \phi_x - \mathfrak{F}\phi_x \mid x \in \mathcal{N} \}$$

Example



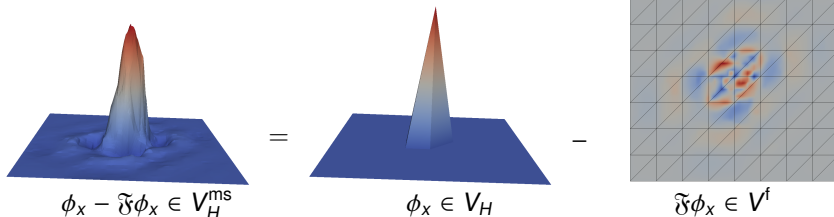
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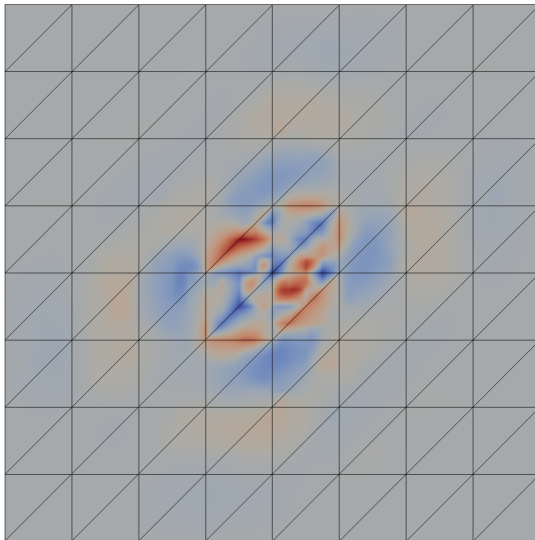
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Example

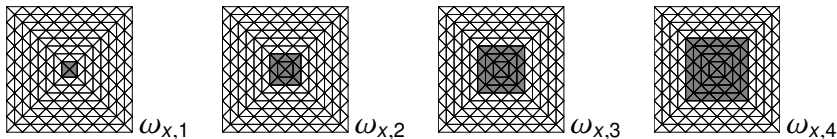


Modified nodal basis



Localization

- Define nodal patches of ℓ -th order $\omega_{x,\ell}$ about $x \in \mathcal{N}$



- Localized corrections $\mathfrak{F}\phi_{x,\ell} \in V^f(\omega_{x,\ell}) := \{v \in V^f \mid v|_{\Omega \setminus \omega_{x,\ell}} = 0\}$
solve

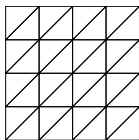
$$a(\mathfrak{F}\phi_{x,\ell}, w) = a(\phi_x, w) \quad \text{for all } w \in V^f(\omega_{x,\ell})$$

Localized multiscale FE spaces

$$V_{H,\ell}^{\text{ms}} = \text{span}\{\phi_x - \mathfrak{F}\phi_{x,\ell} \mid x \in \mathcal{N}\}$$

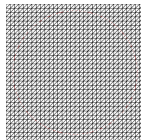
Fine scale discretization

- Finescale mesh



$\mathcal{T}$

mesh refinement



$\mathcal{T}_h$ with $h \leq H$

- Reference FE space

$$V_h := \{v \in V \mid \forall T \in \mathcal{T}(\Omega), v|_T \in P_1(T)\}$$

- Reference FE solution $u_h \in V_h$ and $\lambda_h \in \mathbb{R}$ solves

$$a(u_h, v) = \lambda_h(u_h, v) \quad \text{for all } v \in V_h$$

- Fully discrete corrections $\mathfrak{F}\phi_{x,\ell}^h \in V_h^f(\omega_{x,\ell}) := V^f(\omega_{x,\ell}) \cap V_h$:

$$a(\mathfrak{F}\phi_{x,\ell}^h, w) = a(\phi_x, w) \quad \text{for all } w \in V_h^f(\omega_{x,\ell})$$

Localized Orthogonal Decomposition (LOD)

Fully discrete multiscale FE spaces

$$V_{H,\ell}^{\text{ms},h} = \text{span}\{\phi_x - \mathfrak{F}\phi_{x,\ell}^h \mid x \in \mathcal{N}\}$$

Fully discrete multiscale approximation $u_{H,\ell}^{\text{ms},h} \in V_{H,\ell}^{\text{ms},h}$, $\lambda_{H,\ell}^{\text{ms},h} \in \mathbb{R}$

$$a(u_{H,\ell}^{\text{ms},h}, v) = \lambda_{H,\ell}^{\text{ms},h}(u_{H,\ell}^{\text{ms},h}, v) \quad \text{for all } v \in V_{H,\ell}^{\text{ms},h}$$

Remarks:

- $\dim V_{H,\ell}^{\text{ms},h} = |\mathcal{N}| = \dim V_H$
- The basis functions have local support, with overlap depending on ℓ , and are independent.

A priori error analysis (discrete case)

Lemma (Truncation error)

$$||| \mathfrak{F}\phi_x^h - \mathfrak{F}\phi_{x,\ell}^h ||| \leq C_1 \gamma^\ell ||| \mathfrak{F}\phi_x^h |||.$$

$C_1 < \infty$ and $\gamma < 1$ depends on β/α , not A' .

By choosing $\ell = C_2 \log(H^{-1})$ with appropriate C_2 we guarantee that the truncation leads to a higher order perturbation:

Theorem

$$\begin{aligned} \lambda_h^{(k)} &\leq \lambda_{H,\ell}^{ms,h,(k)} \leq \lambda_h^{(k)} + CH^4, \\ ||| u_h^{(k)} - u_{H,\ell}^{ms,h,(k)} ||| &\leq CH^2, \end{aligned}$$

with C independent of A' and the regularity of the eigenfunctions.

Sketch of proof (ideal method)

For the k :th eigenvalue it holds

Theorem

$$\lambda_h^{(k)} \leq \lambda_H^{ms,(k)} \leq \lambda_h^{(k)} + CH^4,$$

C independent of A' and only H^1 -regularity of the eigenfunctions.

Sketch of proof for the **lowest** eigenvalue $\lambda_h^{(1)}$ with eigenfunction u :

- Let $u = u_c + u_f$ with $u_c \in V_H^{ms}$ and $u_f \in V^f$, such that $\|u\|_{L^2(\Omega)} = 1$. Then

$$\begin{aligned} \lambda_H^{ms,(1)} &\leq \frac{a(u_c, u_c)}{(u_c, u_c)} \leq \frac{a(u, u)}{(u_c, u_c)} = \frac{a(u, u)}{(u - u_f, u - u_f)} \\ &= \frac{\lambda_h^{(1)}}{(u, u) - 2(u, u_f) + (u_f, u_f)} \leq \frac{\lambda_h^{(1)}}{1 - 2(u, u_f)}. \end{aligned}$$

Sketch of proof (ideal method)

For the k :th eigenvalue it holds

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$$\lambda_h^{(k)} \leq \lambda_H^{ms,(k)} \leq \lambda_h^{(k)} + CH^4,$$

C independent of A' and only H^1 -regularity of the eigenfunctions.

Sketch of proof for the **lowest** eigenvalue $\lambda_h^{(1)}$ with eigenfunction u :

- Since $\mathfrak{I}_{\mathcal{T}} u_f = 0$, $(\mathfrak{I}_{\mathcal{T}} u, u_f) = 0$ (weighted Clement, CV99),
 $a(u_c, u_f) = 0$, and $|||u|||^2 = \lambda_h^{(1)}$, we have

$$(u, u_f) = (u - \mathfrak{I}_{\mathcal{T}} u, u_f - \mathfrak{I}_{\mathcal{T}} u_f) \leq CH^2 |||u||| \cdot |||u_f||| \leq C' H^2 |||u_f|||,$$

$$|||u_f|||^2 = a(u, u_f) = \lambda_h^{(1)} (u - \mathfrak{I}_{\mathcal{T}} u, u_f - \mathfrak{I}_{\mathcal{T}} u_f) \leq CH^2 |||u_f|||.$$

- We conclude $\lambda_H^{ms,(1)} \leq \frac{\lambda_h^{(1)}}{1 - CH^4} \leq \lambda_h^{(1)} + 2CH^4$.

Outline

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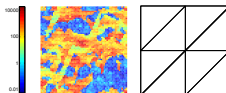
Eigenvalue Problem



k	$\lambda_h^{(k)}$	$e^{(k)}(1/2 \sqrt{2})$	$e^{(k)}(1/4 \sqrt{2})$	$e^{(k)}(1/8 \sqrt{2})$	$e^{(k)}(1/16 \sqrt{2})$
1	9.6436869	0.003494567	0.000034466	0.000000546	0.000000010
2	15.1989274	0.009621397	0.000079887	0.000000845	0.000000010
3	19.7421815	0.023813222	0.000213097	0.000002073	0.000000023
4	29.5281571	0.096910157	0.000724615	0.000006574	0.000000076
5	31.9265496	0.094454625	0.000874659	0.000009627	0.000000138
6	41.4922250	-	0.002395227	0.000019934	0.000000254
7	44.9604884	-	0.002443271	0.000019683	0.000000223
8	49.3631826	-	0.003651870	0.000028869	0.000000308
9	49.3655623	-	0.004266472	0.000032835	0.000000355
10	56.7389993	-	0.006863742	0.000055219	0.000000618
11	65.4085991	-	0.011534878	0.000082414	0.000000856
12	71.0947630	-	0.012596114	0.000090083	0.000001002
13	71.6064671	-	0.014249938	0.000098736	0.000001006
14	79.0043994	-	0.021801461	0.000164436	0.000001605
15	89.3706421	-	0.033550079	0.000211985	0.000002296
16	92.3648207	-	0.040060692	0.000239441	0.000002295
17	97.4459210	-	0.037438984	0.000284936	0.000002724
18	98.7545147	-	0.044544409	0.000269854	0.000002559
19	98.7545639	-	0.047835987	0.000276139	0.000002539
20	101.6755971	-	0.038203654	0.000297356	0.000002909

Table : Errors $e^{(k)}(H) =: \frac{\lambda_H^{\text{ms},(k)} - \lambda_h^{(k)}}{\lambda_h^{(k)}}$ and $h = 2^{-7} \sqrt{2}$.

Eigenvalue Problem



k	$\lambda_h^{(k)}$	$e^{(k)}(1/2\sqrt{2})$	$e^{(k)}(1/4\sqrt{2})$	$e^{(k)}(1/8\sqrt{2})$	$e^{(k)}(1/16\sqrt{2})$
1	21.4144522	5.472755371	0.237181706	0.010328293	0.000781683
2	40.9134676	-	0.649080539	0.032761482	0.002447049
3	44.1561133	-	1.687388874	0.097540102	0.004131422
4	60.8278691	-	1.648439518	0.028076168	0.002079812
5	65.6962136	-	2.071005692	0.247424446	0.006569640
6	70.1273082	-	4.265936007	0.232458016	0.016551520
7	82.2960238	-	3.632888104	0.355050163	0.013987920
8	92.8677605	-	6.850048057	0.377881216	0.049841235
9	99.6061234	-	10.305084010	0.469770376	0.026027378
10	109.1543283	-	-	0.476741452	0.005606426
11	129.3741945	-	-	0.505888044	0.062382302
12	138.2164330	-	-	0.554736550	0.039487317
13	141.5464639	-	-	0.540480876	0.043935515
14	145.7469718	-	-	0.765411709	0.034249528
15	152.6283573	-	-	0.712383825	0.024716759
16	155.2965039	-	-	0.761104705	0.026228034
17	158.2610708	-	-	0.749058367	0.091826207
18	164.1452194	-	-	0.840736127	0.118353184
19	171.1756923	-	-	0.946719951	0.111314058
20	179.3917590	-	-	0.928617606	0.119627862

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The quadratic eigenvalue problem

The vibrations of a spatially discretized structure can mathematically be described as a quadratic eigenvalue problem:

$$Kx + \lambda_h Dx + \lambda_h^2 Mx = 0,$$

where K is stiffness, D is damping, and M is mass matrix. If D is real and symmetric the eigenvalues are complex conjugate (or real).

- We consider the case $K_{ij} = (A \nabla \phi_j, \nabla \phi_i)$ where A varies rapidly in space and $M_{ij} = (\phi_j, \phi_i)$.
- The damping can have different form, we let $D_{ij} = d(\phi_j, \phi_i)$ be real.
- Rayleigh damping $D = \alpha_0 M + \alpha_1 K$ leads to unchanged eigenmodes, systems of Rayleigh damped components can be modelled by letting α_0 and α_1 be functions.

QEP: Linearization and approximation

Linearization: ($y = \lambda_h x$)

$$\begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_h \begin{bmatrix} -D & -M \\ M & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

On weak form: find (u_1, u_2) and $\lambda_h \in \mathbb{C}$ such that,

$$a(u, v) = \lambda_h b(u, v),$$

$$a(u, v) = (A \nabla u_1, \nabla v_1) + (u_2, v_2),$$

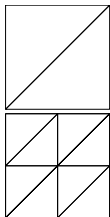
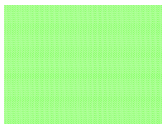
$$b(u, v) = -d(u_1, v_1) - (u_2, v_1) + (u_1, v_2).$$

Multiscale formulation:

Find $u_H^{\text{ms}} \in V_H^{\text{ms}} \times V_H^{\text{ms}}$ and $\lambda_H^{\text{ms}} \in \mathbb{C}$ such that,

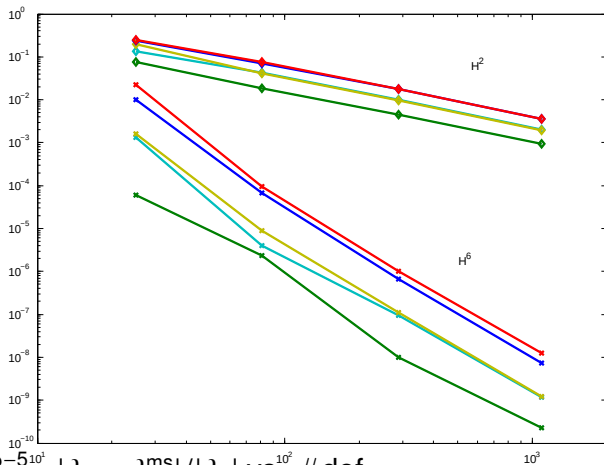
$$a(u_H^{\text{ms}}, v) = \lambda_H^{\text{ms}} b(u_H^{\text{ms}}, v), \quad \forall v \in V_H^{\text{ms}} \times V_H^{\text{ms}}.$$

Numerical experiment



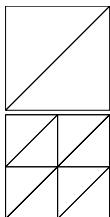
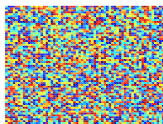
$$H = 2^{-1}, 2^{-2}, \dots, 2^{-5 \cdot 10^1}$$

$$h = 2^{-6}, k = \infty$$

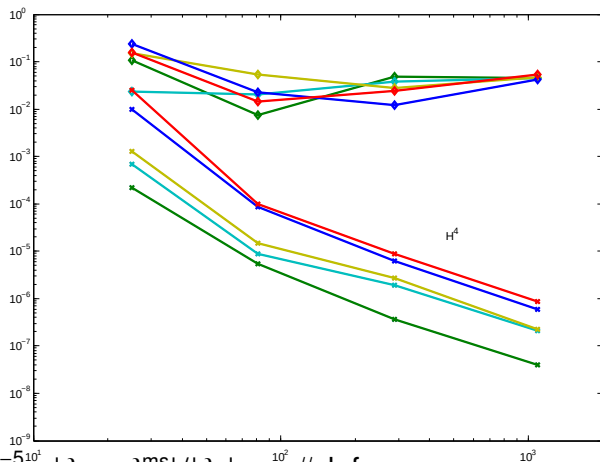


$$A = 1, d(u, v) = \int_{\Omega} (1 + \sin(10x)) u v \, dx$$

Numerical experiment

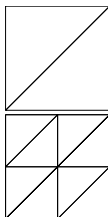
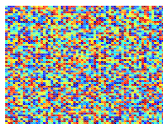


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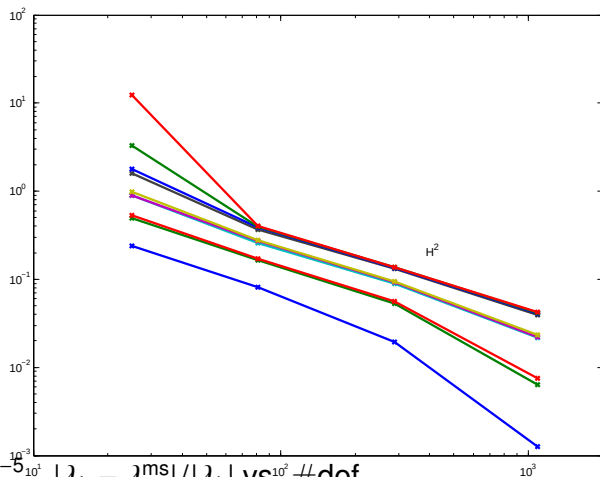
$$A \text{ (pic)}, d(u, v) = \int_{\Omega} (1 + \sin(10x)) u v \, dx$$

Numerical experiment



$$H = 2^{-1}, 2^{-2}, \dots, 2^{-5} \quad |\lambda_h - \lambda_H^{\text{ms}}| / |\lambda_h| \text{ vs. } \# \text{dof}$$

$$h = 2^{-6}, k = \infty$$



$$A \text{ (pic)}, d(u, v) = \int_{\Omega} (2 - x - y) A \nabla u \nabla v \, dx$$

QEP: Analysis

We note that the operator $B_h : V_h \times V_h \rightarrow V_h \times V_h$ defined by,

$$a(B_h u, v) = b(u, v),$$

has eigenvalues $B_h u = \mu_h u = \lambda_h^{-1} u$ since,

$$b(u, v) = a(B_h u, v) = a(\mu_h u, v) = \mu_h a(u, v).$$

The LOD approximation of the eigenvalue can be constructed in a similar way: $B_H : V_H^{\text{ms}} \times V_H^{\text{ms}} \rightarrow V_H^{\text{ms}} \times V_H^{\text{ms}}$,

$$a(B_H u_H, v) = b(u_H, v),$$

has eigenvalues $B_H u_H = \mu_H u_H$.



I. Babuška and J. Osborn.

Eigenvalue problems. Handbook of numerical analysis II, 1991.

Error in eigenvalues

Theorem

Let μ_h be an eigenvalue of B_h with algebraic and geometric multiplicity 1 (simple). If

$$d(v - \mathfrak{I}_{\mathcal{T}} v, w) + d(v, w - \mathfrak{I}_{\mathcal{T}} w) \leq CH^\gamma \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}.$$

$$|\mu_h - \mu_H| \leq CH^{2\gamma},$$

if $H \leq C'$, where C' depends on B_h and γ .

If $d(v, w) = (fv, w) \rightarrow \gamma = 2$ and $d(v, w) = (f\nabla v, \nabla w) \rightarrow \gamma = 0!$

We will consider a new split (with triangular a):

$$a(u, v) = (A\nabla u_1, \nabla v_1) + d(u_2, v_1) + (u_2, v_2),$$

$b(u, v) = -(u_2, v_1) + (u_1, v_2)$ and a basis that depend on d .

- 1 Model problem
- 2 Upscaling technique and error analysis
- 3 Numerical experiments
- 4 Applications to non-linear eigenvalue problems
- 5 **Conclusions**

Conclusion

- The Localized Orthogonal Decomposition (LOD) technique preserves the low spectrum of the operator. In particular the eigenvalue error is proportional to H^4 after postprocessing H^6 .
- Numerical experiments indicates even higher rates possibly due to additional regularity in the solution that is not taken advantage of in the analysis.
- The technique is applicable also for non-linear eigenvalue problems without pre-asymptotic effects in the convergence.
- More work is needed in the analysis to get sharp bounds for the quadratic eigenvalue problem with more challenging damping.

Thank you for your attention!