On Adaptive Finite Element Methods Based on A Posteriori Error Estimates

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Outline

- The Finite Element Method
- A Posteriori Error Estimation
- Adaptive Algorithms
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Abstract Model Problem

Strong form. Find $u \in H_0^1(\Omega)$ such that

$$\mathcal{A}u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

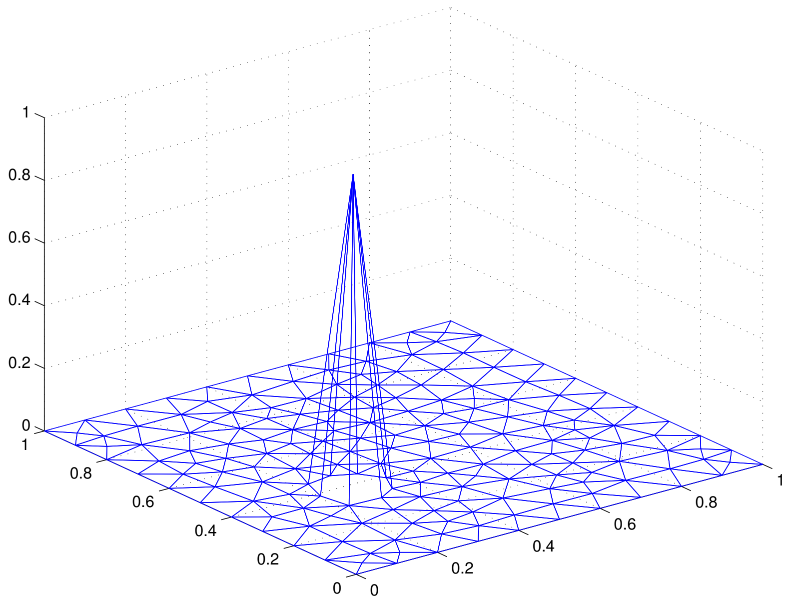
where $\mathcal{A}$ is a second order differential operator.

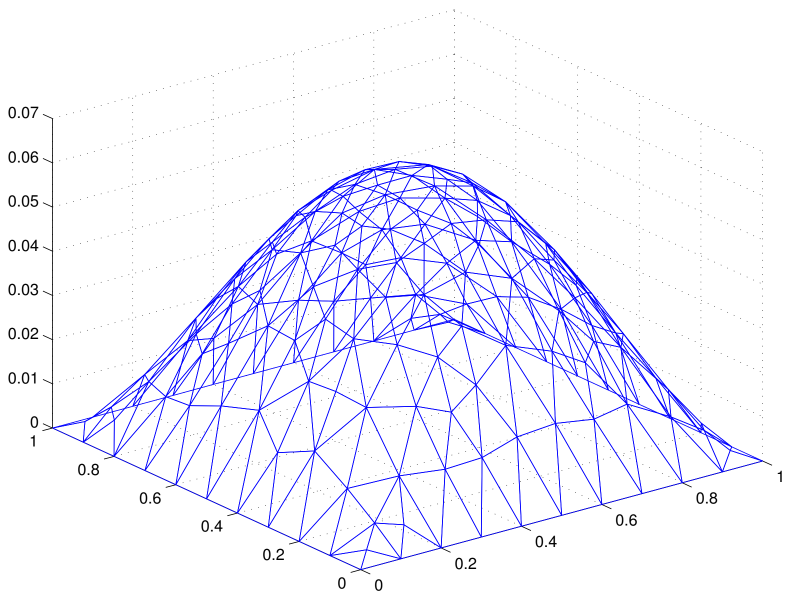
Weak form. Let $a(v, w) = (\mathcal{A}v, w)$ for all $v, w \in H_0^1$. The weak form reads: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega).$$

Discretization

Let $V_h = span\{\varphi_i\} \subset H_0^1(\Omega)$ be the finite dimensional space of piecewise linear polynomials on a triangulation $\mathcal{K} = \{K\}$ with mesh parameter h.





The Finite Element Method

Weak form. Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \text{for all } v \in H_0^1(\Omega).$$

Finite Element Method. Find $U \in V_h$ such that

$$a(U,v) = (f,v) \quad \text{for all } v \in V_h.$$

We subtract the two equations and introduce the error $e = u - U$.

A Posteriori Error Estimation

Galerkin Orthogonality.

$$a(e, v) = 0 \quad \text{for all } v \in V_h.$$

Error Representation Formula. We can proceed with the following calculation for an arbitrary function $\phi \in H_0^1$,

$$a(e, \phi) = a(e, \phi - \pi\phi) = (\mathcal{A}e, \phi - \pi\phi) = (f - \mathcal{A}U, \phi - \pi\phi)$$

Energy Norm Estimate

We choose $\phi = e$ to get,

$$\|e\|_a^2 = a(e,e) = (f - \mathcal{A}U, e - \pi e)$$
$$\leq C\|hR(U)\|\|\nabla e\| \leq C\|hR(U)\|\|e\|_a,$$

which is possible if

$$\|\nabla e\| \leq C\|e\|_a.$$

We get

$$\|e\|_a \leq C\|hR(U)\|$$

Linear Functional Estimate

If we instead let $\phi \in H_0^1$ solve the following dual problem,

$$(v, \mathcal{A}^*\phi) = (v, \psi), \quad \text{for all } v \in H_0^1,$$

we get

$$\begin{aligned}
(e, \psi) &= (e, \mathcal{A}^*\phi) = (\mathcal{A}e, \phi) \\
&= a(e, \phi) = (f - \mathcal{A}U, \phi - \pi\phi) \\
&= (R(U), \phi - \pi\phi)
\end{aligned}$$

Adaptive Algorithm

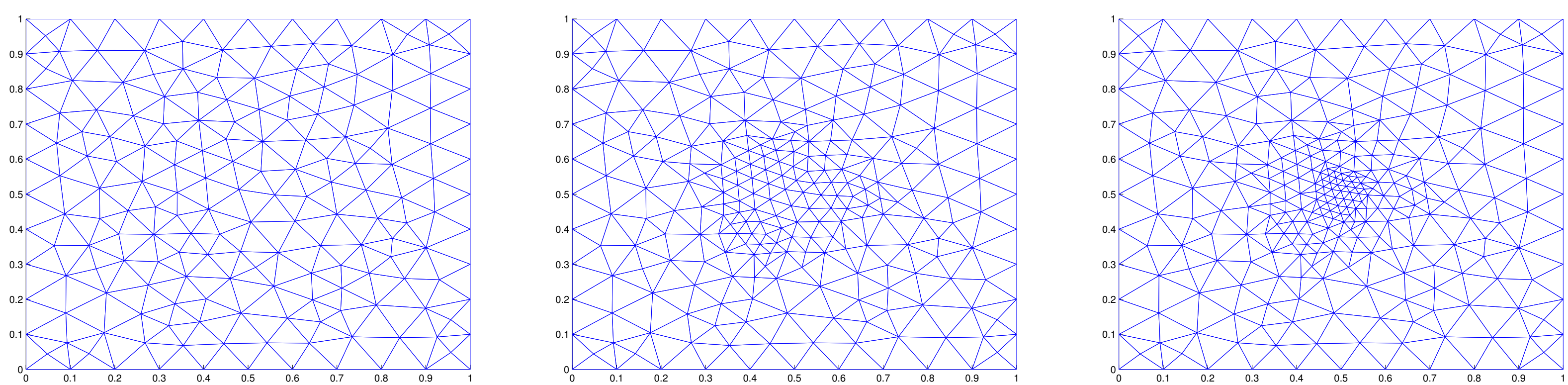
We study the energy norm estimate,

$$\|e\|_a^2 \leq C\|hR(U)\|^2 = C \sum_{K \in \mathcal{K}} \|hR(U)\|_K^2$$
$$= C \sum_{K \in \mathcal{K}} \rho_K(U),$$

where we refer to ρ_K as element indicators.

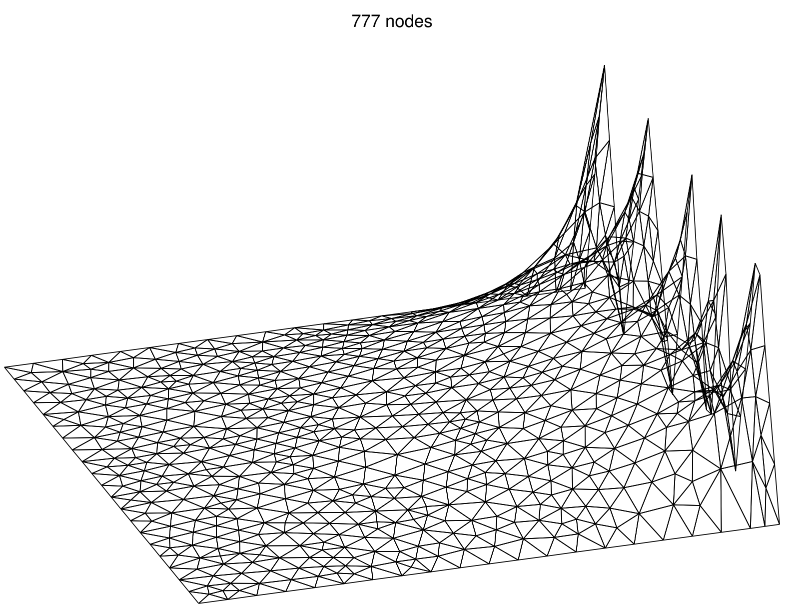
Adaptive Algorithm

- Use FEM to calculate the solution U.
- Calculate the element indicators ρ_K.
- Refine elements where $\rho_K(U)$ is large and return to one, or stop if $\sum_{K\in\mathcal{K}} \rho_K(U)$ is sufficiently small.



Paper I

A Posteriori Error Analysis of the Boundary Penalty Method



The Model Problem

The Dirichlet problem:

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma, \end{cases}$$

Boundary penalty method:

$$\begin{cases} -\triangle u_\epsilon = f & \text{in } \Omega, \\ -\partial_n u_\epsilon = \epsilon^{-1}(u_\epsilon - g) & \text{on } \Gamma. \end{cases}$$

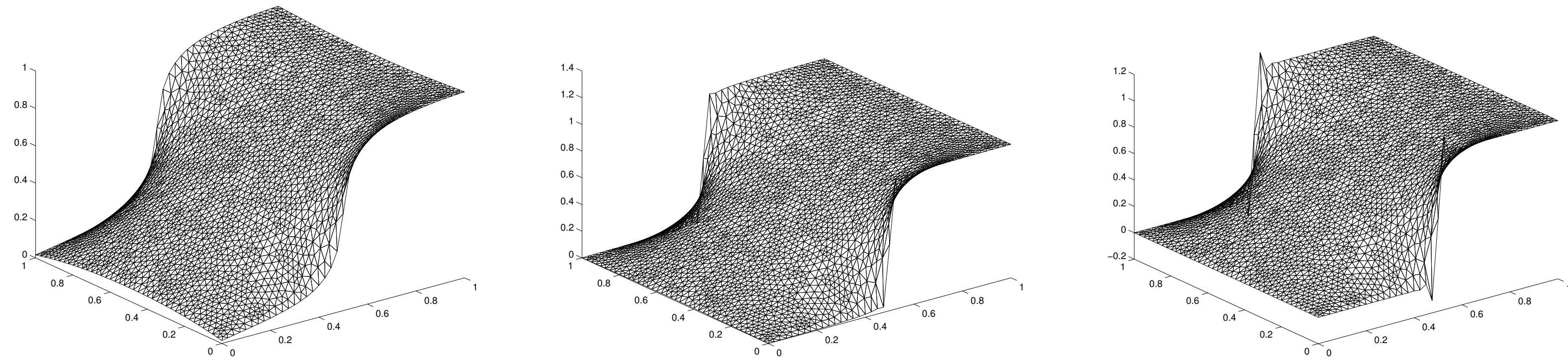
Motivation

$$-\partial_n u = \epsilon(x)^{-1}(u - g_D(x)) + g_N(x)$$

- One form can represent Dirichlet, Neumann and Robin boundary condition simply by changing the parameter $\epsilon(x)$.
- The method is used in various FEM codes.
- It can be used on interior boundaries in non-matching grid problems.
- A simpler compliment to Nitsche's method.

Motivation

How do we choose ϵ to impose Dirichlet conditions weakly?



- To small ϵ will give large condition numbers.
- We need to choose ϵ as large as possible without increasing the error.

Previous Work

Earlier a priori work by I. Babuška, J. W. Barrett and C. M. Elliott among others shows

$$\|u - u_\epsilon\|_0 \leq C\epsilon\|u\|_2$$

and if we refer to the boundary penalty solution as U we have

$$\|e\|_1 = \|u - U\|_1 \leq Ch\|u\|_2$$

for piecewise linears with $\epsilon = h$.

Our Contributions

- A posteriori error estimates in the energy norm.
- A posteriori error estimates in the $L^2(\Omega)$ norm.
- Adaptive strategy to choose h and ϵ.
- Examples with simple and more complicated boundary conditions where this strategy works.

The Boundary Penalty Method

Finite Element Method. Find $U \in V$ such that

$$(\nabla U, \nabla v) + (\epsilon^{-1} U, v)_\Gamma = (f, v) + (\epsilon^{-1} g, v)_\Gamma \quad \forall v \in V.$$

Green's formula yields the following identity,

$$(\nabla u, \nabla v) - (\partial_n u, v)_\Gamma = (f, v) \quad \forall v \in H^1(\Omega),$$

Error Representation Formula.

$$(\nabla e, \nabla v) + (\epsilon^{-1} e, v)_\Gamma = (\partial_n u, v)_\Gamma \quad \forall v \in V.$$

A Posteriori Error Estimate

Energy Norm.

$$\|\nabla e\| \leq C \left(\|hR(U)\| + \|g - U\|_{1/2,\Gamma} \right)$$

$L^2(\Omega)$ **Norm.**

$$\|e\| \leq C \left(\|h^2 R(U)\| + \|g - U\|_{-1/2,\Gamma} \right)$$

A Posteriori Error Estimate

Energy Norm.

$$\|g - U\|_{1/2,\Gamma} \leq C\|g - Pg\|_{1/2,\Gamma} + \epsilon C \left(\|P(\partial_n U)\|_{1/2,\Gamma} + \sum_{\partial K \cap \Gamma \neq \emptyset} \|R(U)\|_K \right)$$

$L^2(\Omega)$ **Norm.**

$$\|g - U\|_{-1/2,\Gamma} \leq C\|g - Pg\|_{-1/2,\Gamma} + \epsilon C \left(\|P(\partial_n U)\|_{-1/2,\Gamma} + \|\nabla e\| + \|hR(U)\| \right)$$

Adaptive Strategy

We have the a posteriori estimate in energy norm with two terms

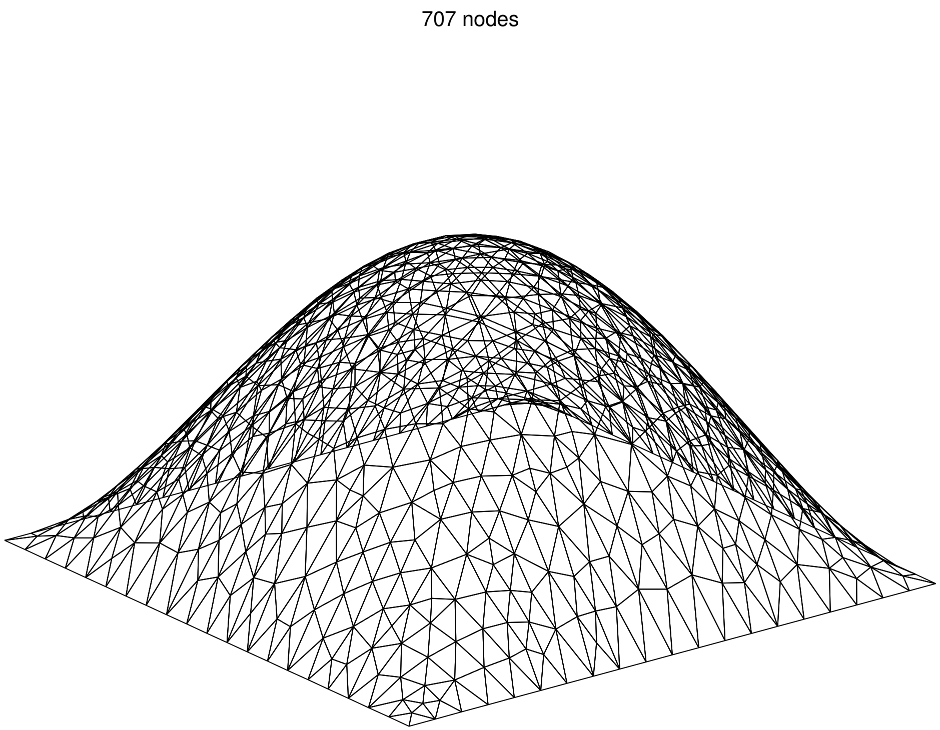
$$\|\nabla e\| \leq C(r_1 + r_2)$$

where $r_1 = (\|hR(U)\| + \|g - Pg\|_\Gamma)$,
$r_2 = \epsilon(\|P(\partial_n U)\|_{1/2,\Gamma} + \sum_{\partial K \cap \Gamma \neq \emptyset} \|R(U)\|_K)$.

- Solve problem with $\epsilon_0 = h$, calculate r_i.
- Do h-refinement if r_1 is to big.
- Let $\epsilon = \epsilon_0 r_1 / r_2$ (weighted if h is refined).

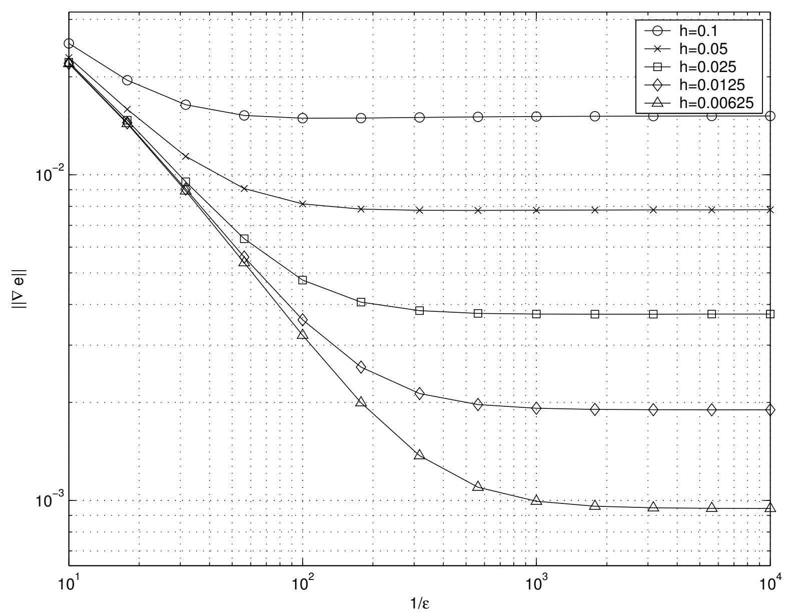
Numerical Examples

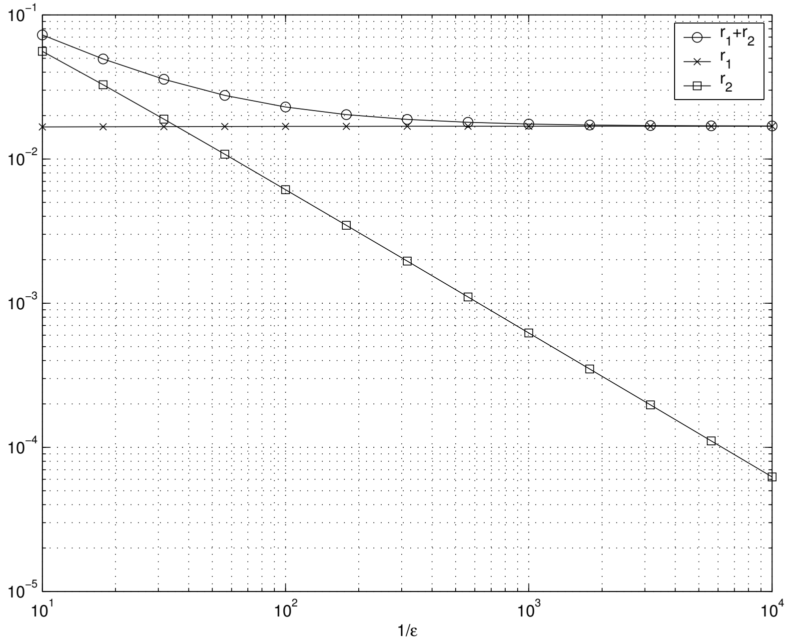
We let $g = 0$ on Γ and we choose f such that $u = x(1-x)y(1-y)$.



Numerical Examples

We compare true and estimated error.

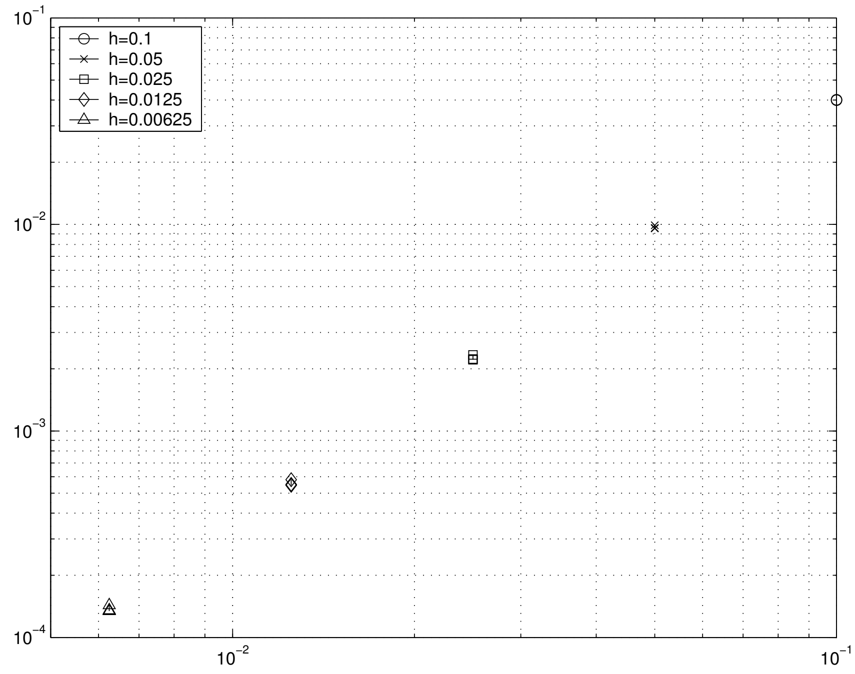




We second plot is for $h = 0.025$ we see a slight over estimate in r_1 which is due to the estimation.

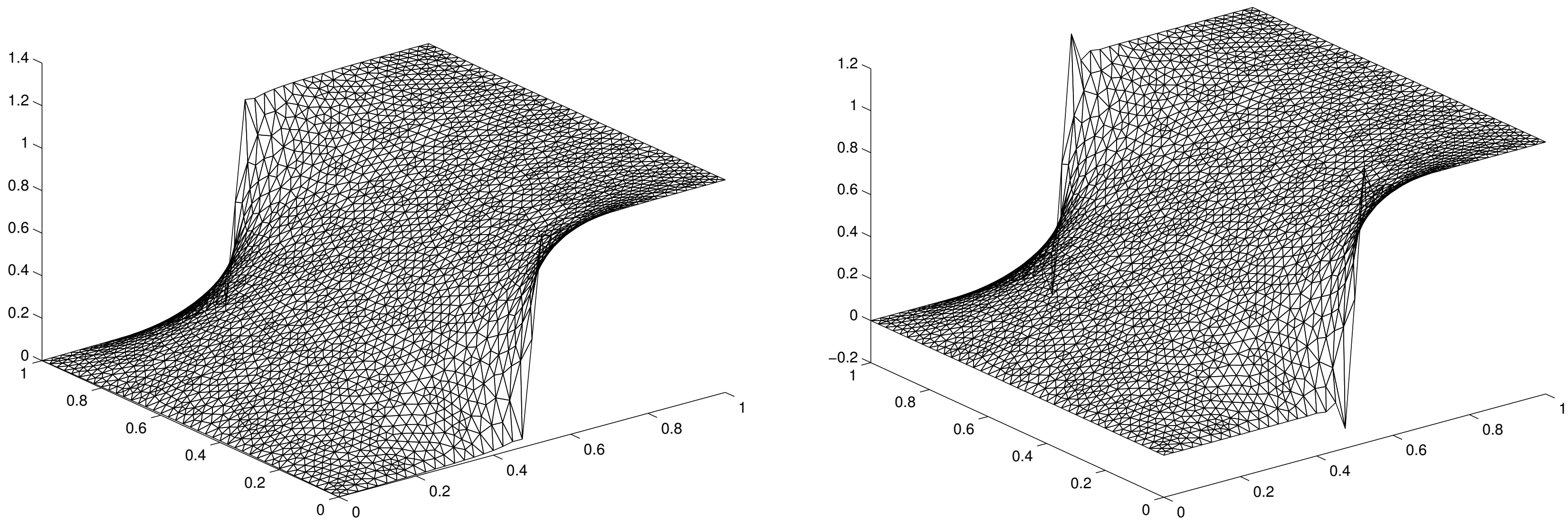
Numerical Examples

We study how $\epsilon_0 \in [10^{-7}, 10^{-1}]$ affects ϵ and how ϵ depends on h.



Numerical Examples

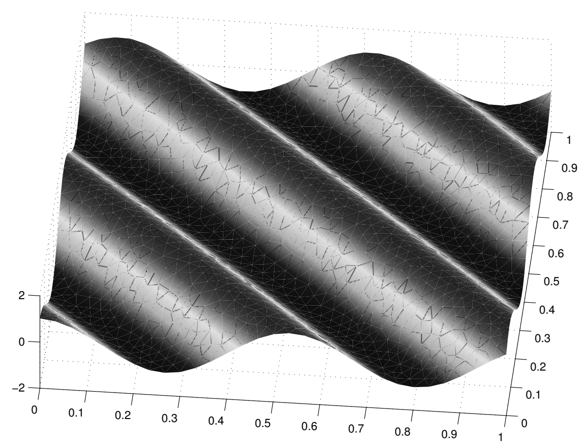
Why did we get oscillations for really small ϵ?



This will make $U \approx Pg$. Solution using adaptive algorithm to the left.

Paper II

A Posteriori Error Analysis of Stabilized Finite Element Approximations of the Helmholtz Equation on unstructured grids



One Dimensional Model Problem

Helmholtz Equation.

$$\begin{cases} -u'' - k^2\, u = 0 \quad \text{in } \Omega, \\ \hfill u'(0) = ik, \\ \hfill u'(\pi) = ik\, u(\pi)\,, \end{cases}$$

where $\Omega = [0, \pi]$ and analytic solution $u(x) = e^{ikx}$.

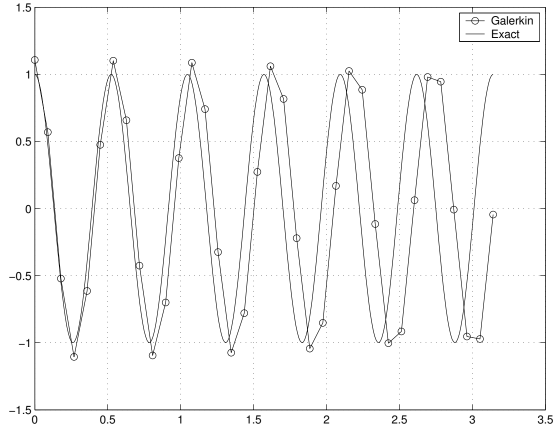
Weak Form. Find $u \in H^1(\Omega)$ such that

$$(u', v') - k^2\,(u, v) - ik\, u(\pi)v(\pi)^* = -ik\, v(0)^*,$$

for all $v \in H^1(\Omega)$,

Motivation

- Discretization gives an inaccurate numerical wave number (pollution).



- The Helmholtz equation is very important in acoustics and electro-magnetics.

Previous Work

- Analysis in one dimension that gives correct numerical wave number using the Galerkin least-squares method by Hughes et. al. and Generalized finite element method by Babuška et. al.
- Analysis in two dimensions on structured grids by Harari et. al.
- A gain in accuracy was detected by Wu et. al. when solving Helmholtz equation in two and three dimensions on unstructured grids.

Our Contributions

- A posteriori error analysis of the GLS method in one and two dimensions.
- Analysis of how stochastic perturbations in the mesh affects the numerical wave number in one and two dimensions.
- Suggestions of how existing method for structured meshes can be modified to suite an unstructured mesh.
- Unstructured meshes are of course of great interest in practice. (isotropic)

Galerkin Least-Squares

Find $u \in H^1(\Omega)$ such that

$$(U', v') - k^2 (U, v) + (\tau \mathcal{A}U, \mathcal{A}v)_{\tilde{\Omega}} - ik\, U(\pi) v(\pi)^* = -ik\, v(0)^*, \text{ for all } v \in V_h,$$

where $\mathcal{A} = -\partial^2/\partial x^2 - k^2$, τ is the method parameter, and $\tilde{\Omega}$ is the union of element interiors.
For piecewise linears $(\mathcal{A}U, \mathcal{A}v)_K = k^4 (U, v)_K$ which gives...

Galerkin Least-Squares

Find $U \in V_h$ such that

$$(U', v') - k^2(1 - \tau\, k^2)\,(U, v) - ik\, U(\pi)v(\pi)^* \\ = -ik\, v(0)^*, \text{ for all } v \in V_h,$$

or with $p = 1 - \tau\, k^2$, find $U \in V_h$ such that

$$(U', v') - pk^2(U, v) - ik\, U(\pi)v(\pi)^* \\ = -ik\, v(0)^*, \text{ for all } v \in V_h.$$

A Posteriori Error Analysis

Error Representation formula.

$$(e,\psi) = (k^2\,U, \phi - \pi\phi) + (\tau\,k^4 U, \pi\phi).$$

We choose τ such that $(e,\psi) = 0$ i.e.

$$\tau = -\frac{(k^2\,U, \phi - \pi\phi)}{(k^4 U, \pi\phi)}$$

$$p = 1 - \tau k^2 = \frac{(U,\phi)}{(U,\pi\phi)}$$

Unstructured Mesh

We introduce perturbations on the mesh.

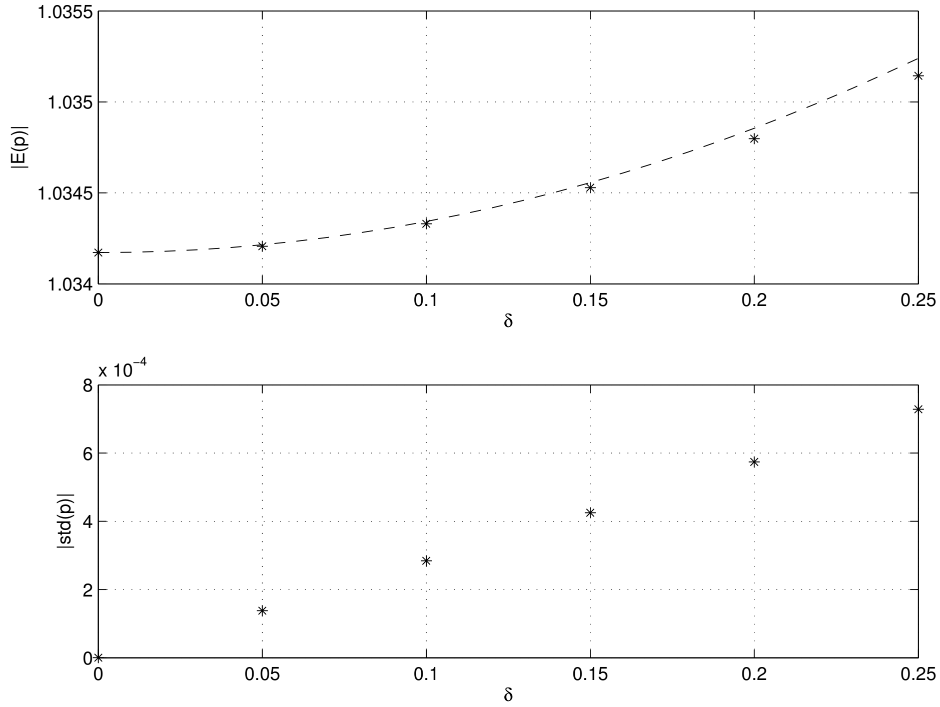
$$\begin{cases} x_0 & = & 0 \\ x_i & = & \frac{i\pi}{n} + \delta_i, \quad \text{for } i = 1, \ldots, n-1, \\ x_n & = & \pi, \end{cases}$$

where $\delta_i \in U([-\frac{\delta\pi}{2n}, \frac{\delta\pi}{2n}])$.

$$\hat{p}(\{\delta_i\}) = 1 - \tau k^2 = \frac{(U, \phi)}{(U, \pi\phi)}$$

Unstructured Mesh

Given δ we show that $E[\hat{p}] = 1 + C(hk)^2(1 + \frac{\delta^2}{2})$ makes $E[\bar{e}_\psi] = 0$ where $\bar{e}_\psi \approx (e, \psi)$.



Error Estimate

With this choice of p we show (Chebyshev gives $P(|e| > \epsilon) \leq \text{Var}(e)/\epsilon^2$) that for each ϵ there exists a constant C such that

$$P(|\bar{e}_\psi| \leq C\delta h^{5/2}k^3) > 1 - \epsilon.$$

Numerical tests gives the even better

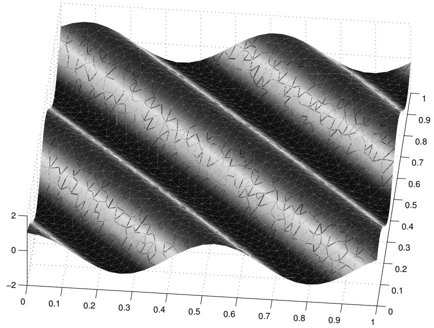
$$P(|\bar{e}_\psi| \leq C\delta h^{7/2}k^4) > 1 - \epsilon.$$

This means that the mean i correct but the variance grows with k.

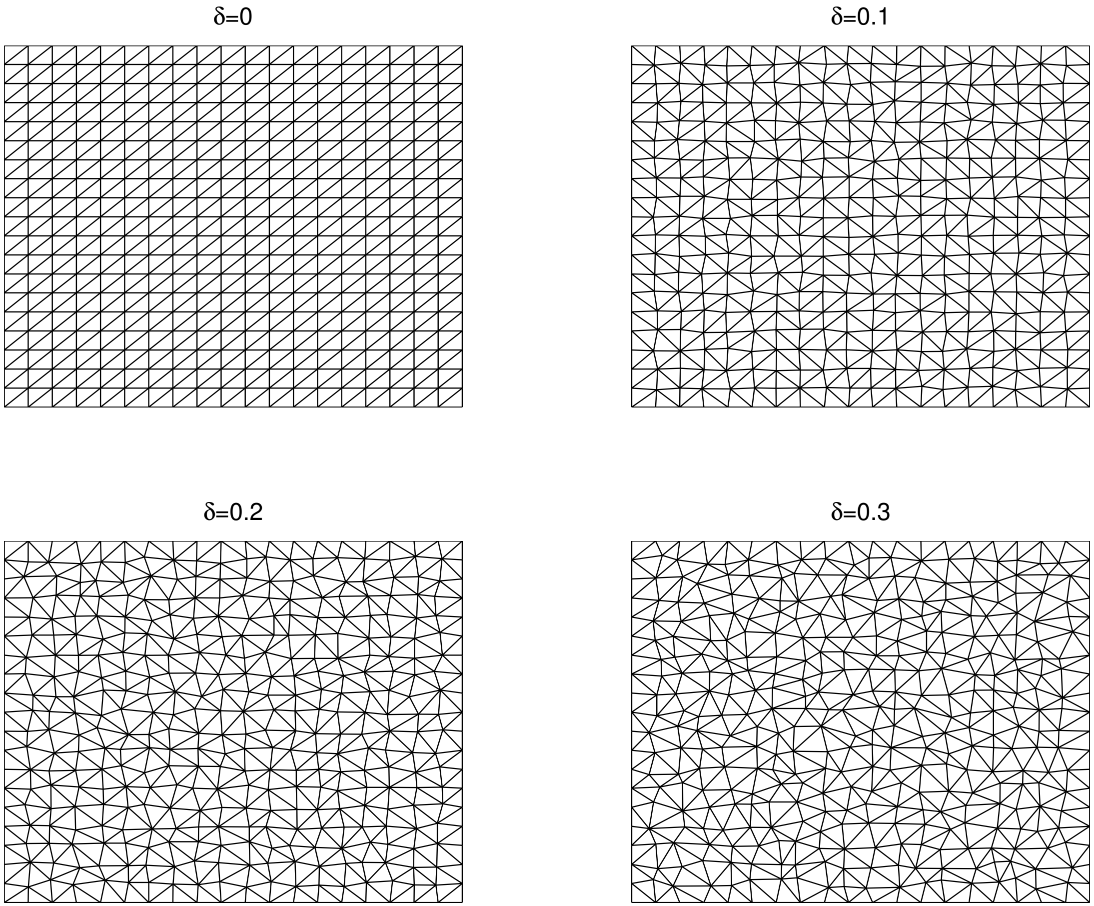
Two Dimensional Model Problem

We use a model problem from Harari with inhomogeneous Robin boundary conditions chosen such that the solution u is equal to $e^{i\mathbf{k}\cdot x}$.

$$\begin{cases} -\triangle u - k^2\, u = 0 & \text{in } \Omega, \\ -\partial_n u = -ik(u-g) & \text{on } \Gamma, \end{cases}$$



Unstructured Grid



Error Estimate

A similar calculation as in one dimension gives

$$(e, \psi_\Omega) - ik(e, \psi_\Gamma)_\Gamma = (R_\Omega(U), \phi - \pi\phi) \\ - (R_\Gamma(U), \phi - \pi\phi)_\Gamma + (\tau \mathcal{A}U, \mathcal{A}\pi\phi)_{\tilde{\Omega}},$$

again we choose τ such that $(e, \psi_\Omega) - ik(e, \psi_\Gamma)_\Gamma = 0$ i.e.

$$\tau = -\frac{(R_\Omega(U), \phi - \pi\phi) - (R_\Gamma(U), \phi - \pi\phi)_\Gamma}{(\mathcal{A}U, \mathcal{A}\pi\phi)_{\tilde{\Omega}}}$$

Two Dimensional Results

For plane waves on stochastically perturbated grids we numerically detect that

$$P\left(|(e, I_{\Gamma_o})_\Gamma| \leq C(hk)^5\right) \geq 1 - \epsilon,$$

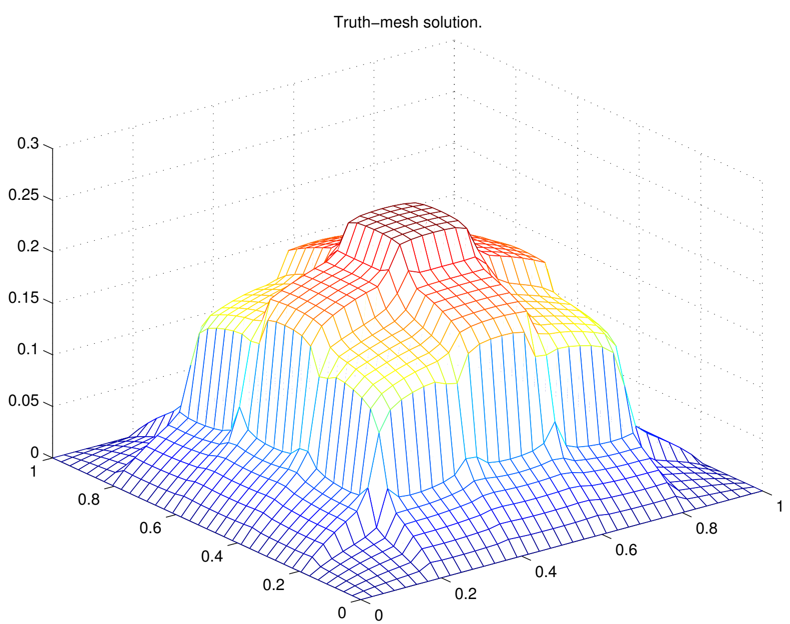
where I_{Γ_o} is the indicator function on the out flow boundary.
Again the mean is correct and this time we detect no pollution.
We also get $E[\tau] \sim 1 + C\delta^2$ both in theory and numerics.

Conclusions

- We derive a posteriori error estimates based on duality arguments for the GLS method.
- We explain how perturbations in the mesh affects the optimal numerical wavenumber.
- We note that for plane waves in two dimensions the contributions to the error seems to "even out" over the boundary so we do not get any pollution. ($\text{Var}(\int_{I_o} e) \approx \text{Var}(\sum_i^n e_i h) \sim h^2 n \text{Var}(e_i) \sim h \text{Var}(e_i)$).

Paper III

Adaptive Variational Multiscale Method Based on A Posteriori Error Estimates



The Model Problem

Poisson Equation. Find $u \in H_0^1(\Omega)$ such that

$$-\nabla \cdot a\nabla u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

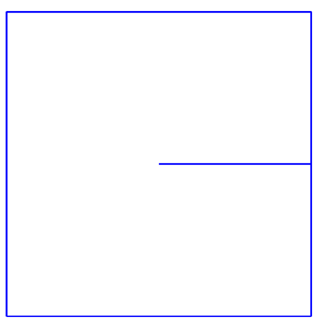
where $f \in H^{-1}(\Omega)$, $a > 0$ bounded, and Ω is a domain in $\mathbf{R}^d$, $d = 1, 2, 3$.

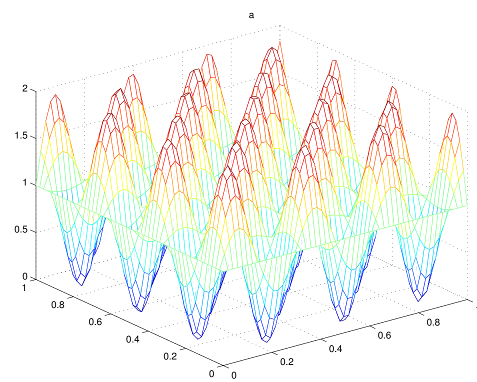
Weak form. Find $u \in H_0^1(\Omega)$ such that

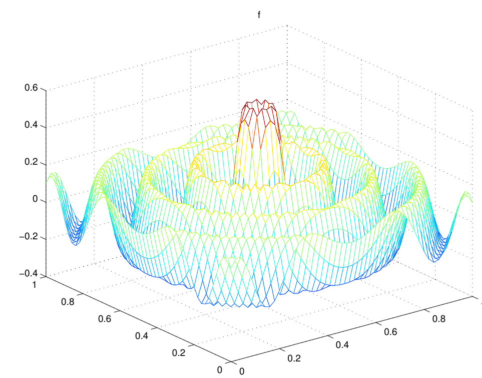
$$a(u, v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega).$$

Multiscale Problems

Below are three examples of multiscale problems.







The first one represents difficulties in the domain (cracks, holes, ...) the second one oscillations in a and the third one oscillations in f.

Motivation

- Very important applications.
- The problems are very computationally challenging so error estimation and efficient algorithms are crucial.
- Attempts on using adaptive algorithms are not common in the literature.

Variational Multiscale Method

- See for instance T.J.R. Hughes (1995).
- $H_0^1 = V_c \oplus V_f$, $u = u_c + u_f$, and $v = v_c + v_f$.

Find $u_c \in V_c$ and $u_f \in V_f$ such that

$$
\begin{aligned}
a(u_c, v_c) + a(u_f, v_c) &= (f, v_c) \quad \text{for all } v_c \in V_c, \\
a(u_f, v_f) &= (f, v_f) - a(u_c, v_f) \\
&:= (R(u_c), v_f) \quad \text{for all } v_f \in V_f.
\end{aligned}
$$

Variational Multiscale Method

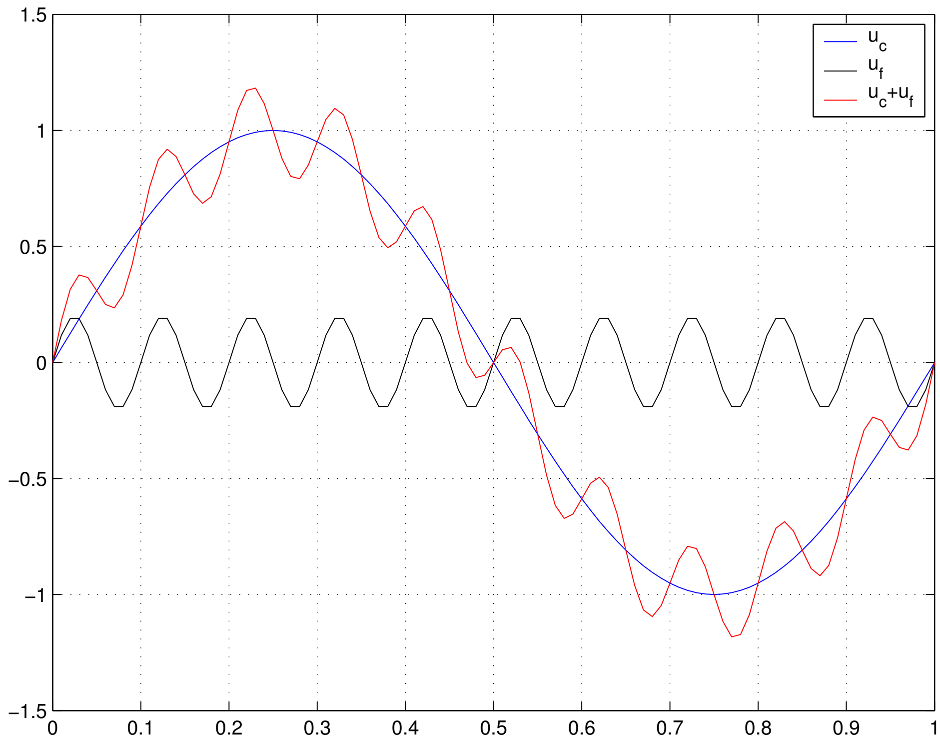


Figure 1: u_c, u_f, and $u_c + u_f$.

Variational Multiscale Method

- The fine scale is driven by the coarse scale residual.
- Approximation to fine scale solution solved on each element analytically (Green's functions).
- Fine scale information is then used to modify the coarse scale equation.

$$a(u_c, v_c) + a(\hat{A}_f^{-1} R(U_c), v_c) = (f, v_c) \quad \forall v_c \in V_c.$$

Our Basic Idea

- Discretization of V_f by (W)HB-functions (V_f^h).
- Solve localized fine scale problems for each coarse node (or some coarse nodes).
- Possibility to do this in parallel.
- A posteriori error estimation framework.
- Adaptive strategy for this setting.

Decouple Fine Scale Equations

Remember the fine scale equations:

$$a(U_f, v_f) = (R(U_c), v_f), \quad \text{for all } v_f \in V_f^h.$$

Include a partition of unity,

$$a(U_f, v_f) = (R(U_c), v_f) \quad = \sum_{i=1}^{n} (R(U_c), \varphi_i v_f),$$

let $U_f = \sum_i^n U_{f,i}$ where $a(U_{f,i}, v_f) = (R(U_c), \varphi_i v_f)$.

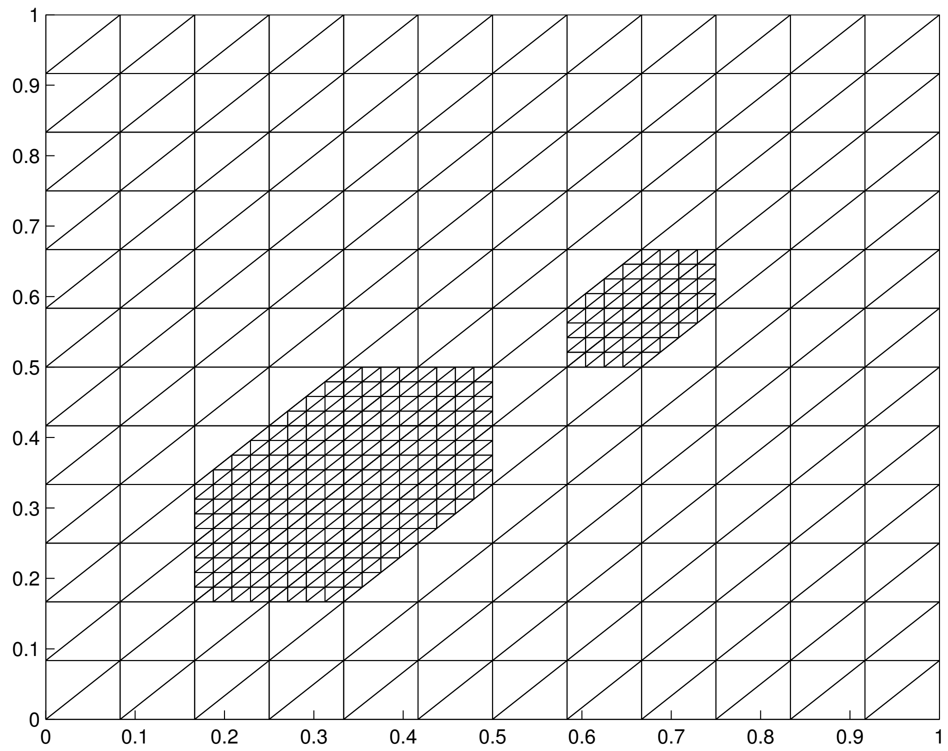
Approximate Solution

Find $U_c \in V_c$ and $U_f = \sum_i^n U_{f,i}$ where $U_{f,i} \in V_f^h(\omega_i)$ such that

$$a(U_c, v_c) + a(U_f, v_c) = (f, v_c) \quad \text{for all } v_c \in V_c,$$
$$a(U_{f,i}, v_f) = (R(U_c), \varphi_i v_f) \quad \text{for all } v_f \in V_f^h(\omega_i).$$

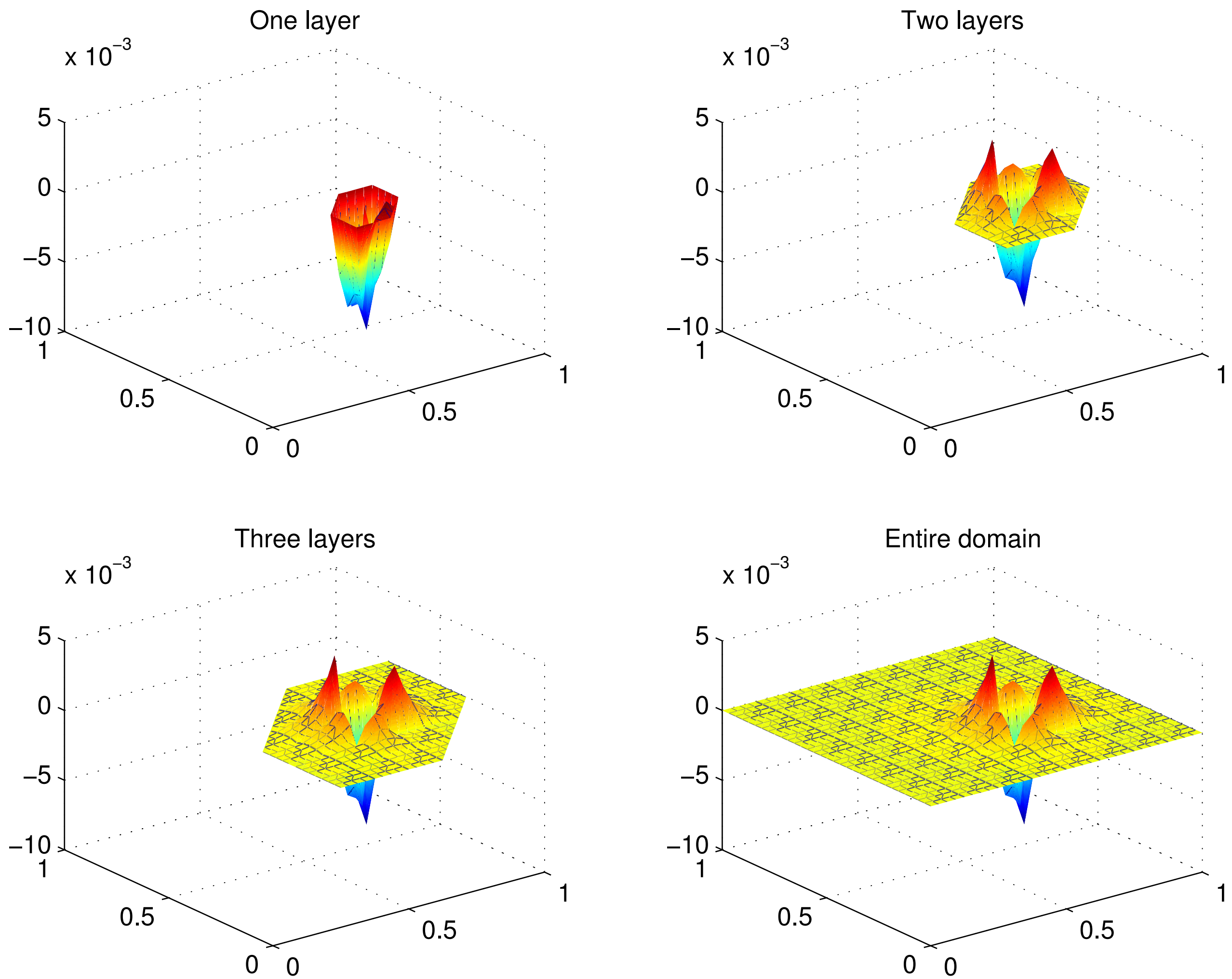
- Since φ_i has support on a star S_i^1 in node i we solve the fine scale equations approximately on ω_i with $U_{f,i} = 0$ on $\partial\omega_i$.

Refinement and Layers



One and two layer stars.

Localized Fine Scale Solution



Energy Norm Estimate

$$\|\sqrt{a}\nabla e\| \leq \sum_{i\in\mathcal{C}} C_i \|H\mathcal{R}(U_c)\|_{\omega_i}$$
$$+ \sum_{i\in\mathcal{F}} C_i \left(\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i} \right)$$

- The first term is coarse mesh error.
- The second term is the normal derivative of the fine scale solutions on $\partial\omega_i$.
- The third term is fine scale error.

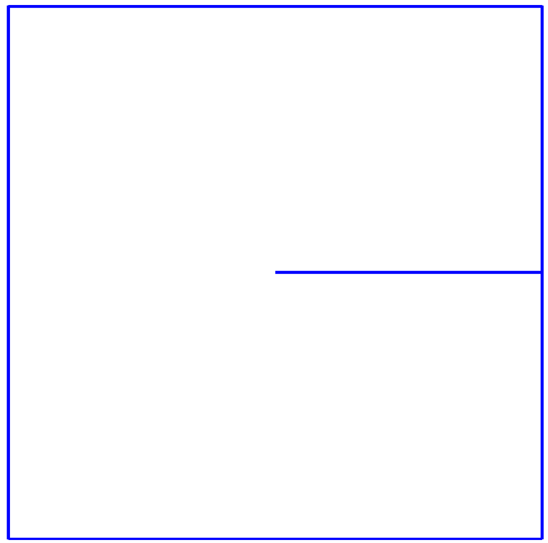
Adaptive Strategy

$$\|\sqrt{a}\nabla e\| \leq \sum_{i\in\mathcal{C}} C_i \|H\mathcal{R}(U_c)\|_{\omega_i} + \sum_{i\in\mathcal{F}} C_i \left(\|\sqrt{H}\Sigma(U_{f,i})\|_{\partial\omega_i} + \|h\mathcal{R}_i(U_{f,i})\|_{\omega_i} \right)$$

- We calculate these for each $i \in \{$coarse fine$\}$.
- Large values $i \in$ coarse $\rightarrow$ more local problems.
- Large values $i \in$ fine $\rightarrow$ more layers or smaller h.

Numerical Examples

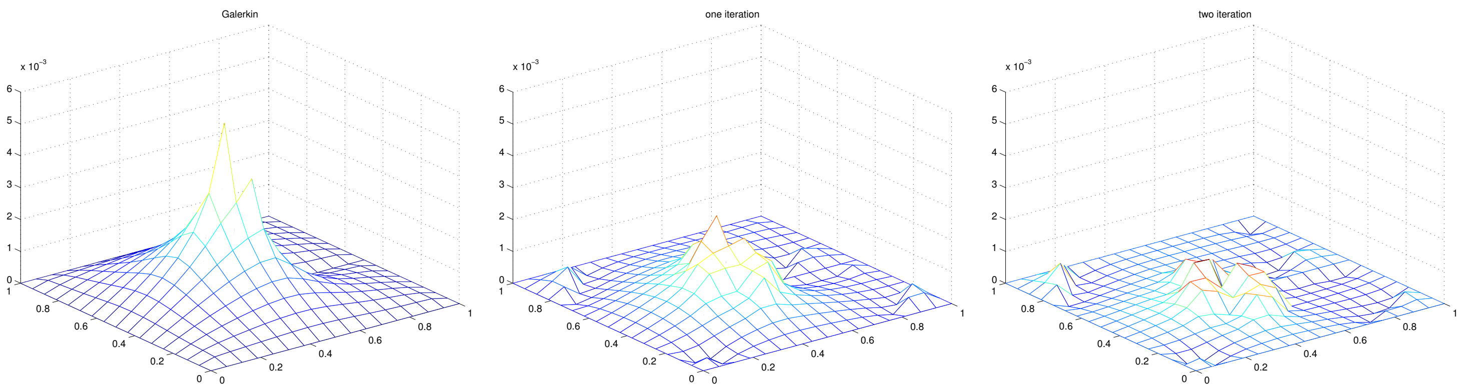
We start with a unit square containing a crack.



We let the coefficient $a = 1$ and solve, $-\triangle u = f$ with $u = 0$ on the boundary including the crack.

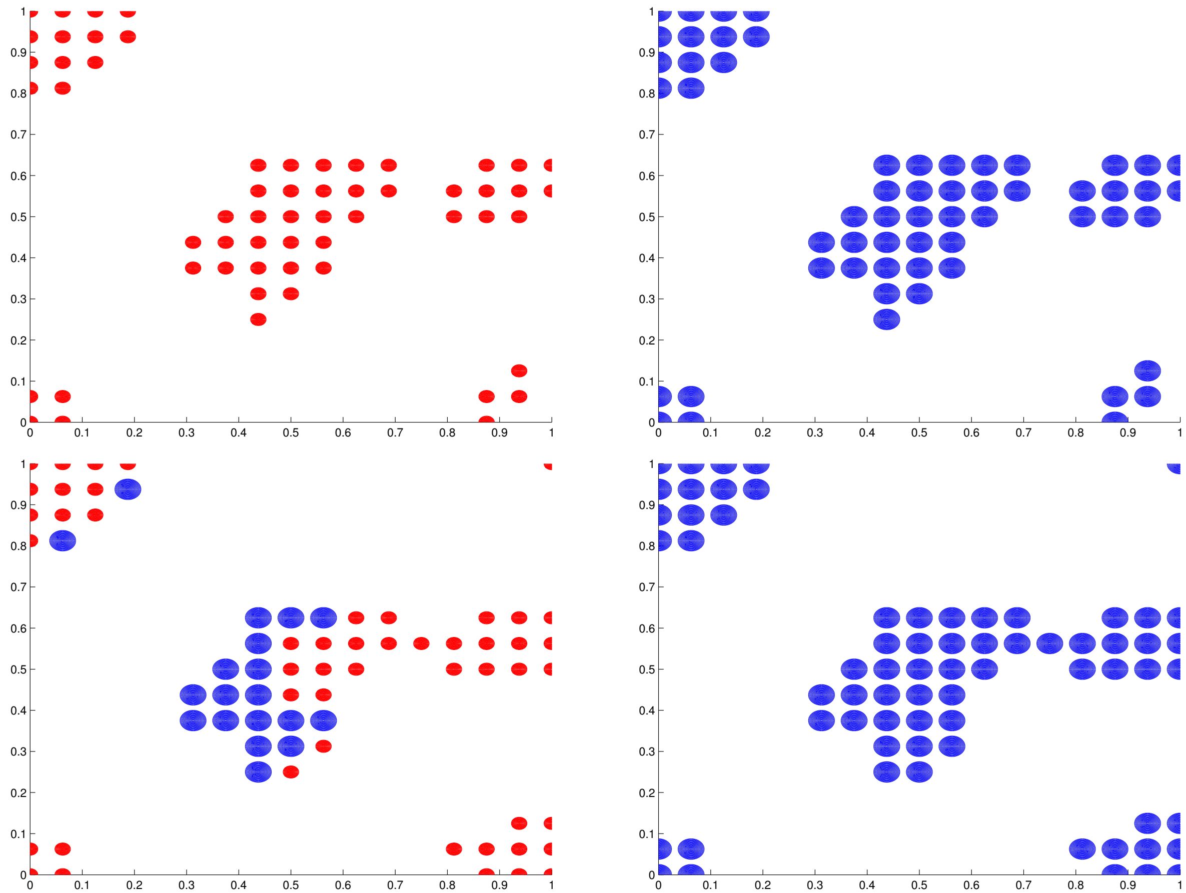
Numerical Examples

We solve the problem by using the adaptive algorithm.



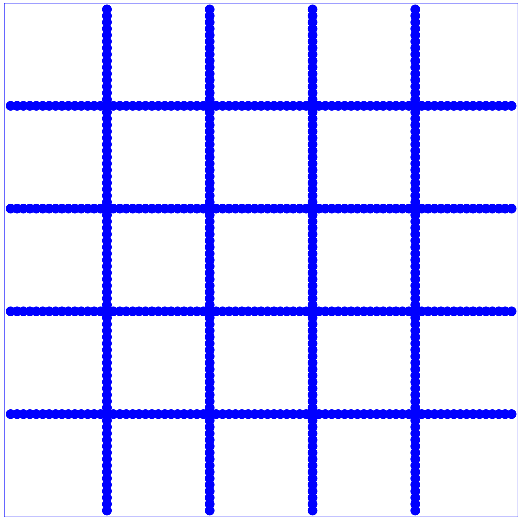
We plot the difference between our solution and a reference solution.

Numerical Examples

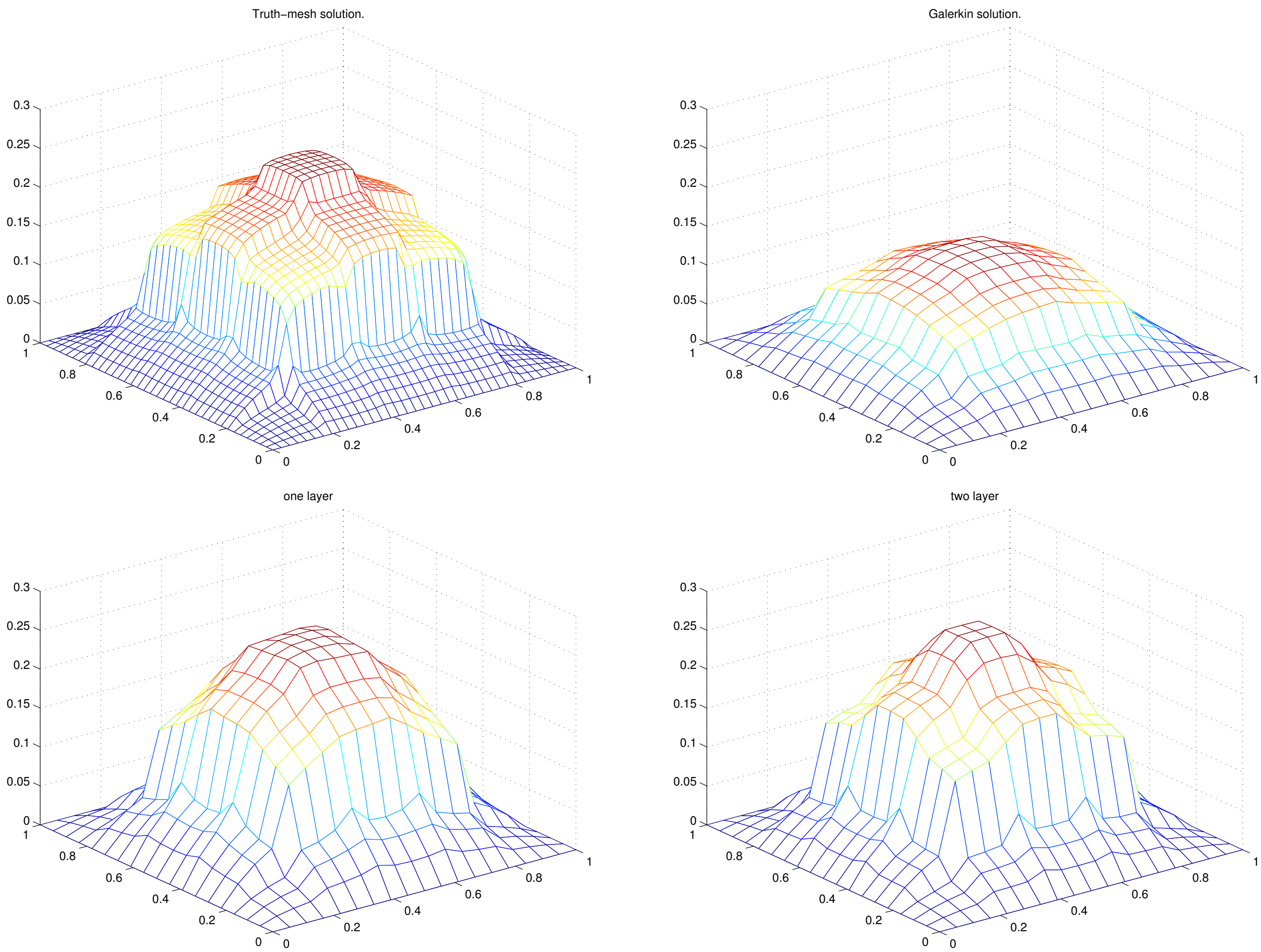


Numerical Examples

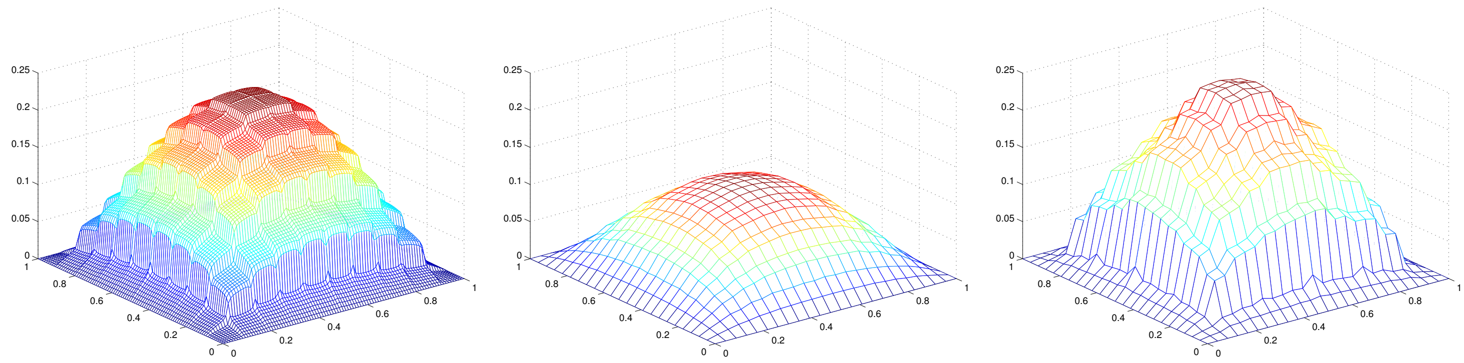
In this example we study a discontinuous coefficient a in $-\nabla \cdot a\nabla u = f$. $a = 1$ (white) and $a = 0.05$ (blue).



Numerical Examples



Numerical Examples



The number of layers seems to depend on the fine scale structure rather that the domain size.

Outlook

- Extended numerical tests in both 2D and 3D.
- Mixed formulation.
- Other equations (convection-diffusion, ...).
- More scales.
- Comparing results with classical Homogenization theory.