Splitting techniques for solving PDEs

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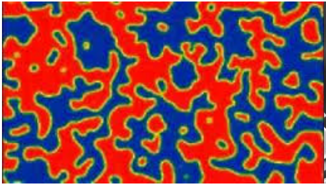
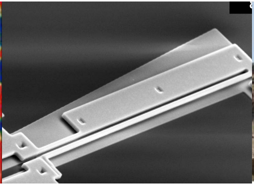
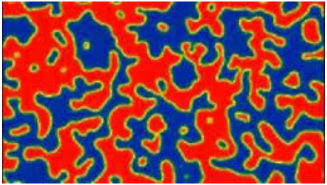
Three examples


1. A non-linear evolution problem. Splitting is used to reduce the computational effort in each time step.
2. A multiphysics problem solved using optimized single physics solvers. Splitting allows well established single physics codes to be used.
3. A multiphysics problem where the character of the involved physical processes calls for different numerical treatment. Implicit-Explicit approach is often used in porous media flow simulations.

Non-linear evolution problems



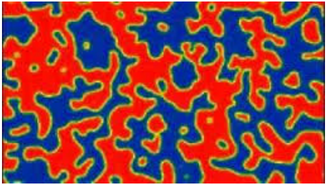
We consider the semi-linear problem,

$$\dot{y} - \Delta y = F(y), \qquad t \in (0, T]$$

one application is the bistable equation.

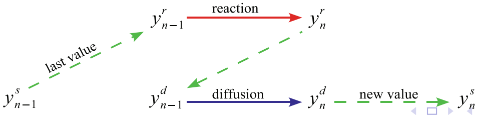
- Diffusion induces stability and changes over long time scales.
- Reaction may induce instability and changes over short time scales.

Non-linear evolution problems

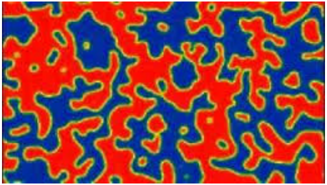


We discretize in time $0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T$.
Given $y^s_{n-1} \approx y(t_{n-1})$, on (t_{n-1}, t_n), we solve

$$\begin{cases} \dot{y}^r = F(y^r) \\ y^{r+}_{n-1} = y^s_{n-1} \end{cases} \qquad \begin{cases} \dot{y}^d = \Delta y^d \\ y^{d+}_{n-1} = y^{r-}_n \end{cases} \qquad y^s_n = y^{d-}_n$$

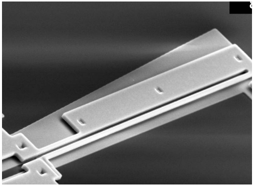


Non-linear evolution problems



- An advantage is that the reaction equation do not contain spatial derivatives and can be solved cheaply.
- The diffusion equation is the standard heat equation for which very efficient methods exists.
- We can allow different time steps for the different subproblems.
- However, error analysis is crucial since the splitting can induce numerical instability.

Multiphysics problems

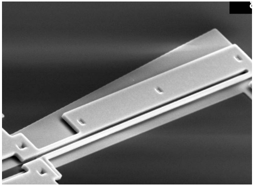


We consider temperature (T), electric potential (Φ), and displacement ($\mathbf{u}$) in a micro-electro mechanical system,

$$\begin{cases} \dot{T} - \Delta T &= \kappa(T)|\nabla\Phi|^2 - \nabla \cdot \dot{\mathbf{u}} \\ -\nabla \cdot \kappa(T)\nabla\Phi &= 0 \\ \ddot{\mathbf{u}} - \nabla \cdot (\epsilon(\mathbf{u}) - \alpha T\mathbf{I}) &= \mathbf{f}. \end{cases}$$

The equations are solved with different optimized single physics solvers. This approach naturally leads to splitting i.e. the single physics simulations are done sequentially in each time step.

Multiphysics problems



- Plenty of error sources in addition to splitting, discretization in time, space, error in data transfer between solvers (possibly different grids, methods), ...
- Errors may trigger numerical instability which destroys convergence.
- Error analysis is very challenging, already existence and uniqueness of solution and regularity of solutions is difficult.
- This situation is typical in applications.

Multiphysics problems

We consider a simplified model for secondary oil recovery

$$\begin{cases} -\nabla \cdot v & = & 0 \\ v & = & \kappa\lambda(s)\nabla p \\ \dot{s} - v \cdot \nabla f(s) & = & 0 \end{cases}$$

where s is concentration of water ($1 - s$ is oil), p global pressure, κ permeability, λ mobility, and v velocity.

Its solved using (IMPES) IMplicit Pressure Explicit Saturation.

Multiphysics problems

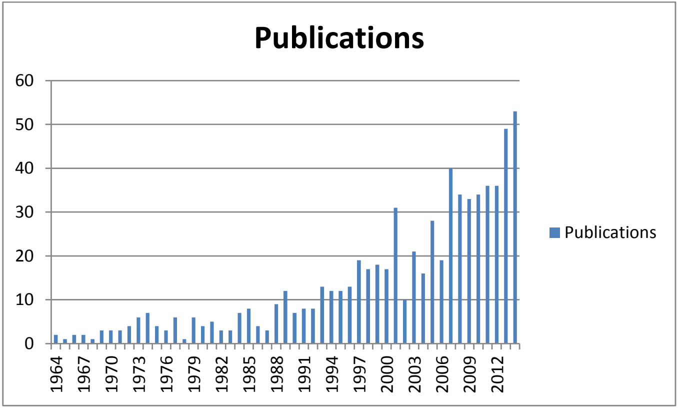
We consider a simplified model for secondary oil recovery

$$\begin{cases} -\nabla \cdot v &= 0 \\ v &= \kappa\lambda(s)\nabla p \\ \dot{s} - v \cdot \nabla f(s) &= 0 \end{cases}$$

- The elliptic pressure equation is solved implicitly.
- The hyperbolic saturation equation is solved explicitly.
- This simplifies the computations but the effect on the approximation error is important to analyze.

Publications

MathSciNet search on ”splitting schemes” or ”splitting methods”.



First entry was in 1964 we see an increase in the activity the last eight years with major contributions from Lund University.

Conclusion and Erik's contribution

- Splitting schemes has been used for a long time and the interest in the scientific community has increased over the years, in particular the last ten years.
- Splitting schemes are very useful and often necessary in real applications (multi-physics).
- The mathematical analysis of the effect of splitting methods is difficult since many error sources are present at the same time.
- Erik's contribution is a theoretical framework for analyzing the error due to splitting for a large class of problems, e.g. my first example (Paper I).
- An analysis of the combined effect of splitting an discretization error (Paper II).
- A real application where splitting reduces the computational work (Paper III + Chapter 3).